An introduction to noncommutative topology

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ABSTRACT. C*–algebras provide an elegant setting for many problems in mathematics and physics. In view of Gelfand duality, their study is often referred to as *noncommutative topology*: general noncommutative C*–algebras are interpreted as noncommutative spaces. These form an established research field within mathematics, with applications to quantum theory and other areas where deformations play a role. Many classical geometric and topological concepts can be translated into operator algebraic terms, leading to the so-called noncommutative geometry (NCG) dictionary.

These lectures aim to provide the participants with the tools to understand, consult and use the NCG dictionary, covering the basic theory of C*-algebra and their modules. Focus will be given on examples, especially those that come from deformation theory and quantisation.

Contents

CHAPTER 1

An introduction to commutative C*-algebras

Motivation

Over the past decades, the term *noncommutative topology* has come to indicate the study of C*-algebras. In this lecture series, I have tried to stay as closed as possible to the original motivation: I will make use of many results and definition stemming from the theory of C*-algebras, the focus will always be on the *topological* aspects. In the spirit of Connes' noncommutative geometry, I will think of a C*-algebra as some sort of generalised topological space.

The goal of this lecture series is to provide the reader with some basic tools to understand the so-called *NCG dictionary*, or at least its topological entries:

operator algebra	**topology**
commutative C* algebra	locally compact Hausdorff space
unitality	compactness
* homomorphisms	continuous proper functions
* automorphisms	homeomorphisms
separability	metrizability
finitely generated projective module	vector bundle
equivalence bimodule	line bundle
f.g.p. C*-module	Hermitian vector bundle
self-Morita equivalence	Hermitian line bundle
quantum group coaction	Lie group action
Hopf–Galois extension	principal bundle

1. Trading spaces for algebras

Let X be a compact, Hausdorff topological space. The set of continuous functions $X \to \mathbb{C}$ (with respect to the standard topology on the set of complex numbers $\mathbb{C}$) will be denoted by $C(X)$. We are interested in the properties of this set, which, as we will see, gives a complete algebraic characterisation of the underlying topological space.

(1) It inherits a complex vector space structure from $\mathbb{C}$: for all $f, g \in C(X)$, and for every $\lambda \in C$ we can define the function $f + \lambda g$ as

$$(f + \lambda g)(x) = f(x) + \lambda g(x).$$

(2) It is a *commutative, unital algebra*:

(a) multiplication between functions is defined point-wise, namely $(fg)(x) := f(x)g(x)$ for all $x \in X$, and this defines a commutative product, namely $fg = gf$ for all $f, g \in C(X)$;
(b) the constant function $1(x) \equiv 1$ is the unit element for this multiplication;
(c) the product is compatible with the vector space structure;

(3) There is a natural *norm* on the algebra $C(X)$

$$\|f\| := \sup_{x \in X} |f(x)|, \tag{1}$$

and the norm is compatible with the algebra structure, i.e. $\|fg\| \leq \|f\| \, \|g\|$. This makes $(C(X), \|\cdot\|)$ into a *normed algebra*.

(4) $C(X)$ is *complete* with respect to the norm defined in (1), i.e. every Cauchy sequence in $C(X)$ converges in the topology defined by the norm (1). One says that $(C(X), \|\cdot\|)$ is a Banach algebra.

(5) The Banach algebra $C(X)$ possesses a $*$-*structure*, namely an antilinear map $^* \colon C(X) \to C(X)$ such $(f^*)^* = f$ (* is thus called an antilinear *involution*): this is defined by $f^*(x) := \overline{f(x)}$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. It is compatible with the $*$-structure: $\|f^*\| = \|f\|$.

(6) The norm satisfies the *C*-property*, namely

$$\|f^* f\| = \|f\|^2 \quad \text{for all } f \in C(X).$$

Indeed

$$\|f^* f\| = \sup_{x \in X} |\overline{f(x)} f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|^2.$$

The above list of properties can be summarized by saying that $C(X)$ is a *commutative unital C*-algebra*

Definition 1.1. *An (abstract) C*-algebra is a Banach algebra $(A, \|\cdot\|)$ with the property that*

$$\|a^* a\| = \|a\|^2 \tag{2}$$

for every $a \in A$.

The C*-property (2) is crucial, as it ties together analytical and algebraic properties and has profound implications, giving C*-algebras a very special place within the zoo of Banach algebras. The most striking one, perhaps, is that, as we will see, the norm of a (commutative) C*-algebra can be given in purely algebraic terms.

Our goal is to prove that, actually, all commutative C*-algebras are of the form $C(X)$ for some compact Hausdorff space X:

Theorem 1.1 (Gel'fand–Naĭmark). *Let A be a commutative, unital C*-algebra. Then there exists a compact Hausdorff space $X = \sigma(A)$, called the* spectrum of the algebra A, *such that A is* **isometrically $*$-isomorphic** *to the algebra $C(\sigma(A))$ of continuous functions on $\sigma(A)$.*

Exercise 1.1. *Show that the same definitions as above make the space*

$$C_b(X) := \left\{ f \in C(X) \;\middle|\; \sup_{x \in X} |f(x)| < \infty \right\}$$

of bounded continuous functions on X into a unital, commutative C-algebra.*

1.1. Spectral theory for Banach algebras. In this Section, we recall some basic facts from the spectral theory of Banach algebras.

1.1.1. *Invertibility and the spectrum.* Recall that an element a of a unital algebra (or ring) A is called *invertible* if there is an element $b \in A$ such that $ab = ba = 1$.

EXERCISE 1.2. *Let A be a unital ring.*

(1) *Prove that if $a \in A$ is invertible, then there is a* unique $b \in A$ *satisfying* $ab = ba = 1$. *This b is usually denoted a^{-1}.*
(2) *Given $a \in A$, suppose that there exist elements $b, c \in A$ such that $ab = ca = 1$. Prove that $b = c$ (and hence that a is invertible).*
(3) *Show that if a and b are invertible, then ab is invertible with $(ab)^{-1} = b^{-1}a^{-1}$. (So the set of invertible elements of A forms a group under multiplication.)*
(4) *Show that if a and b commute (i.e., $ab = ba$), then ab is invertible if and only if both a and b are invertible.*
(5) *Find an example of a unital C^*-algebra A and elements $a, b \in A$ such that $ab = 1$ but a and b are not invertible.*
(6) *Now suppose that A is a C^*-algebra, and that $a \in A$ is invertible. Show that a^* is invertible, with $(a^*)^{-1} = (a^{-1})^*$.*

DEFINITION 1.2. *Let A be a Banach algebra with unit element 1. The* spectrum *of an element $a \in A$ is the set*

$$\sigma(a) := \{z \in \mathbb{C} \mid z1 - a \textit{ is not invertible in } A\}.$$

The resolvent *of a is the complement of the spectrum:*

$$\mathbb{C} \setminus \sigma(a) = \{z \in \mathbb{C} \mid z1 - a \text{ is } \textit{invertible in } A\},$$

and the resolvent function *for a is the function*

$$R_a : \mathbb{C} \setminus \sigma(a) \to A, \qquad R_a(z) := (z1 - a)^{-1}.$$

The notion of spectrum generalises both the range of a function, and the set of eigenvalues of a matrix.

EXERCISE 1.3. *Consider the C^*-algebra $C(X)$ of continuous functions on a compact Hausdorff space X. Prove that for each $f \in C(X)$ one has*

$$\sigma(f) = \mathrm{range}(f) = \{f(x) \in \mathbb{C} \mid x \in X\}.$$

EXERCISE 1.4. *Consider the C^*-algebra $M_n(\mathbb{C})$ of $n \times n$ matrices. Prove that for each $a \in M_n(\mathbb{C})$ one has*

$$\sigma(a) = \{\textit{eigenvalues of } a\}.$$

EXERCISE 1.5. *Show that in a unital C^*-algebra one has $\sigma(a^*) = \overline{\sigma(a)} = \{\overline{z} \mid z \in \sigma(a)\}$.*

EXERCISE 1.6 (Properties of the Spectrum). *Let A be a unital Banach algebra.*

(1) *Show that for all $a, b \in A$ one has*

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}.$$

Hint: *prove that if $z1 - ba$ is invertible, then $(1 + a(z1 - ba)^{-1}b)(z1 - ab) = z1$.*
(2) *Find an example where $\sigma(ab) \neq \sigma(ba)$.*
(3) *Show that for each $w \in \mathbb{C}$ and each $a \in A$ one has $\sigma(w1 - a) = w - \sigma(a)$, where the right-hand side is defined to be the set $\{w - z \mid z \in \sigma(a)\}$.*

(4) *Show that for each polynomial* $p = \sum_{i=0}^{n} p_i x^i \in \mathbb{C}[x]$ *and each* $a \in A$ *one has*

$$\sigma(p(a)) = p(\sigma(a))$$

where we define $p(a) = \sum_{i=0}^{n} p_i a^i$*, and* $p(\sigma(a)) = \{p(z) \in \mathbb{C} \mid z \in \sigma(a)\}$*. This equality is (an example of) the* spectral mapping property*. (*Hint*: write the polynomial* $z1 - p$ *as a product of linear factors.)*

Exercise 1.7. *Let* $\phi : A \to B$ *be a homomorphism of unital Banach algebras (i.e., a linear map satisfying* $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$ *for all* $a_1, a_2 \in A$*, and* $\phi(1_A) = 1_B$*). Prove that* $\sigma(\phi(a)) \subseteq \sigma(a)$ *for every* $a \in A$*.*

We shall need the following fact about convergence of series in Banach spaces (which you might have seen in a course on functional analysis).

Proposition 1.2. *Let* x_n *be a sequence of elements of a Banach space* X *such that the series* $\sum_{n=0}^{\infty} \|x_n\|$ *converges in* $\mathbb{R}$*. Show that the series* $\sum_{n=0}^{\infty} x_n$ *converges in* X*.*

Lemma 1.3. *Let* A *be a unital Banach algebra, and let* $a \in A$ *be an element with* $\|1 - a\| < 1$*. Then* a *is invertible, with*

$$a^{-1} = \sum_{n=0}^{\infty} (1 - a)^n.$$

(As usual, we define $x^0 \coloneqq 1$ for all $x \in A$.)

Proof. Since A is a Banach algebra we have $\|(1-a)^n\| \le \|1-a\|^n$ for every $n \ge 0$. Since $\|1-a\| < 1$, comparison with a geometric series shows that the series $\sum_{n=0}^{\infty} \|(1-a)^n\|$ converges. Exercise 1.2 then implies that the series $\sum_{n=0}^{\infty} (1-a)^n$ converges to some element $b \in A$.

Since multiplication in a Banach algebra is continuous, we have

$$\begin{aligned} ab &= (1 - (1-a))b \\ &= (1 - (1-a)) \cdot \lim_{N\to\infty} \left(1 + (1-a) + (1-a)^2 + \cdots + (1-a)^N\right) \\ &= \lim_{N\to\infty} (1 - (1-a))(1 + (1-a) + (1-a)^2 + \cdots + (1-a)^N) \\ &= \lim_{N\to\infty} (1 - (1-a)^{N+1}) \\ &= 1 \end{aligned}$$

(because $\|(1-a)^N\| \le \|1-a\|^N \to 0$). A similar computation, left to the reader, shows that $ba = 1$, and so a is invertible with $a^{-1} = b$. □

Corollary 1.4. *Let* A *be a unital Banach algebra. Suppose that* $a \in A$ *is invertible, and that* $b \in A$ *satisfies* $\|a - b\| < \|a^{-1}\|^{-1}$*. Then* b *is invertible.*

Proof. We have

$$\|1 - a^{-1}b\| = \|a^{-1}(a-b)\| \le \|a^{-1}\| \cdot \|a - b\| < 1$$

by our assumption on $\|a - b\|$, and so $a^{-1}b$ is invertible by Lemma 1.3. Thus the product $b = a(a^{-1}b)$ is invertible by Exercise 1.2. □

Exercise 1.8. *Use Corollary 1.4 to prove that the set of invertible elements in a unital Banach algebra* A *is an open subset of* A *(in the norm topology); and that for each* $a \in A$ *the spectrum* $\sigma(a)$ *is a closed subset of* $\mathbb{C}$*.*

LEMMA 1.5. *Let A be a unital Banach algebra. For each $a \in A$, and each complex number $w \in \mathbb{C} \setminus \sigma(a)$, the resolvent function R_a is given on the disk $\{z \in \mathbb{C} \mid |w - z| < \|R_a(w)\|^{-1}\}$ by the convergent power series*

$$R_a(z) = \sum_{n=0}^{\infty} (w - z)^n R_a(w)^{n+1}.$$

We are going to use these facts (Lemmas 1.3 and 1.5) to prove:

THEOREM 1.6. *For each element a of a unital Banach algebra A, the spectrum $\sigma(a)$ is a nonempty, compact subset of $\mathbb{C}$.*

PROOF. We first show that $\sigma(a)$ is compact. If $z \in \mathbb{C}$ has $|z| > \|a\|$, then

$$\|1 - (1 - z^{-1}a)\| = \|z^{-1}a\| = |z|^{-1}\|a\| < 1,$$

and so Lemma 1.7 implies that $1 - z^{-1}a$ is invertible, with

$$(1 - z^{-1}a)^{-1} = \sum_{n=0}^{\infty} z^{-n}a^n.$$

Therefore $z(1 - z^{-1}a) = z1 - a$ is invertible too, with

$$R_a(z) = (z1 - a)^{-1} = \sum_{n=0}^{\infty} z^{-n-1}a^n. \tag{3}$$

This shows that

$$\sigma(a) \subseteq \{z \in \mathbb{C} \mid |z| \le \|a\|\},$$

and so $\sigma(a)$ is a bounded subset of $\mathbb{C}$. Since the spectrum is also closed, as we saw in the last lecture, it is compact (by the Heine–Borel theorem).

It remains to show that $\sigma(a)$ is nonempty. Suppose that $\sigma(a)$ is empty, so that $z1 - a$ is invertible for every $z \in \mathbb{C}$. We are going to arrive at a contradiction to Liouville's theorem, which asserts that the only bounded analytic functions on $\mathbb{C}$ are the constants. (Compare with the proof that every $a \in M_n(\mathbb{C})$ has an eigenvalue: one uses Liouville's theorem to prove that every complex polynomial has a root.)

The element $R_a(0) \in A$ is invertible, hence nonzero, and so the Hahn-Banach theorem implies that there is a bounded linear functional $\phi \in A^*$ for which the function

$$F_\phi : \mathbb{C} \to \mathbb{C}, \qquad F_\phi(z) := \phi\left(R_a(z)\right)$$

has $F_\phi(0) \neq 0$. Since ϕ is linear and continuous, Lemma 6 from Monday 4/9 implies that F_ϕ is given in a neighbourhood of each point $w \in \mathbb{C}$ by a convergent power series, and is thus an analytic function. On the other hand, (3) implies that for $|z| > \|a\|$ we have

$$\|R_a(z)\| \le \sum_{n=0}^{\infty} |z|^{-n-1}\|a\|^n = \frac{1}{|z| - \|a\|} \to 0 \quad \text{as} \quad |z| \to \infty,$$

and so

$$|F_\phi(z)| \le \|\phi\|\|R_a(z)\| \to 0 \quad \text{as} \quad |z| \to \infty.$$

Now Liouville's theorem implies that $F_\phi(z) = 0$ for every $z \in \mathbb{C}$, whereas we chose ϕ in such a way that $F_\phi(0) \neq 0$. Having arrived at a contradiction, we conclude that $\sigma(a)$ is nonempty. □

COROLLARY 1.7 (Gelfand–Mazur). *Let A be a unital Banach algebra in which every nonzero element is invertible. Then $A \cong \mathbb{C}$.*

Proof. Exercise. □

Definition 1.3. *The* spectral radius $\mathrm{radius}(a)$ *of an element a of a unital Banach algebra A is defined to be the number*

$$\mathrm{radius}(a) := \sup_{z \in \sigma(a)} |z|.$$

Exercise 1.9. *Prove that* $\mathrm{radius}(a^n) = \mathrm{radius}(a)^n$ *for every* $n \geq 0$.

We saw in the proof of Theorem 1.6 that

$$\mathrm{radius}(a) \leq \|a\| \tag{4}$$

for every a. This inequality can be strict:

Exercise 1.10. *Find an example of a unital Banach algebra A and an element $a \in A$ such that* $\mathrm{radius}(a) \neq \|a\|$.

Nevertheless, the spectral radius can be expressed in terms of the norm:

Lemma 1.8. *Let A be a unital Banach algebra. For each $a \in A$ one has*

$$\mathrm{radius}(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

For C^*-algebras, we have the following improvement of the spectral radius formula:

Proposition 1.9. *If A is a C^*-algebra and $a \in A$ is* self-adjoint *(i.e., $a = a^*$), then we have* $\mathrm{radius}(a) = \|a\|$.

Proof. If b is any self-adjoint element of A then $\|b^2\| = \|b^*b\| = \|b\|^2$. Applying this inductively shows that $\|a^{2^n}\| = \|a\|^{2^n}$ for every $n \geq 0$. Thus the sequence $\|a^n\|^{1/n}$, whose limit is $\mathrm{radius}(a)$, has a constant subsequence $\|a^{2^n}\|^{1/2^n} = \|a\|$. Thus $\mathrm{radius}(a) = \|a\|$. □

Exercise 1.11. *Let A be a unital C^*-algebra.*

(1) *Prove that if $a \in A$ is* normal *(i.e., $aa^* = a^*a$), then* $\mathrm{radius}(a) = \|a\|$.
(2) *Prove that for every $a \in A$ one has* $\|a\| = \mathrm{radius}(a^*a)^{1/2}$.

This fact provides our first illustration of the way that the algebraic structure, the involution, and the norm in a C^*-algebra are all intimately linked. In particular, the norm is completely determined by the $*$-algebra structure!

1.2. The Gelfand transform and commutative C^*-algebras. Our goal is to realise any commutative unital C^*-algebra as the algebra of continuous functions on some compact Hausdorff space. The first step is to extract a compact Hausdorff space from any commutative unital Banach algebra.

Definition 1.4. *A* multiplicative linear functional *on a commutative Banach algebra A is a* nonzero *algebra homomorphism $\phi : A \to \mathbb{C}$: that is, a nonzero linear map satisfying $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$. The set of all multiplicative linear functionals on A is denoted $\sigma(A)$.*

Exercise 1.12. *Show that $\phi(1_A) = 1_{\mathbb{C}}$ for every $\phi \in \sigma(A)$.*

Lemma 1.10. *Let A be a unital commutative Banach algebra. Then every multiplicative linear functional on A is bounded with norm 1.*

Proof. If $a \in A$ is such that $|\phi(a)| > \|a\|$, then the element $1 - \phi(a)^{-1}a \in A$ is invertible, by Lemma 1.3; let b be its inverse. We then have

$$1 = \phi(1) = \phi((1 - \phi(a)^{-1}a)b) = \big(\phi(1) - \phi(a)^{-1}\phi(a)\big)\,\phi(b) = 0,$$

and this contradiction implies that we must have $|\phi(a)| \leq \|a\|$ for every $a \in A$. Thus ϕ is bounded with $\|\phi\| \leq 1$. Since $|\phi(1)| = 1 = \|1\|$ we in fact have $\|\phi\| = 1$. □

Recall that the *weak* topology* on A^* (or on the dual of any Banach space) is defined by declaring that a net $(\phi_\lambda) \subset A^*$ converges to $\phi \in A^*$ if and only if the net of complex numbers $(\phi_\lambda(a))$ converges to $\phi(a)$ for every $a \in A$.

Exercise 1.13. *Suppose that a net $(\phi_\lambda) \subset \sigma(A)$ of multiplicative linear functionals converges, in the weak*-topology on A^*, to some bounded linear functional $\phi \in A^*$. Show that ϕ is itself a multiplicative linear functional.*

Corollary 1.11. *If A is a unital commutative Banach algebra then $\sigma(A)$ is compact and Hausdorff in the weak* topology.*

Proof. Lemma 1.10 asserts that $\sigma(A)$ is a subset of the unit ball in the Banach space A^*, while Exercise 1.13 asserts that $\sigma(A) \subset A^*$ is closed in the weak* topology. The unit ball in A^* is compact and Hausdorff in the weak* topology, by the Banach-Alaoglu theorem. Thus the closed subset $\sigma(A)$ is compact and Hausdorff as well. □

So we have successfully associated a compact Hausdorff space to each commutative unital Banach algebra. The next step is to realise the given algebra as an algebra of continuous functions on the space.

Definition 1.5. *Let A be a commutative unital Banach algebra. The* Gelfand transform *for A is the map*

$$\Gamma : A \to C(\sigma(A)), \qquad \Gamma(a) : \phi \mapsto \phi(a).$$

Exercise 1.14. *Show that the Gelfand transform Γ is well-defined (i.e., that $\Gamma(a)$ is a* continuous *function on $\sigma(A)$ for every $a \in A$), and that Γ is a homomorphism of unital algebras (i.e., Γ is linear, $\Gamma(ab) = \Gamma(a)\Gamma(b)$ for all $a, b \in A$, and $\Gamma(1) = 1$).*

For general Banach algebras, the Gelfand transform might fail to be injective and/or surjective (see exercises). But for C^*-algebras:

Theorem 1.12 (Gelfand-Naimark). *If A is a commutative unital C^*-algebra, then the Gelfand transform is an isometric $*$-isomorphism $\Gamma : A \xrightarrow{\cong} C(\sigma(A))$.*

Lemma 1.13. *The Gelfand transform for a commutative unital Banach algebra A preserves spectra:*

$$\sigma(a) = \sigma(\Gamma(a)) = \{\phi(a) \mid \phi \in \sigma(A)\}.$$

for every $a \in A$.

Proof. First notice that our computation of spectra in algebras of the form $C(X)$ (see corresponding Exercise) immediately gives

$$\sigma(\Gamma(a)) = \mathrm{image}(\Gamma(a)) = \{\phi(a) \mid \phi \in \sigma(A)\}.$$

Since Γ is a unital homomorphism, we get the inclusion $\sigma(\Gamma(a)) \subseteq \sigma(a)$. So we are left to show that $\sigma(a) \subseteq \sigma(\Gamma(a))$.

Suppose then that $z \in \sigma(a)$, so that $z1 - a$ is not invertible in A. Then $z1 - a$ is contained in a maximal ideal J of A by a standard Zorn's lemma argument [1]. Hence, by the bijection between spectrum and maximal ideals, there is a $\phi \in \sigma(A)$ with $z - \phi(a) = \phi(z1 - a) = 0$. Then $\Gamma(a)(\phi) = z$, showing that $z \in \text{image}(\Gamma(a)) = \sigma(\Gamma(a))$. □

For general Banach algebras, the Gelfand transform might fail to be injective and/or surjective.

EXERCISE 1.15. *Let $A \subset M_2(\mathbb{C})$ be the Banach *-algebra*

$$A := \left\{ \begin{bmatrix} w & z \\ 0 & w \end{bmatrix} \middle| w, z \in \mathbb{C} \right\}.$$

Show that the Gel'fand transform $\Gamma : A \to C(\sigma(A))$ is not injective.

From now on we are going to focus on C^*-algebras, where the situation is much better.

Recall from Proposition 1.9 that for A a C^*-algebra and $a \in A$ is self-adjoint, we have

$$\text{radius}(a) = \|a\|. \tag{5}$$

Morevover, that if a is not self-adjoint we have

$$\|a\|^2 = \text{radius}(a^*a).$$

As a consequence, we have uniqueness of the C*-norm on a C*-algebra.

The following easy exercise will be useful in the proof of the next lemma, which asserts that if A is a C^*-algebra, then every algebra homomorphism $A \to \mathbb{C}$ is automatically a $*$-homomorphism.

EXERCISE 1.16. *Show that every element in a C^*-algebra can be written as $a = a_1 + ia_2$ with $a_i \in A$ self-adjoint.*

LEMMA 1.14. *Let A be a unital C^*-algebra, and let $\phi : A \to \mathbb{C}$ be an algebra homomorphism. Then $\phi(a^*) = \overline{\phi(a)}$ for every $a \in A$.*

PROOF. We first let a be a *self-adjoint* element of A (i.e., $a = a^*$), and write $\phi(a) = x + iy$ (for $x, y \in \mathbb{R}$). For each $t \in \mathbb{R}$ we use the fact that linear functionals are bounded with norm one and the C^*-identity to estimate

$$\begin{aligned} x^2 + (y+t)^2 &= |x + iy + it|^2 = |\phi(a + it1)|^2 \leq \|a + it1\|^2 \\ &= \|(a+it1)^*(a+it1)\| = \|(a - it1)(a + it1)\| = \|a^2 + t^2 1\| \\ &\leq \|a\|^2 + t^2. \end{aligned}$$

Subtracting t^2 from the first and the last expressions gives the inequality

$$x^2 + y^2 + 2ty \leq \|a\|^2$$

for every $t \in \mathbb{R}$. If $y \neq 0$ then the left-hand side can be made arbitrarily large by judicious choice of t, so we must have $y = 0$. Thus $\phi(a) = x$ is real for every self-adjoint element $a \in A$. Now if $a \in A$ is arbitrary, write $a = a_1 + ia_2$ with a_1, a_2 self-adjoint (see Exercise 1.16), and note that

$$\phi(a^*) = \phi(a_1 - ia_2) = \phi(a_1) - i\phi(a_2) = \overline{\phi(a_1) + i\phi(a_2)} = \overline{\phi(a)}.$$ □

[1] Indeed, since $z1 - a$ is not invertible, it is contained in a proper ideal. The collection of all maximal ideal containing a proper ideal, with partial order given by inclusion, is such that every partially ordered chain has an upper bound. Hence we can apply Zorn's lemma.

Remark 1.1. *In the proof we made again use of the C*-property. The previous Lemma is not true for Banach *-algebras, and motivates the following:*

Definition 1.6. *A commutative Banach *-algebra is called* symmetric *if* $\phi(a^*) = \overline{\phi(a)}$ *for all* $a \in A,\ \phi \in \sigma(a)$.

Thefore, by Lemma 1.14, every commutative unital C*-algebra is symmetric.

Theorem 1.15 (The Gelfand-Naimark theorem for unital C^*-algebras). *The Gelfand transform* $\Gamma : A \to C(\sigma(A))$ *is an isometric unital* $*$*-isomorphism for every unital commutative* C^**-algebra* A.

Proof. We already know that Γ is a unital algebra homomorphism (see Exercise) It remains to show that Γ is isometric (and hence injective), surjective, and satisfies $\Gamma(a^*) = \Gamma(a)^*$.

Recalling how the $*$ operation is defined in the C^*-algebra $C(\sigma(A))$, and applying Lemma 1.14, it is a simple matter to check that $\Gamma(a^*) = \Gamma(a)^*$: for each $\phi \in \sigma(A)$ we have

$$\Gamma(a^*)(\phi) = \phi(a^*) = \overline{\phi(a)} = \overline{\Gamma(a)(\phi)} = \Gamma(a)^*(\phi).$$

So Γ is a $*$-homomorphism.

Next we show that Γ is isometric. If $a \in A$ is self-adjoint then we have

$$\|a\| = \text{radius}(a) = \text{radius}(\Gamma(a)) = \|\Gamma(a)\|,$$

where the first and the last equalities hold by (5), and the middle equality holds by Lemma 1.13. Now if $a \in A$ is arbitrary then a^*a is self-adjoint and so

$$\|a\| = \|a^*a\|^{1/2} = \|\Gamma(a^*a)\|^{1/2} = \|\Gamma(a)^*\Gamma(a)\|^{1/2} = \|\Gamma(a)\|.$$

Thus Γ is isometric.

Finally, to show that Γ is surjective, we use the Stone–Weierstrass theorem, recalled below. The image $\Gamma(A)$ of Γ is a $*$-subalgebra of $C(X)$, because Γ is a $*$-homomorphism. Moreover, $\Gamma(A)$ is closed because Γ is isometric. The algebra $\Gamma(A)$ also separates points, because if $\phi, \psi \in \sigma(A)$ are two distinct linear functionals, then there is some element $a \in A$ such that $\Gamma(a)(\phi) = \phi(a) \neq \psi(a) = \Gamma(a)(\psi)$. So the Stone-Weierstrass theorem implies that either $\Gamma(A) = C(X)$, or that there is a point $x_0 \in X$ such that $\Gamma(A) = \{f \in C(X) \mid f(x_0) = 0\}$. This second possibility cannot hold, because $\Gamma(A)$ contains the constant function 1. Thus $\Gamma(A) = C(X)$, and this completes the proof that Γ is an isometric $*$-isomorphism. □

Here is the version of the Stone-Weierstrass theorem that we are using. See [Folland, Real Analysis, 4.51] for a proof.

Theorem 1.16 (The Stone-Weierstrass theorem). *Let* X *be a compact Hausdorff space, and suppose that* $B \subset C(X)$ *is a norm-closed* $*$*-subalgebra which* separates points*: i.e., for every pair of distinct points* $x_1 \neq x_2 \in X$ *there is a function* $f \in B$ *such that* $f(x_1) \neq f(x_2)$. *Then either* $B = C(X)$, *or there is a point* $x_0 \in X$ *such that* $B = \{f \in C(X) \mid f(x_0) = 0\}$. □

In the case of Banach algebra, there is not much we can say, since we do not have injectivity non surjectivity. We can, however, still apply the Stone-Weierstrass theorem and obtain the following result

Proposition 1.17. *Let* A *be a symmetric unital Banach *-algebra. Then* $\Gamma(A)$ *is a dense subalgebra of* $C(\sigma(A))$.

Theorem 1.15 says that every commutative, unital C^*-algebra A is isomorphic to one of the form $C(X)$. What does the theorem say when A is already given in the form $C(X)$?

EXERCISE 1.17. *For each $x \in X$ consider the map* $\mathrm{eval}_x : C(X) \to \mathbb{C}$ *of evaluation at* x. *Prove that the map* $x \mapsto \mathrm{eval}_x$ *is a homeomorphism* $X \xrightarrow{\cong} \sigma(C(X))$.

Identifying $\sigma(C(X))$ *with* X *as in the previous part, show that the Gelfand transform* $\Gamma : C(X) \to C(\sigma(C(X))) \cong C(X)$ *is the identity.*

1.3. Maximal Ideals. There is an alternative description of the set $\sigma(A)$, in terms of ideals of A. For this we need the following construction.

EXERCISE 1.18. *Let A be a Banach algebra, and let $J \subseteq A$ be a closed, two-sided ideal. (So J is a norm-closed linear subspace of A such that $aj \in J$ and $ja \in J$ for all $j \in J$ and all $a \in A$.) Equip the quotient vector space A/J with the multiplication*

$$(a + J)(b + J) \coloneqq ab + J$$

and the norm

$$\|a + J\| \coloneqq \inf_{j \in J} \|a + j\|.$$

Prove that these definitions make A/J into a Banach algebra, which is unital/commutative if A is unital/commutative.

PROPOSITION 1.18. *Let A be a commutative unital Banach algebra. The map $\phi \mapsto \ker \phi$ is a bijection between the set $\sigma(A)$ of multiplicative linear functionals on A, and the set of maximal ideals in A.*

Recall that a *maximal ideal* in A is a proper ideal (i.e., one that is not equal to A itself) that is not contained in any strictly larger proper ideal. In view of this result, $\sigma(A)$ is sometimes called the *maximal ideal space* of A.

PROOF. First note that the kernel of any algebra homomorphism is an ideal; and that the kernel of any $\phi \in \sigma(A)$ is a proper ideal, since ϕ is by definition nonzero. To see that $\ker \phi$ is a maximal ideal, note that if $a \notin \ker \phi$ then we can write $1 = (1 - \phi(a)^{-1}a) + \phi(a)^{-1}a$, where $1 - \phi(a)^{-1}a \in \ker \phi$. Thus any ideal containing both a and $\ker \phi$ must contain 1, and clearly the only ideal of A containing 1 is A itself. Thus $\ker \phi$ is a maximal ideal in A.

For the converse, suppose that J is a maximal ideal of A. Then the closure $\overline{J}$ is again an ideal of A, because multiplication is continuous. Since J is a proper ideal, J does not contain any invertible elements of A, and hence cannot be dense in A (since the invertible elements constitute an open subset of A). Thus $\overline{J}$ is a proper ideal of A, containing J. Since J is maximal we must have $J = \overline{J}$: i.e., J is closed in A. Now the quotient A/J is a commutative unital Banach algebra (Exercise 1.18), and every nonzero element of A/J is invertible (because J is a maximal ideal). Corollary 1.7 then gives an isomorphism of algebras $\psi : A/J \to \mathbb{C}$. Composing this ψ with the quotient mapping $A \to A/J$, we obtain a multiplicative linear functional $\phi : A \to \mathbb{C}$ whose kernel is precisely J.

So far we have shown that $\ker \phi$ is a maximal ideal for every $\phi \in \sigma(A)$, and that every maximal ideal of A has the form $\ker \phi$. It remains to show that the map $\phi \mapsto \ker \phi$ is one-to-one. Suppose that $\phi, \psi \in \sigma(A)$ have the same kernel. This kernel is a codimension-one subspace of A, so in order to show that $\phi = \psi$ it will suffice to show that the two functionals agree on a single element of $A \setminus \ker \phi$. Since $\phi(1) = \psi(1) = 1$, we do indeed have $\phi = \psi$. $\square$

Exercise 1.19. *Let X be a compact Hausdorff space, and consider the unital commutative C^*-algebra $C(X)$. For each point $x \in X$, let*

$$J_x := \{f \in C(X) \mid f(x) = 0\}$$

denote the set of functions vanishing at x.

(1) *Prove that for each $x \in X$ the restriction map $f \mapsto f|_{X\setminus\{x\}}$ is an isometric $*$-isomorphism $J_x \xrightarrow{\cong} C_0(X \setminus \{x\})$.*
(2) *Prove that for each $x \in X$, the subset J_x is a maximal ideal of $C(X)$.*
(3) *Prove that every maximal ideal of $C(X)$ is of the form J_x for some $x \in X$.*

1.3.1. *Proof of Lemma 1.13.* First notice that our computation of spectra in algebras of the form $C(X)$ (see corresponding Exercise) immediately gives

$$\sigma(\Gamma(a)) = \text{image}(\Gamma(a)) = \{\phi(a) \mid \phi \in \sigma(A)\}.$$

Since Γ is a unital homomorphism, we get the inclusion $\sigma(\Gamma(a)) \subseteq \sigma(a)$. So we are left to show that $\sigma(a) \subseteq \sigma(\Gamma(a))$.

Suppose then that $z \in \sigma(a)$, so that $z1 - a$ is not invertible in A. Then $z1 - a$ is contained in a maximal ideal J of A by a standard Zorn's lemma argument [2]. Hence, by the bijection between spectrum and maximal ideals, there is a $\phi \in \sigma(A)$ with $z - \phi(a) = \phi(z1 - a) = 0$. Then $\Gamma(a)(\phi) = z$, showing that $z \in \text{image}(\Gamma(a)) = \sigma(\Gamma(a))$.

1.4. An equivalence of categories.

Theorem 1.19. *There is a equivalence of categories between the category $\mathsf{Top_{cpt}}$ of compact Hausdorff topological spaces with continuous maps, and the opposite category of commutative unital C^*-algebras with *-homomorphisms $\mathsf{C^*alg^{op}_{com,u}}$.*

Proof. Given a continuous mapping $f : X \to Y$, between two compact Hausdorff spaces, we consider the map $Cf : C(Y) \to C(X)$ given by pullback, i.e. $Cf(g)(y) = g(f(y))$. Then it is easy to check that Cf is a unital *-homomorphism. Indeed

$$Cf(g + h) = (g + h) \circ f = g \circ f + h \circ f = Cf(g) + Cf(h), \quad \forall g, h \in C(Y),$$

and since the adjoint is given by point-wise conjugation, we have

$$Cf(g^*) = g^* \circ f = (g \circ f)^* = Cf(g)^*.$$

Finally, for the constant function equal to one, we have $C(1) = 1 \circ f = 1$.

If now we have continuous maps $f : X \to Y$ and $g : Y \to Z$, then we have a C*-homomorphism

$$C(g \circ f) = Cf \circ Cg : C(Z) \to C(Y).$$

Moreover, $C\,\text{Id}_X = 1_{C(X)}$.

We summarise this by saying that $X \to C(X), f \to Cf$ is a *contravariant* functor from the category $\mathsf{Top_{cpt}}$ of compact Hausdorff topological spaces with continuous maps to the category of commutative unital C*-algebras with *-homomorphisms $\mathsf{C^*alg_{com,u}}$. The functor is called contravariant since it "reverses the arrows".

The Gel'fand transform provides a contravariant functor going the other way. Let $\phi : A \to B$ a *-homomorphism between two commutative unital Banach algebras. Then we denote by

[2]Indeed, since $z1 - a$ is not invertible, it is contained in a proper ideal. The collection of all maximal ideal containing a proper ideal, with partial order given by inclusion, is such that every partially ordered chain has an upper bound. Hence we can apply Zorn's lemma.

$\sigma(\phi) : \sigma(B) \to \sigma(A)$ given by $\phi(x) = x \circ \phi$ for $x \in \sigma(B)$. Clearly this map is continuous, since for every $a \in A$, the map $\Gamma(a) \circ \sigma(\phi) = \Gamma(\phi(a))$ is continuous for each $a \in A$, where $\Gamma(a)$ is the Gel'fand transform. If $\psi : B \to C$ is another unital *-homomorphism, then

$$\sigma(\psi \circ \phi) = \sigma(\phi) \circ \sigma(\psi) : \sigma(C) \to \sigma(A).$$

Recall that the map $\epsilon_X : X \to \sigma(C(X))$ which assigns to every point in X, the evaluation functional $\mathrm{eval}_x : C(X) \to \mathbb{C}$, is a homeomorphsim of topological spaces. We have a *natural transformation* between the identity functor and the functor σC on the category of compact topological spaces, i.e.,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} \\ \sigma(C(X)) & \xrightarrow{\sigma(C(f))} & \sigma(C(Y)) \end{array}$$

whenever $f : X \to Y$ is continuous.

The natural transformation between the identity functor and the functor $C\sigma$ on the category of commutative unital C*-algebras is given, again, by the Gel'fand–Naĭmarktheorem. Indeed, the theorem states that $a \to \Gamma(a)$ is an isomorphism betwenn A and $C(\sigma(A))$. Morever, if $\phi : A \to B$ is a unital *-homomorphism, we have

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle \Gamma_A} & & \downarrow{\scriptstyle \Gamma_B} \\ C(\sigma(A)) & \xrightarrow{C(\sigma(\phi))} & C(\sigma(B)), \end{array}$$

i.e., for $\Gamma(a) \in C(\sigma(A))$ and $y \in \sigma(B)$ we have

$$(C(\sigma(\phi)\Gamma(a))(y) = \Gamma(a)((\sigma(\phi))y) = \Gamma(a)(y \circ \phi) = y(\phi(a)) = \Gamma(\phi(a))(y).$$

In particular, all unital *-homomorphisms from $C(Y) \to C(X)$ come from continuous maps $X \to Y$. This concludes the proof of the theorem. □

This equivalence of categories has important consequences, including the following:

EXERCISE 1.20. *Prove that two unital commutative C*-algebras are isomorphic if and only if their spectra are homeomorphic.*

1.5. Beyond the compact case. It turns out that a variant of the Gel'fand–Naĭmarktheorem holds also in the case where the topological space X is non-compact, but only *locally* compact (i.e. each point has a system of compact neighbourhoods).

THEOREM 1.20 (Gel'fand–Naĭmark). *Let A be a commutative C*-algebra. Then there exists a locally compact Hausdorff space X such that A is isometrically $*$-isomorphic to the algebra $C_0(X)$ of continuous functions on X vanishing at infinity.*

The proof of the above theorem relies on the concept of *unitalization* of the C*-algebra A. Before we start, let us now recall the definition of direct sum of C*-algebras.

DEFINITION 1.7. *Let A and B two C*-algebras. Their vector space $A \oplus B$, with componentwise multiplication, and norm given by*

$$\|(a, b)\| := \max\{\|a\|, \|b\|\}$$

is a C-algebra, called the* direct sum.

Exercise 1.21. *Prove that $A \oplus B$ is indeed a C*-algebra.*

Exercise 1.22. *Let A be a* non-unital *C^*-algebra, and consider the vector space $A^+ := A \oplus \mathbb{C}$ equipped with the product*

$$(a, z)(b, w) := (ab + zb + wa, zw),$$

involution

$$(a, z)^* := (a^*, \overline{z}),$$

and norm

$$\sup_{b \in A,\ \|b\| \leq 1} \|ab + zb\|,$$

Prove that A^+ is a unital C^*-algebra, and that the embedding $A \to A^+$ given by $a \mapsto a^+ := (a, 0)$ is an isometric $*$-homomorphism.

If A is a unital C-algebra, we define A^+ to be the direct sum algebra $A \oplus \mathbb{C}$, with the usual structures.*

Definition 1.8. *The C^*-algebra A^+ is called the* minimal unitisation *of A.*

The term "minimal" is explained by the following universal property:

Exercise 1.23. *Let A and B be C^*-algebras, with B unital, and let $\phi : A \to B$ be a $*$-homomorphism. Prove that there exists a unique unital $*$-homomorphism $\phi^+ : A^+ \to B$ satisfying $\phi^+(a^+) = \phi(a)$ for every $a \in A$.*

Let us emphasize that, while the A^+ is isomorphic, as vector space, to the direct sum $A \oplus \mathbb{C}$, the product structure and the norm are, in general different.

The notion of unitisation allows us to extend some of the definitions and results we proved for unital Banach and C*-algebras to the nonunital case. The first example is the notion of spectrum of an element.

Definition 1.9. *If A is a Banach algebra without unit, we define the spectrum of an element a as the set*

$$\sigma(a) = \sigma_{A^+}(a).$$

With this convention, previous results like the spectral radius formula have a natural nonunital version.

Recall from the first lecture that, given a Hausdorff topological space X, a function $f : X \to \mathbb{C}$ is said to *vanish at infinity* if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq X$ such that

$$|f(x)| < \varepsilon, \qquad \forall x \in X \setminus K_\varepsilon.$$

Then $C_0(X)$, the space of all functions on X vanishing at infinity, is a commutative C*-algebra with pointwise multiplication and supremum norm

$$\|f\| := \sup_{x \in X} |f(x)|.$$

If X is compact, then clearly $C_0(X) = C(X)$.

Let us recall the construction of the one point compactification, also called *Alexandrov compactification* X^∞ of the space X. Let ∞ denote a new point not belonging to X, then as a set $X^\infty = X \cup \{\infty\}$. X^∞ is a topological space with respect to the following topology: a set $U \subseteq X^\infty$ is called *open* if either $U \subseteq X$ and it is open, or if $\infty \in U$ and $X \setminus U$ is compact (hence closed because X is Hausdorff).

Then every continuous function on $C(X^\infty)$ defines, by restriction, a function in $C_0(X)$. One can identify the functions in $C_0(X)$ with continuous fucntions in $C(X^\infty)$ that satisfy $f(\infty) = 0$. This justifies the notion of *vanishing at infinity.*

EXERCISE 1.24. *Let A be a commutative C^*-algebra, and consider the minimal unitisation A^+ as in Exercise 1.22. For each $\phi \in \sigma(A)$, define a map $\phi^+ : A^+ \to \mathbb{C}$ by $\phi^+(a, z) := \phi(a) + z$.*

(1) *Show that $\phi^+ \in \sigma(A^+)$. Conclude that every $\phi \in \sigma(A)$ is bounded with norm ≤ 1.*
(2) *Show that the topology on $\sigma(A)$ inherited from the weak* topology on $(A^+)^*$ via the embedding*
$$\sigma(A) \to \sigma(A^+) \subset (A^+)^*, \qquad \phi \mapsto \phi^+$$
coincides with the weak topology that $\sigma(A)$ inherits from A^*.*
(3) *Show that $\sigma(A^+) = \{\phi^+ \mid \phi \in \sigma(A)\} \cup \{\phi_\infty\}$, where ϕ_∞ is the multiplicative linear functional on A^+ defined by $\phi_\infty(a, z) = z$.*
(4) *Conclude that $\sigma(A)$, equipped with the weak* topology, is a locally compact Hausdorff space.*
(5) *For each $f \in C_0(\sigma(A))$, define a function f^+ on $\sigma(A^+)$ by*
$$f^+(\phi^+) := f(\phi) \text{ for } \phi \in \sigma(A), \qquad f^+(\phi_\infty) := 0.$$
Show that the map $f \mapsto f^+$ is an isometric $*$-homomorphism $C_0(\sigma(A)) \to C(\sigma(A^+))$, whose image is the set
$$\{f \in C(\sigma(A^+)) \mid f(\phi_\infty) = 0\}.$$

CHAPTER 2

Noncommutative C*-algebas, representations, and the GNS construction

1. Representation of C*-algebras

Our main goal is to prove that every C*-algebra, commutative or not, *can be realised* as a C*-subalgebra of $B(H)$.

1.1. Basic definitions. Throughout this section we let A be a C^*-algebra: not assumed commutative, not assumed unital.

DEFINITION 1.1. *A* $*$-representation *of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathrm{B}(H)$ into the C^*-algebra of bounded operators on some Hilbert space.*

It is common to just say "representation" in place of "$*$-representation", when it is clear what is meant. It is also common to write H_π to indicate the Hilbert space involved in the representation π.

There is a corresponding notion of morphisms between representations:

DEFINITION 1.2. *Let $\pi_1 : A \to \mathrm{B}(H_1)$ and $\pi_2 : A \to \mathrm{B}(H_2)$ be $*$-representations. An* intertwining operator *between π_1 and π_2 is a bounded linear operator $t \in \mathrm{B}(H_1, H_2)$ which satisfies $t \circ \pi_1(a) = \pi_2(a) \circ t$ for all $a \in A$. The representations π_1 and π_2 are* isomorphic *if there exists an invertible intertwining operator $t : H_1 \to H_2$.*

The Hilbert space H in a $*$-representation can be regarded as a module over A, via the action $a \cdot h \coloneqq \pi(a)(h)$. In algebra, when working with modules M over unital rings R, one usually imposes the condition that $1_R \cdot m = m$ for every $m \in M$. The following is a useful replacement for this condition in the setting of representations of (possibly non-unital) C^*-algebras.

DEFINITION 1.3. *A $*$-representation $\pi : A \to \mathrm{B}(H)$ is* nondegenerate *if the linear subspace* $\mathrm{span}\{\pi(a)\xi \mid a \in A,\ \xi \in H\}$ *is dense in H.*

REMARK 1.1. *If A is a* unital *C^*-algebra, then a $*$-representation $\pi : A \to \mathrm{B}(H)$ is nondegenerate if and only if $\pi(1_A) = 1_{\mathrm{B}(H)}$.*

REMARK 1.2. *Remarkably, if π is nondegenerate then in fact the set $\{\pi(a)(\xi) \mid a \in A,\ \xi \in H\}$ is already equal to H. This follows from the Cohen–Hewitt factorisation theorem* [**5**, II.5.3.7]*, which states that every element x of a nondegenerate (right) Banach A-module over a C^*-algebra A can be factored as $\xi = a\eta$.*

EXERCISE 1.1. *Let A be a C^*-algebra, $\pi : A \to \mathrm{B}(H)$ a $*$-representation. Show that the following two conditions are equivalent:*

(1) $\mathrm{span}\{\pi(a)\xi \mid a \in A,\ \xi \in H\}$ *is dense in H;*
(2) *For every $\xi \in H, \xi \neq \underline{0}$, there exists $a \in A$ such that $\pi(a)\xi \neq 0$.*

In other words, a representation is nondegenerate in the sense of Definition 1.3 if and only if (2) *holds.*

DEFINITION 1.4. *Let $\pi : A \to \mathrm{B}(H)$ be a $*$-representation. A linear subspace $K \subseteq H$ is A*-invariant *if $\pi(a)(k) \in K$ for every $k \in K$. The representation π is called* topologically irreducible *if H has no* closed *invariant subspaces besides $\{0\}$ and H itself. The representation π is called* algebraically irreducible *if H has no invariant subspaces, closed or not, besides $\{0\}$ and H.*

Algebraically irreducible representations are obviously topologically irreducible. It is far from obvious, but the converse is true as well:

THEOREM 1.1. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

In view of Theorem 1.1 we shall from now on just say "irreducible".

1.2. Schur's lemma. A useful characterization of irreducible representations is provided by *Schur's lemma.*

EXERCISE 1.2. *Given a subset $M \subset \mathrm{B}(H)$ its* commutant *is defined by*

$$M' := \{t \in \mathrm{B}(H) : mt = tm \text{ for all } m \in M\}.$$

- *Show that, even if M is just a* sub*set of* $\mathrm{B}(H)$*, its commutant M' is always a* sub*algebra of* $\mathrm{B}(H)$*;*
- *Prove that if M is also closed under the * operation (namely $A^* \in M$ whenever $A \in M$), then M' is a $*$-subalgebra of* $\mathrm{B}(H)$*;*
- *Prove that*
$$\pi(A)' := \{T \in \mathrm{B}(H) \mid t\pi(a) = \pi(a)t \ \forall a \in A\}$$
is a unital C^*-subalgebra of $\mathrm{B}(H)$*.*

LEMMA 1.2 (Schur). *Let $\pi \colon A \to \mathrm{B}(H)$ be a representation of the C^*-algebra A. Then*

$$\pi \text{ is irreducible} \iff \pi(A)' = \mathbb{C}1.$$

PROOF. Assume that π is irreducible, and let $T \in \pi(A)'$. Pick any λ in the spectrum of T, and define $H_1 := \mathrm{Ran}(T - \lambda 1) \subset H$ (the image of the operator $T - \lambda 1$). Then, by definition, for each $\eta \in H_1$ there exists a vector $\xi \in H$ such that

$$\eta = (T - \lambda 1)\xi. \tag{6}$$

By applying the operator $\pi(a) \in \mathrm{B}(H)$, for $a \in A$, to both sides of this equation, we obtain

$$\pi(a)\eta = \pi(a)(T - \lambda 1)\xi = (T - \lambda 1)\underbrace{\pi(a)\xi}_{=:\xi'}$$

because $T - \lambda 1$ commutes with $\pi(a)$. Hence also $\pi(a)\eta$ is of the form (6), and thus H_1 is an invariant subspace under the representation of A on H. By the irreducibility of π, we must have either $H_1 = \{0\}$ or $H_1 = H$.

If $H_1 = \{0\}$, then $(T - \lambda 1)\xi = 0$ for all $\xi \in H$, and hence $T = \lambda 1$, and we deduce that $\pi(a)' = \mathbb{C}1$ as we wanted. We claim that $H_1 = H$ leads to a contradiction. Indeed, if $H_1 = H$ then the operator $T - \lambda 1$ cannot be injective, otherwise the inverse $(T - \lambda 1)^{-1}$ would be well-defined on H contradicting the fact that $\lambda \in \sigma(T)$. Hence we have that $H_2 := \ker(T - \lambda 1) \neq \{0\}$, and also H_2 is a proper subspace of H since $T - \lambda 1$ is not identically zero. Let now $\xi \in H_2$: then

$$0 = \pi(a)(T - \lambda 1)\xi = (T - \lambda 1)\pi(a)\xi \quad \text{for all } a \in A,$$

and hence also $\pi(a)\xi \in H_2$. In other words, H_2 is a non-trivial subspace of H which is invariant under the action of A, contradicting our hypothesis that π is irreducible.

Conversely, suppose that $\pi(A)' = \mathbb{C}1$. Let $H_1 \subset H$ be an invariant subspace for A. Notice that if $\eta \in H_1^{\perp}$ (namely $\langle \eta, \xi \rangle = 0$ for all $\xi \in H_1$), then for all $a \in A$

$$\langle \pi(a)\eta, \xi \rangle = \langle \eta, \pi(a^*)\xi \rangle = 0 \quad \text{for all } \xi \in H_1,$$

since $\pi(A)H_1 \subset H_1$. Thus, $\pi(a)\eta \in H_1^{\perp}$, and also $H_1^{\perp}$ is an invariant subspace.

Now, let $P \in \mathcal{B}(H)$ be the orthogonal projection onto H_1. Since by definition we have that $P\xi = \xi$ for all $\xi \in H_1$ and $P\eta = 0$ for all $\eta \in H_1^{\perp}$, we deduce that for all $a \in A$

$$P\pi(a)\xi = \pi(a)\xi = \pi(a)P\xi \qquad \text{because } \pi(a)H_1 \subset H_1,$$
$$P\pi(a)\eta = 0 = \pi(a)P\eta \qquad \text{because } \pi(a)H_1^{\perp} \subset H_1^{\perp}.$$

Hence P commutes with all the operators in $\pi(A)$, and thus by our hypothesis $P = \lambda 1$ for some $\lambda \in \mathbb{C}$. Since P is a projection, we have that $P^2 = P$, which in turn gives $\lambda^2 = \lambda$, or equivalently either $\lambda = 0$ or $\lambda = 1$. If $\lambda = 0$, then $P = 0$, and hence $H_1 = \{0\}$; while if $\lambda = 1$, then $P = 1$, and hence $H_1 = H$. □

EXERCISE 1.3 (Schur's alternative). *Two representations are said to be* unitarily equivalent *if there exists a unitary intertwiner $u^*u = 1 = uu^*$, between π_1 and π_2. If the only unitary intertwiner is the zero operator, we say that π_1 and π_2 are* disjoint.

Let $\pi_i : A \to \mathrm{B}(H_i)$, $i = 1, 2$, be irreducible representations of A. Prove that π_1 and π_2 are not disjoint if and only if they are unitarily equivalent.

DEFINITION 1.5. *Let $\pi_1 : A \to \mathrm{B}(H_1)$ and $\pi_2 : A \to \mathrm{B}(H_2)$ be $*$-representations of a C^*-algebra. The* direct sum *of π_1 and π_2 is the $*$-representation $\pi_1 \oplus \pi_2 : A \to \mathrm{B}(H_1 \oplus H_2)$ defined by*

$$(\pi_1 \oplus \pi_2)(a) : (h_1, h_2) \mapsto (\pi_1(a)h_1, \pi_2(a)h_2).$$

In algebra one encounters a separate notion of "indecomposability": a representation is indecomposable if it is not isomorphic to a direct sum of two proper subrepresentations. Irreducible modules are obviously indecomposable, but the converse is not always true. (Consider for instance $\mathbb{C}^2$ as a module over the algebra of upper-triangular 2×2 matrices.) For $*$-representations of C^*-algebras, however, every indecomposable representation is irreducible:

REMARK 1.3. *Let $\pi : A \to \mathrm{B}(H)$ be a $*$-representation of a C^*-algebra. We saw in the proof of Schur's lemma that if $K \subset H$ is a closed A-invariant subspace, then the orthogonal complement $K^{\perp} \subset H$ is also a closed A-invariant subspace. We conclude that π is irreducible if and only if π is not isomorphic to a direct sum of nonzero representations.*

To conclude this Section, let us introduce some more terminology. Let $B \subset \mathrm{B}(H)$ be a $*$-subalgebra of $\mathrm{B}(H)$. We say that a vector $\xi \in H$ is *cyclic* for B if $B\xi$ is dense in H; we say that ξ is *separating* for B if $T\xi = 0$ for $T \in B$ implies that $T = 0$.

EXERCISE 1.4. *Let $B \subset \mathrm{B}(H)$ be a $*$-subalgebra. Show that if $\xi \in H$ is cyclic for B, then it is separating for the commutant B'.*

DEFINITION 1.6. *A representation is called cyclic if it has a cyclic vector.*

Let A be a C*-algebra and $\pi : A \to B(H)$ an irreducible representation. Then either

(1) every $\xi \in H$ is a cyclic vector;
(2) $\pi(A) = \{0\}$ and $H = \mathbb{C}$.

Clearly, a cyclic representation is always nondegenerate. By Schur's Lemma every irreducible representation is cyclic. Therefore we have the following chain of implications for nonzero $*$-representations

$$\pi \text{ irreducible} \Rightarrow \pi \text{ cyclic} \Rightarrow \pi \text{ nondegenerate}.$$

The key notion linking C^*-algebras with operators on Hilbert space is that of *positivity.*

Definition 1.7. *An element a of a C^*-algebra A is called* positive *(notation: $a \geq 0$) if a is self-adjoint and has $\sigma(a) \subset [0, \infty)$.*

Note that, by spectral permanence (and by the definition of spectra for elements of nonunital algebras), the property of positivity does not depend on the containing C^*-algebra. In particular, an element $a \in A$ is positive if and only if the corresponding element $a^+ = (a, 0) \in A^+$ is positive.

Exercise 1.5. *Show that a function $f \in C_0(X)$ is positive (in the sense of Definition 1.7) if and only if* $\text{image}(f) \subset [0, \infty)$.

Exercise 1.6. *Prove that if a is normal in A, and $f \geq 0$ in $C_0(\sigma(a) \setminus \{0\})$, then $f(a) \geq 0$ in A.*

The most important fact about positivity is:

Theorem 1.3. *Let A be a C^*-algebra. An element $a \in A$ is positive if and only if $a = b^*b$ for some $b \in A$.*

Lemma 1.4. *If $a \in A$ is positive, then $a = b^*b$ for some $b \in A$.*

2. States and the GNS construction

In this subsection presents some basic topics regarding *states* over a C*-algebra. This notion is tightly related to that of a representation and the relation is contained in the GNS (Gel'fand–Naĭmark–Seagal) construction.

Definition 2.1. *Let A be a C^*-algebra. A linear functional $\phi : A \to \mathbb{C}$ is* positive *(denoted $\phi \geq 0$) if $\phi(a) \geq 0$ whenever $a \geq 0$; and ϕ is a* state *if $\phi \geq 0$ and $\|\phi\| = 1$.*

Exercise 2.1. *Prove that every multiplicative linear functional is a state. Give an example of a state that is not a multiplicative linear functional.*

Exercise 2.2. *Let $\pi : A \to \mathrm{B}(H)$ be a $*$-representation. Prove that for each $\xi \in H$ the map*

$$\phi_\xi : A \to \mathbb{C}, \qquad \phi_\xi(a) := \langle \pi(a)\xi, \xi \rangle$$

is a positive linear functional. Prove that if π is nondegenerate and $\|\xi\| = 1$, then ϕ_ξ is a state.

The set of states of A will be denoted by $\mathcal{S}(A)$.

The question we would like to ask is whether any state $\phi \in \mathcal{S}(A)$ is of the above form? This question is positively answered by the GNS construction.

2.1. The GNS construction.

EXERCISE 2.3. *Let $\phi : A \to \mathbb{C}$ be a positive linear functional and define*

$$\langle a, b\rangle := \phi(b^* a). \tag{7}$$

Check that

1. *$\langle \cdot, \cdot\rangle$ is linear in the first variable, conjugate linear in the second;*
2. *$\langle a, a\rangle \geq 0$ for all $a \in A$;*
3. *$\langle \cdot, \cdot\rangle$ satisfies the Cauchy–Schwartz inequality*

$$|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle. \tag{8}$$

Deduce that (7) *defines a positive semidefinite sesquilinear form on A.*

LEMMA 2.1. *Positive linear functionals are continuous. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit on A, then $\|\phi\| = \lim_{\lambda\in\Lambda} \phi(e_\lambda)$. In particular, if A is unital, then $\|\phi\| = \phi(1_A)$.*

We are now ready to state and prove the Theorem that goes under the name of GNS construction.

THEOREM 2.2. *Let A be a C^*-algebra and ϕ a positive linear functional on A. Then there exist a Hilbert space H_ϕ, a representation $\pi_\phi : A \to \mathrm{B}(H_\phi)$ and a vector $\xi_\phi \in H$ which is cyclic for π such that*

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle \tag{9}$$

for all $a \in A$.

Conversely, if ξ is a unit cyclic vector for the representation $\pi : A \to \mathrm{B}(H)$, then $\phi(a) := \langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle$ gives a state on A and the map $a \mapsto \pi(a)\xi$ induces a unitary isomorphism U from the Hilbert space H_ϕ to H such that

$$\pi(a) = U\pi_\phi(a)U^*.$$

PROOF. The first ingretient we need is the Hilbert space on which to define the representation. Let

$$N_\phi := \{a \in A \mid \phi(a^*a) = 0\}.$$

Since ϕ induces the sesquilinear form (7), using the Cauchy–Schwartz inequality (8) we see that N_ϕ can also be characterised as

$$N_\phi := \{a \in A \mid \phi(b^*a) = 0 \ \forall b \in A\}.$$

It follows that N_ϕ is a closed subspace. Moreover, it is a left ideal, since for all $n \in N_\phi$ and $a \in A$ we have

$$\phi(b^*(an)) = \phi((a^*b)^*n) = 0.$$

Therefore, it makes sense to consider the quotient A/N_ϕ, with the inner product

$$\langle [x], [y]\rangle := \phi(y^*x). \tag{10}$$

This inner product is well defined, indeed, for n_1, n_2 we have

$$\phi(y + n_2)^*(x + n_1)) = \phi(y^*x) + \phi((y + n_2)^*n_1) + \phi(n_2^*x) = \phi(y^*x),$$

where we have used the fact that $\phi(n_2^*x) = \overline{\phi(x^*n_2)}$.

We define the Hilbert space H_ϕ to be the completion of A/N with respect to the norm coming from the inner product.

To construct the representation π_ϕ we start from the left regular representation $\pi_0 : A \to A/N$ given by

$$\pi_0(a)([x]) = [ax].$$

This is well defined: indeed, for $n \in N$ we have

$$\pi_0(a)(x+n) = ax + an$$

which is a representative for $[ax]$ since N is an ideal.

To see that the left regular representation is indeed a *-representation, let us consider the inner product

$$\begin{aligned} \pi_0(a)([x]), [y]\rangle &= \langle [ax], [y]\rangle = \phi(y^*ax) = \phi((a^*y)^*x) \\ &= \langle [x], [a^*y]\rangle = \langle [x], \pi_0(a^*)[y]\rangle, \end{aligned}$$

hence, by definition of adjoint operator, we have that $\pi_0(a)^* = \pi_0(a^*)$.

Moreover, $\|\pi_0\| \leq 1$. This can be checked using the operator norm

$$\|\pi_0(a)\|^2 := \sup_{\|x\|\leq 1} \|\pi_0(a)x\|^2 = \sup_{\|x\|\leq 1} \phi(x^*a^*ax) \leq \|a\|^2\phi(x^*x) \leq \|a\|^2.$$

Note that we have used the fact that $\|a\|^2x^*x - x^*a^*ax$ is positive. Therefore π_0 extends by continuity to a representation π_ϕ of A on H.

Finally, to construct a cyclic vector for π_ϕ, we first consider the case of A unital. If we define $\xi_\phi := [1_A]$, we have that

$$\langle \pi_0(a)[1], [1]\rangle = \phi(a),$$

and $\pi_\phi(A)\xi = A/N$, which is dense in H by construction.

Moreover, by Lemma 2.1, we have that

$$\|\phi\| = \|\phi(1)\| = \|1\| = \|\xi_\phi\|.$$

If the algebra A is not unital, we need to consider approximate identities.

□

Given a state ϕ on a C^*-algebra A, we showed that there is a triple $(\pi_\phi, H_\phi, \xi_\phi)$, where H_ϕ is a Hilbert space, $\pi_\phi : A \to \mathrm{B}(H_\phi)$ is a representation, and ξ_ϕ is a cyclic unit vector in H_ϕ, satisfying $\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle$ for every $a \in A$.

We want to show that for every C^*-algebra A there is a *faithful* (i.e., injective) representation $A \to \mathrm{B}(H)$. To see how the GNS construction helps us to do this, observe the following:

Theorem 2.3. *Let A be a C^*-algebra, and let $a \in A$ be self-adjoint. There is a state ϕ on A such that $\phi(a) = \|a\|$.*

Theorem 2.4. *Every C^*-algebra is isomorphic to a C^*-subalgebra of $\mathrm{B}(H)$, for some Hilbert space H.*

Proof. Consider the direct sum

$$\pi := \bigoplus_{\phi\in\mathcal{S}(A)} \pi_\phi : A \to \mathrm{B}\left(\bigoplus_{\phi\in\mathcal{S}(A)} H_\phi\right)$$

of all of the GNS representations of A. Being a direct sum of $*$-homomorphisms, π is itself a $*$-homomorphism. To see that π is injective, take an arbitrary nonzero $a \in A$ and use the previous theorem to find a state ϕ having $\phi(a^*a) \neq 0$. Then $\pi_\phi(a) \neq 0$ (see exercise), so $\pi(a) \neq 0$. Now

π is an injective $*$-homomorphism, so it is an isometric isomorphism $A \cong \text{image}(\pi)$ (see exercise from last Thursday). $\square$

REMARK 2.1. *The set $\mathcal{S}(A)$ is generally very big (uncountably infinite), and so the Hilbert space H constructed in the above proof is also very big. If A is* separable, *meaning that it has a countable dense subset, then an easy approximation argument shows that the Hilbert space H can be taken to be separable as well.*

3. Notable C*-algebras

3.1. Universal C*-algebras. Many interesting C*-algebras are described in terms of a set of generation and relations.

Unlike what happens in algebra, where one can consider quotients of free rings by universal object, C*-algebras are always realised as algebras of bounded operators on some Hilbert space, so the relations must prescribe uniform bonds on the norms of the generators. As a consequence, depending on the generators and on the shape of the relations, such a universal C*-algebra may not exist. We will see a non-example later.

In order to tackle the problem of describing C*-algebras subject to generation and relations, we follow [**4**].

We would like to define a universal C*-algebra starting from a set $\mathcal{G} = \{x_\alpha\}$ subject to a set of relations $\mathcal{R}$. For simplicity, we consider relations of the form

$$\|p(x_{\alpha_1}, \dots, x_{\alpha_n}, x^*_{\alpha_1}, \dots, x^*_{\alpha_n})\| \leq \eta, \tag{11}$$

where p is a complex polynomial in $2n$ non-commuting variables, $x_{\alpha_1}, \dots, x_{\alpha_n} \in \mathcal{G}$ and $\eta \geq 0$. Note that for $\eta = 0$, relations of the form (11) may be rewritten as algebraic relations among $x_{\alpha_1}, \dots, x_{\alpha_n} \in \mathcal{G}$ and the scalars.

DEFINITION 3.1. *A representation of $(\mathcal{G}, \mathcal{R})$ is a set of bounded operators $\{y_\alpha\}$ on a Hilbert space $\mathcal{H}$ that satisfy*

$$\|p(y_{\alpha_1}, \dots, y_{\alpha_n}, y^*_{\alpha_1}, \dots, y^*_{\alpha_n})\| \leq \eta,$$

whenever $\big(\|p(x_{\alpha_1}, \dots, x_{\alpha_n}, x^*_{\alpha_1}, \dots, x^*_{\alpha_n})\| \leq \eta\big) \in \mathcal{R}$.

A representation of $(\mathcal{G}, \mathcal{R})$ extends uniquely to a *-homomorphism ρ from the free $*$-algebra $\mathcal{F}(\mathcal{G})$ generated by $\mathcal{G}$ to the C*-algebra $\mathrm{B}(\mathcal{H})$.

DEFINITION 3.2. *A set $(\mathcal{G}, \mathcal{R})$ is called* admissible *if*

(1) *there exists a representation of $(\mathcal{G}, \mathcal{R})$;*
(2) *whenever $\{y^\beta_\alpha\}$ is a representation of $(\mathcal{G}, \mathcal{R})$ on a Hilbert space $\mathcal{H}^\beta$, then $\oplus y^\beta_\alpha \in \mathrm{B}(\bigoplus \mathcal{H}^\beta)$ (and $\oplus y^\beta_\alpha$ is a representation of $(\mathcal{G}, \mathcal{R})$).*

The two conditions in the definition imply that

$$\|z\| := \sup\{\|\rho(z)\| \ : \ \rho \text{ is a representation of } (\mathcal{G}, \mathcal{R})\}$$

is a well defined complex number, and that $\|\cdot\|$ is a seminorm on the free algebra $\mathcal{F}(\mathcal{G})$.

We are now ready to give the definition we were after:

DEFINITION 3.3. *The completion of the quotient algebra $\mathcal{F}(\mathcal{G})/\{z : \|z\| = 0\}$ under $\|\cdot\|$ is called the* universal C*-algebra *of $(\mathcal{G}, \mathcal{R})$, and it is denoted by $C^*(\mathcal{G}, \mathcal{R})$. the algebra $C^*(\mathcal{G}, \mathcal{R})$ has the property that any representation of $(\mathcal{G}, \mathcal{R})$ extends uniquely to a representation of $C^*(\mathcal{G}, \mathcal{R})$, and conversely every representation of $C^*(\mathcal{G}, \mathcal{R})$ gives a representation of $(\mathcal{G}, \mathcal{R})$.*

EXAMPLE 3.1. *Let A be a C*-algebra, let $\mathcal{G} = A$ and $\mathcal{R}$ the set of all *-algebraic relations in A. Then $C^*(\mathcal{G}, \mathcal{R}) = A$.*

3.1.1. *Commutative examples.* As mentioned earlier, many well known C*-algebras can be naturally realised as universal C*-algebras. We start with some commutative examples.

EXERCISE 3.1. (1) *Let* $\mathcal{G} = \{x\}$ *and*

$$\mathcal{R} = \{x = x^*,\ \|x\| \leq 1,\ \|1 - x^2\| \leq 1\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C_0((0, 1])$.

(2) *Let* $\mathcal{G} = \{x, 1\}$ *and*

$$\mathcal{R} = \{x = x^*,\ \|x\| \leq 1,\ \|1 - x^2\| \leq 1,\ 1 = 1^2 = 1^*,\ x1 = x = 1x\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C([0, 1])$.

(3) *Let* $\mathcal{G} = \{x\}$ *and*

$$\mathcal{R} = \{xx^* = 1 = x^*x\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C(S^1)$.

(4) *Let* $\mathcal{G} = \{x\}$ *and*

$$\mathcal{R} = \{xx^* = 1 = x^*x\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C(S^1)$.

(5) *Let* $\mathcal{G} = \{x\}$ *and*

$$\mathcal{R} = \{xx^* = x^*x,\ \|x\| \leq 1\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C_0(D_0)$, *where* D_0 *is the punctured unit disk.*

(6) *Let* $\mathcal{G} = \{x\}$ *and*

$$\mathcal{R} = \{xx^* = x^*x,\ \|x\| \leq 1\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C_0(D_0)$, *where* D_0 *is the punctured unit disk.*

(7) *Let* $\mathcal{G} = \{x, 1\}$ *and*

$$\mathcal{R} = \{xx^* = x^*x,\ \|x\| \leq 1,\ 1 = 1^2 = 1^*,\ x1 = x = 1x\}\}.$$

Prove that $C^*(\mathcal{G}, \mathcal{R}) = C(D)$, *where* D *is the unit disk.*

3.1.2. *Cuntz and Cuntz-Krieger algebras.* Cuntz algebras and Cuntz-Krieger algebras are universal C*-algebras generated by (partial) isometries subject to certain additional conditions. They naturally appear in the study of dynamical systems.

Before describing these algebras in detail, one needs to recall some facts about isometries and partial isometries.

DEFINITION 3.4. *Let* S *be a bounded linear operator on a Hilbert space.* S *is an isometry if and only if* $S^*S = \mathrm{Id}$.

DEFINITION 3.5. *Let* $n \geq 2$. *The Cuntz algebra is the universal C*-algebra generated by* n*-isometries* $S_1, \ldots, S_n$ *subject to the additional condition*

$$\sum_{i=1}^{n} S_i {S_i}^* = \mathrm{Id}\,. \tag{12}$$

Note that condition (12) implies in particular that the corresponding range projections are orthogonal, i.e. $S_i^* S_j = \delta_{ij}\,\mathrm{Id}_n$ for all i, j.

THEOREM 3.1 ([**7**, Theorem 3.1]). *Suppose that* S_i *and* T_i, $i = 1, \ldots, n$ *are two families of non-zero partial isometries satisfying* (12). *Then the map* $S_i \mapsto T_i$ *extendes to an isomorphism* $C^*(S_1, \ldots, S_n) \simeq C^*(T_1, \ldots, T_n)$.

Cuntz algebras admit an extension by compact operators:

PROPOSITION 3.2 ([**7**, Proposition 3.1]). *Let* $V_1, \dots, V_n$ *be isometries on a Hilbert space* $\mathcal{H}$ *such that* $\sum_{i=1}^n V_i V_i^* \leq 1$. *Then the projection* $P := \mathrm{Id} - \sum_{i=1}^n V_i V_i^*$ *generates a closed two sided ideal* $\mathcal{I}$ *in* $C^*(V_1, \dots, V_n)$ *which is isomorphic to* $\mathcal{K}(\mathcal{H})$ *and contains* P *as a minimal projection. The quotient* $C^*(V_1, \dots, V_n)/\mathcal{I}$ *is isomorphic to* $\mathcal{O}_n$.

The Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(V_1, \dots, V_n) \longrightarrow \mathcal{O}_n \longrightarrow 0. \tag{13}$$

The Cuntz algberas are pairwise non isomorphic, i.e. $\mathcal{O}_n \simeq \mathcal{O}_m$ if and only if $n = m$.

3.1.3. *Cuntz–Krieger algebras.* Cuntz–Krieger algebras are a generalization of the above construction.

THEOREM 3.3. *Let* S *be a bounded linear operator on a Hilbert space. The following are equivalent:*

1. S *is a partial isometry;*
2. S^*S *is a projection;*
3. SS^* *is a projection;*
4. $SS^*S = S$*;*
5. $S^*SS^* = S^*$.

In this case, S^*S *is the projection on* $(\ker S)^\perp$ *and* SS^* *is the projection on the range of* S.

DEFINITION 3.6. *Let* $\mathbf{A}$ *be a matrix with entries in* $\{0, 1\}$ *with no rows or columns equal to zero.*

The Cuntz-Krieger algebra is the universal C^**-algebra generated by partial isometries* S_i *with pairwise orthogonal range projections, subject to the relations*

$$\begin{aligned} \sum_{i=1}^n S_i {S_i}^* &= \mathrm{Id} \\ {S_i}^* S_i &= \sum_{j=1}^n \mathbf{A}_{ij} S_j {S_j}^*. \end{aligned} \tag{14}$$

Clearly for $\mathbf{A}$ the matrix with all entries equal to one, one gets back the Cuntz algebras $\mathcal{O}_n$.

Any family $\{S_i\}_{i=1}^n$ of partial isometries satisfying the conditions in (14) is called a *Cuntz-Krieger* $\mathbf{A}$*-family*, and there is a uniqueness results similar to Theorem 3.1. We state it here under mildly stronger assumptions than those of the original paper.

THEOREM 3.4 (*cf.*[**8**, Theorem 2.13]). *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with entries in* $\{0, 1\}$ *which is irreducible and not a permutation matrix, and assume that* S_i *and* T_i, $i = 1, \dots, n$ *are Cuntz-Krieger* $\mathbf{A}$*-families. Then the map* $S_i \mapsto T_i$ *extends to an isomorphism* $C^*(S_1, \dots, S_n) \simeq C^*(T_1, \dots, T_n)$.

3.1.4. *Crossed Products by the integers.* Crossed products are the basic tool used to study groups acting on a C*-algebra. They provide a larger C*-algebra which encodes information both on the original algebra and on the group action. Although crossed products can be defined for any locally compact group, we will focus here on the particular case of crossed products by the integers.

DEFINITION 3.7. *Let* B *be a unital* C^**-algebra and* $\alpha \in \mathrm{Aut}(B)$. *The crossed product* C^**-algebra* $B \rtimes_\alpha \mathbb{Z}$ *is realised as the universal* C^**-algebra generated by* B *and a unitary* u *satisfying the covariance condition*

$$\alpha^n(b) = u^n b u^{*n}, \quad \forall b \in B, n \in \mathbb{Z}.$$

CHAPTER 3

Modules as bundles

1. Modules and fiber bundles

The duality between spaces and algebras can be extended to the case of vector bundles by considering modules over algebras. We will see that the finitely generated projective ones can be thought of as *noncommutative vector bundles.* We will state all results for modules over general complex algebras $\mathcal{A}$, and will only need to use C*-algebras in the formulation of the Serre-Swan theorem.

1.1. Modules over algebras. Let $\mathcal{A}$ be a complex algebra. A right $\mathcal{A}$-module is a vector space $\mathcal{E}$ together with a right action $\mathcal{E} \times \mathcal{A} \ni (\xi, a) \mapsto \xi a \in \mathcal{E}$ satisfying

$$\begin{aligned} \xi(ab) &= (\xi a)b, \\ \xi(a+b) &= \xi a + \xi b, \\ (\xi + \eta)a &= \xi a + \eta a, \end{aligned} \tag{15}$$

for all $\xi, \eta \in \mathcal{E}$ and $a \in \mathcal{A}$.

A family $\{\eta_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathcal{E}$, with Λ any directed set, is called a *generating set* for the right module $\mathcal{E}$ if any element of $\xi \in \mathcal{E}$ can be written (not necessarily in a unique way) as a combination

$$\xi = \sum_{\lambda \in \Lambda} \eta_\lambda a_\lambda, \tag{16}$$

with only a finite number of elements $a_\lambda \in \mathcal{A}$ different from zero.

A family $\{\eta_\lambda\}$ is called *free* if its elements are linearly independent over $\mathcal{A}$. It is called a *basis* if it is a free generating set, so that every element $\xi \in \mathcal{E}$ can be written uniquely as a combination of the form (16).

A module is called *free* whenever it admits a basis, and it is called *finitely generated* if it admits a finite generating set. Later on, we will refer to these objects as *algebraically finitely generated modules*, to distinguish them from the topologically finitely generated ones.

From now on the *-algebra $\mathcal{A}$ is assumed to be unital.

Example 1.1. *The module $\mathcal{A}^n$, given by the direct sum of $\mathcal{A}$ with itself n-times, is a free finitely generated module for every n. The collection $\{\xi_j\}_{j=1}^n$, where ξ_j is the vector with one in the i-th entry and zeroes elsewhere, is a basis, called the* standard basis of $\mathcal{A}^n$.

If a module $\mathcal{E}$ is finitely generated, there is always a positive integer n and a module surjection $\rho : \mathcal{A}^n \to \mathcal{E}$, such that the image of the standard basis of $\mathcal{A}^n$ is a generating set for $\mathcal{E}$ (not necessarily free).

Definition 1.1. *A right $\mathcal{A}$-module $\mathcal{E}$ is said to be* projective *if it direct summand of a free module, that is there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ such that*

$$\mathcal{F} \simeq \mathcal{E} \oplus \mathcal{E}'. \tag{17}$$

For the proof of equivalence of the above conditions, we refer the reader to [**21**, Section 2.2] (see also the discussion after [**13**, Definition 7]).

Now suppose that $\mathcal{E}$ is both projective and finitely generated, with surjection $\rho : \mathcal{A}^n \to \mathcal{E}$. Then there exists a lift $s : \mathcal{E} \to \mathcal{A}^n$ such that $s \circ \rho = \mathrm{Id}_{\mathcal{E}}$. Then $\mathbf{e} := \rho \circ s$ is an idempotent in $\mathrm{Mat}_n(\mathcal{A})$ satisfying $\mathbf{e^2} = \mathbf{e}$ and $E = \mathbf{e}\mathcal{A}^n$.

The interest in finitely generated projective modules comes from the Serre-Swan theorem, which establishes a correspondence between vector bundles over a compact topological space X and finitely generated projective modules over the dual C*-algebra of continuous functions $C(X)$. This correspondence is actually an equivalence of categories, realized in terms of the functor of sections Γ.

1.2. The Γ-functor. Recall that a vector bundle E over X, in symbols $E \to X$ is a *locally trivial* continuous family of *finite dimensional* vector spaces indexed by X.

The collection of continuous sections of a complex vector bundle $E \to X$, denoted by $\Gamma(E, X)$ or simply $\Gamma(E)$, is naturally a module for the commutative algebra $C(X)$ of continuous functions on the base space, the right action given by scalar multiplication in each fiber:

$$(s \cdot a)(x) = s(x)a(x), \quad \text{for all} \quad s \in \Gamma(E), a \in C(X). \tag{18}$$

If $\tau : E \to E'$ is a bundle map, then there exists a natural map $\Gamma_\tau : \Gamma(E) \to \Gamma(E')$ given by

$$\Gamma_\tau(s) = \tau \circ s,$$

which is $C(X)$-linear by linearity of each fiber map: $\tau_x : E_x \to E'_x$.

Lemma 1.1 ([**11**, Lemma 2.5]). *The correspondence $E \to \Gamma(E), \tau \to \Gamma_\tau$ is a functor from the category of complex vector bundles over X to the category of $C(X)$-modules.*

The Γ-functor carries the operations of duality, Whitney sum and tensor product of bundles, which are defined fiber-wise, to analogous operations on $C(X)$-modules. Indeed, one can easily check the following module isomorphisms:

$$\Gamma(E^*) \simeq \mathrm{Hom}_{C(X)}(\Gamma(E), C(X)),$$
$$\Gamma(E) \oplus \Gamma(E') \simeq \Gamma(E \oplus E').$$

Moreover, by [**11**, Proposition 2.6] one has:

$$\Gamma(E) \otimes_{C(X)} \Gamma(E') \simeq \Gamma(E \otimes E'),$$

where the tensor product on the left hand side makes sense because each $\Gamma(E)$ is a $C(X)$-bimodule, by commutativity.

Let $E \to X$ be any complex vector bundle over X. By [**11**, Proposition 2.9] the $C(X)$-module $\Gamma(E)$ is finitely generated projective. More explicitely, for every complex vector bundle $E \to X$ there exists an idempotent $\mathbf{e}$ in the matrix algebra $\mathrm{Mat}_n(C(X))$ for some n such that $\Gamma(E) \simeq \mathbf{e}C(X)^n$ as modules over $C(X)$. Conversely, any $C(X)$-module of the form $\mathbf{e}C(X)^n$ is the module of sections of some vector bundle over X.

This correspondence is actually an equivalence of categories, as stated in the Serre-Swan theorem of [**19, 20**], that we present here in the same form of [**11**, Theorem 2.10].

Theorem 1.2 (Serre-Swan). *The Γ-functor from vector bundles over a compact space X to finitely generated projective modules over $C(X)$ is an equivalence of categories.*

1.3. Hermitian structures over projective modules. Hermitian vector bundles, that is bundles with a fiber-wise Hermitian product, correspond to finitely generated projective $\mathcal{A}$-modules endowed with an $\mathcal{A}$-valued sesquilinear form. For C*-algebras, the appropriate framework is that of Hilbert C*-modules,

1.4. Hilbert C*-modules and correspondences.

DEFINITION 1.2. *A* pre-Hilbert module *over a C*-algebra A is a right A-module $\mathcal{E}$ with a A-valued Hermitian product, i.e. a map $\langle\cdot,\cdot\rangle_A : \mathcal{E}\times\mathcal{E}\to A$ satisfying*

$$\begin{aligned}\langle\xi,\eta\rangle_A &= \langle\eta,\xi\rangle_A{}^*;\\ \langle\xi,\eta a\rangle_A &= \langle\xi,\eta\rangle_A a;\\ \langle\xi,\xi\rangle_A &\geq 0;\\ \langle\xi,\xi\rangle_A &= 0 \Rightarrow \xi = 0\end{aligned}$$

for all $\xi,\eta\in\mathcal{E}$ and for all $a\in A$.

To lighten notation we shall write $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_A$ whenever possible.

For a pre-Hilbert module $\mathcal{E}$, one can define a scalar valued norm $\|\cdot\|$ using the C*-norm on A:

$$\|\xi\|^2 = \|\langle\xi,\xi\rangle_A\|_A. \tag{19}$$

DEFINITION 1.3. *A* Hilbert C*-module *is a pre-Hilbert module that is complete in the norm* (19)*.*

If one defines $\langle\mathcal{E},\mathcal{E}\rangle$ to be the linear span of elements of the form $\langle\xi,\eta\rangle$ for $\xi,\eta\in\mathcal{E}$, then its closure its a two-sided ideal in A. We say that the Hilbert module $\mathcal{E}$ is *full* whenever $\langle\mathcal{E},\mathcal{E}\rangle$ is dense in A.

EXAMPLE 1.2. *Clearly, a Hilbert C*-module over the field of complex numbers $\mathbb{C}$ is nothing but a Hilbert space.*

EXAMPLE 1.3. *The simplest (in general) noncommutative example of Hilbert C*-module is given by the algebra A itself, with Hermitian product*

$$\langle a,b\rangle = a^*b, \tag{20}$$

and right action given by the algebra product. This module will be denoted by A_A. By the existence of approximate units for C-algebras (cf.* [**9**, Theorem 1.4.8]*), the module A_A is automatically full.*

EXAMPLE 1.4. *Since every C*-algebra is naturally a Hilbert module over itself, one can use this fact to define, for any natural number n, the A module A^n, in analogy with Example 1.1. It consists of n-tuples of elements $\xi_i\in A$, with component-wise right multiplication and well-defined Hermitian product*

$$\langle\xi,\eta\rangle = \sum_{i=1}^{n}\langle\xi_i,\eta_i\rangle_A, \tag{21}$$

for $\xi=(\xi_i)$ and $\eta=(\eta_i)$, which turn it into a Hilbert C-module.*

EXAMPLE 1.5. *Generalizing the previous example, one can start from $\{\mathcal{E}_i\}_{i=1}^n$ a finite set of Hilbert C*-modules over A. The direct sum $\bigoplus_{i=1}^n\mathcal{E}_i$ is a A-module in the obvious way (pointwise multiplication) with inner product defined, as in* (21)*. If all $\mathcal{E}_i=\mathcal{E}$, then $\bigoplus_i^n\mathcal{E}_i$ will be denoted by $\mathcal{E}^n$.*

Things become subtler if $\{\mathcal{E}_i\}_{i\in I}$ is an infinite set of Hilbert A-modules. One defines $\bigoplus_{i\in I}\mathcal{E}_i$ as the set of sequences (ξ_i), with $\xi_i\in\mathcal{E}_i$ and such that $\sum_{i\in I}\langle\xi_i,\xi_i\rangle$ converges in A. Then for $\xi=(\xi_i)$ and $\eta=(\eta_i)$ the inner product

$$\langle\xi,\eta\rangle = \sum_{i\in I}\langle\xi_i,\eta_i\rangle$$

is well-defined and the module $\bigoplus_{i\in I}\mathcal{E}_i$ is complete, hence a Hilbert C-module.*

EXAMPLE 1.6. *If $\mathcal{H}$ is a Hilbert space, the algebraic tensor product $\mathcal{H}\otimes_{\mathrm{alg}} A$ has a natural structure of right A-module and it can be endowed with a A-valued Hermitian product*

$$\langle \xi\otimes a, \eta\otimes b\rangle := \langle \xi,\eta\rangle a^* b,$$

which turns it into a pre-Hilbert module. We denote its completion with $\mathcal{H}\otimes A$. If the Hilbert space $\mathcal{H}$ has an orthonormal basis, then $\mathcal{H}\otimes A$ can be naturally identified with the direct sum module $\bigoplus_i A$ defined previously. If $\mathcal{H}$ is a separable, finite dimensional Hilbert space, then $\mathcal{H}\otimes A$ is denoted with $\mathcal{H}_A$ and it is referred to as the standard Hilbert module.

A Hilbert C*-module $\mathcal{E}$ is *topologically finitely generated* if there exists a finite set $\{\eta_1,\ldots,\eta_n\}$ of elements of $\mathcal{E}$ such that the A-linear span of the η_i's is dense in $\mathcal{E}$. It is said to be *algebraically* finitely generated if every element $\xi\in\mathcal{E}$ is of the form $\sum_{i=1}^n b_i\eta_i$ for some b_i.

DEFINITION 1.4 (*cf.* Definition 1.1). *Let A a unital C*-algebra. A Hilbert C*-module $\mathcal{E}$ is* projective *if it is a direct summand in the free module A^n for some n.*

PROPOSITION 1.3 ([**22**, Corollary 15.4.8]). *Every algebraically finitely generated Hilbert C*-module over a unital algebra is projective.*

Let now $\mathcal{E},\mathcal{F}$ be two Hilbert C*-modules over the same C*-algebra A.

DEFINITION 1.5. *A map $T:\mathcal{E}\to\mathcal{F}$ is said to be an* adjointable operator *if and only if there exists another map $T^*:\mathcal{F}\to\mathcal{E}$ with the property that*

$$\langle T\xi,\eta\rangle = \langle \xi, T^*\eta\rangle \qquad \text{for all}\quad \xi\in\mathcal{E},\eta\in\mathcal{F}\,.$$

Every adjointable operator is automatically linear, and by the Banach–Steinhaus theorem, it is bounded. However, the converse is in general not true: a bounded linear map between Hilbert modules need not be adjointable. To see this, let B be a unital C*-algebra, A a proper ideal and $i:A\to B$ the inclusion. If i were adjointable, we would have $i^*(1)=1$, which is not the case, since $1\notin A$.

The collection of adjointable operators from $\mathcal{E}$ to $\mathcal{F}$ is denoted $\mathcal{L}_A(\mathcal{E},\mathcal{F})$. When $\mathcal{E}=\mathcal{F}$, the adjointable operators form a C*-algebra, that is denoted by $\mathcal{L}_A(\mathcal{E})$.

Inside the space of adjointable operators one can single out a particular class, which is analogous to that of finite rank operators on a Hilbert space. More precisely, for every $\xi\in\mathcal{F},\eta\in\mathcal{E}$ one defines the operator $\theta_{\xi,\eta}:\mathcal{E}\to\mathcal{F}$ as

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle, \qquad \forall\zeta\in\mathcal{E} \tag{22}$$

This is an adjointable operator, with adjoint $\theta^*_{\xi,\eta}:\mathcal{F}\to\mathcal{E}$ given by $\theta_{\eta,\xi}$.

We denote by $\mathcal{K}_A(\mathcal{E},\mathcal{F})$ the closed linear subspace of $\mathcal{L}_A(\mathcal{E},\mathcal{F})$ spanned by

$$\{\theta_{\xi,\eta}\,|\,\xi,\eta\in\mathcal{E}\}, \tag{23}$$

which we refer to as the space of *compact adjointable operators*.

In particular, $:=\mathcal{K}_A(\mathcal{E},\mathcal{E})\subseteq\mathcal{L}_A(\mathcal{E})$ is an ideal, hence a C*-subalgebra, whose elements are referred to as *compact endomorphisms*. For any C*-algebra A, seen as a Hilbert module over itself, we have $\mathcal{K}_A(A)\simeq A$.

Finally, the C*-algebraic dual of $\mathcal{E}$, is defined as the space

$$\mathcal{E}^* := \{\phi\in\mathcal{L}_A(\mathcal{E},A)\mid \exists\,\xi\in\mathcal{E}\text{ with }\phi(\eta)=\langle\xi,\eta\rangle\ \forall\eta\in\mathcal{E}\}\,. \tag{24}$$

Thus, with $\xi\in\mathcal{E}$, if $\lambda_\xi:\mathcal{E}\to A$ is the operator defined by $\lambda_\xi(\eta)=\langle\xi,\eta\rangle$, for all $\eta\in\mathcal{E}$, every element of $\mathcal{E}^*$ is of the form λ_ξ for some $\xi\in\mathcal{E}$.

We finish by describing how finite projective modules can be characterized in terms of their algebra of compact operators.

THEOREM 1.4 ([**11**, Proposition 3.9]). *Let A be a unital C*-algebra and $\mathcal{E}$ a right A-module. Then $\mathcal{E}$ is finitely generated projective if and only if*

$$1 = \mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_A(\mathcal{E}).$$

1.5. Correspondences. Note that whenever we have a Hilbert A-module $\mathcal{E}$, by its very definition, the algebra $\mathcal{L}_A(\mathcal{E})$ and its subalgebra $\mathcal{K}_A(\mathcal{E})$ act adjointably on $\mathcal{E}$ from the left. More generally, whenever one has a map $\phi : A \to \mathcal{L}_A(\mathcal{E})$, it is possible to endow the Hilbert module $\mathcal{E}$ with a left A-module structure:

$$a \cdot \xi = \phi(a)(\xi) \tag{25}$$

for all $\xi \in \mathcal{E}$ and $a \in A$. This motivates the following:

DEFINITION 1.6. *An (A, B)-*correspondence*, also called a* C*-correspondence from A to B, *is a right Hilbert B-module $\mathcal{E}$ endowed with a *-homomorphism $\phi : A \to \mathcal{L}_B(\mathcal{E})$. If $A = B$ we refer to $(\mathcal{E}, \phi)$ as a* C*-correspondence over A.

When no confusion arises, we will refer to the pair $(\mathcal{E}, \phi)$ by using the compact notation $\mathcal{E}_\phi$.

Two C*-correspondences $\mathcal{E}_\phi$ and $\mathcal{F}_\psi$ over the same algebra A are called *isomorphic* if and only if there exists a unitary $U \in \mathcal{L}_A(\mathcal{E}, \mathcal{F})$ intertwining ϕ and ψ.

C*-correspondences can be *composed*: given an (A, B)-correspondence $\mathcal{E}_\phi$ and a (B, C)-correspondence $\mathcal{F}_\psi$, one can construct an (A, C)-correspondence, named the *interior tensor product* of $\mathcal{E}_\phi$ and $\mathcal{F}_\psi$.

As a first step, one constructs the *balanced tensor product* $\mathcal{E} \otimes_B \mathcal{F}$ which is a quotient of the algebraic tensor product $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ by the subspace generated by elements of the form

$$\xi b \otimes \eta - \xi \otimes \psi(b)\eta, \tag{26}$$

for all $\xi \in \mathcal{E},\ \eta \in \mathcal{F},\ b \in A$.

This has a natural structure of right module over C given by

$$(\xi \otimes \eta)c = \xi \otimes (\eta c)$$

and a C-valued inner product defined on simple tensors as

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_C := \langle \eta_1, \psi(\langle \xi_1, \xi_2 \rangle_A)\eta \rangle_C, \tag{27}$$

and extended by linearity.

The inner product is well-defined and has all required properties; in particular, the null space $N = \{\zeta \in \mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}\,;\ \langle \zeta, \eta \rangle = 0\}$ is shown to coincide with the subspace generated by elements of the form in (26).

One then defines $\mathcal{E} \widehat{\otimes}_\psi \mathcal{F}$ to be the right Hilbert module obtained by completing $\mathcal{E} \otimes_B \mathcal{F}$ in the norm induced by (27).

Moreover for every $T \in \mathcal{L}_B(\mathcal{E})$, the operator defined on simple tensors by

$$\xi \otimes \eta \mapsto T(\xi) \otimes \eta$$

extends to a well-defined operator $\phi_*(T) := T \otimes \mathrm{Id}$. It is adjointable with adjoint given by $T^* \otimes \mathrm{Id} = \phi_*(T^*)$. In particular, this means that there is a left action of A defined on simple tensors by

$$(\phi \otimes_\psi \mathrm{Id})(a)(\xi \otimes \eta) = \phi(a)\xi \otimes \eta,$$

and extended by linearity to a map

$$\phi \otimes_\psi \mathrm{Id} : A \to \mathcal{L}_C(\mathcal{E} \widehat{\otimes}_\psi \mathcal{F}),$$

thus turning $\mathcal{E}\widehat{\otimes}_\psi\mathcal{F}$ into an (A, C)-correspondence.

PROPOSITION 1.5 ([**3**, Proposition 5.2]). *Given two *-algebra homomorphisms* $\phi : A \to B$ *and* $\psi : B \to C$*, one has an isomorphism of* (A, C)*-correspondences.*

$$\mathcal{E}_\phi\widehat{\otimes}_\psi\mathcal{E}_\psi = \mathcal{E}_{\psi\circ\phi}.$$

Taking the interior tensor power is an associative operation on isomorphism classes of C*-correspondences.

1.6. Morita equivalence.

DEFINITION 1.7. *An* (A, B)-equivalence bimodule *is a full* (A, B)*-correspondence* $\mathcal{E}_\phi$ *where the left action* $\phi : A \to \mathcal{L}_B(E)$ *is an isomorphism onto* $\mathcal{K}_B(E)$*. One says that two C*-algebras* A *and* B *are Morita equivalent if such an* (A, B)*-equivalence bimodule exists.*

Every full Hilbert A-module $\mathcal{E}$ is a $(\mathcal{K}_A(\mathcal{E}), A)$-equivalence bimodule.

Morita equivalence is a weaker equivalence relation that isomorphism. Indeed, given an isomorphism $\phi : A \to B$, the C*-correspondence (B_B, ϕ) is an (A, B)-equivalence bimodule.

REMARK 1.1. *Let* (A, B) *two unital C*-algebras. Any C*-correspondence* $\mathcal{E}_\phi$ *implementing the Morita equivalence between the two algebras is finitely generated projective as an A-module. To see this, one can use the fact that* $A \simeq \mathcal{K}_B(E)$ *is unital and Theorem 1.4 to obtain the claim.*

THEOREM 1.6. *Morita equivalence is an equivalence relation.*

Morita equivalence is a purely noncommutative notion. Indeed, Morita equivalent algebras have isomorphic centers (*cf.* [**15**, Section 2.3]), and therefore two commutative C*-algebras are Morita equivalent if and only if they are isomorphic.

In noncommutative topology Morita equivalence is the most natural equivalence relation to consider: Morita equivalent C*-algebras have, among other things, the same representation theory and the same K-theory and K-homology (and also bivariant K-theory) groups.

2. Line bundles, Self-Morita equivalence bimodules, and the Picard group

Let A be a C*-algebra. A self-Morita equivalence bimodule over A is any C*-correspondence $\mathcal{E}_\phi$ over A which implements the reflexivity of Morita equivalence for a C*-algebra A. The simplest example of self-Morita equivalence bimodule is the algebra A itself together with the identity map. Equivalently, self-Morita equivalence bimodules can be defined in terms of (A, A)-bimodules, but we prefer adopting the former approach.

DEFINITION 2.1. *A* self-Morita equivalence bimodule *over A is a C*-correspondence* $(\mathcal{E}, \phi)$*, where* $\mathcal{E}$ *is full and* $\phi : A \to \mathcal{K}_A(\mathcal{E})$ *is an isomorphism.*

Two self-Morita equivalence bimodules are isomorphic if and only if they are isomorphic as C*-correspondences; by [**1**, Corollary 1.2] the left inner product is automatically preserved.

EXAMPLE 2.1. *The prototypical commutative example of a self-Morita equivalence bimodule is provided by* $A = C(X)$*, the C*-algebra of continuous functions on a compact topological space* X*,* $\mathcal{E} = \Gamma(X)$ *the* $C(X)$*-module of sections of a Hermitian line bundle* $L \to X$ *and* ϕ *the trivial action. For this reason, one is led to think of self-Morita equivalence bimodules as* noncommutative *line bundles.*

Similarly to line bundles in classical geometry, self-Morita equivalence bimodules are *invertible* in some sense. More precisely, if $(\mathcal{E}, \phi)$ is a self-Morita equivalence bimodule over A, the dual Hilbert module $\mathcal{E}^*$ as defined in (24), can be made into a self-Morita equivalence bimodule over A as well.

First of all, $\mathcal{E}^*$ is given the structure of a (right) Hilbert C^*-module over A by means of the map ϕ. Recall that the elements of $\mathcal{E}^*$ are of the form λ_ξ for some $\xi \in \mathcal{E}$, with $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in \mathcal{E}$. The right action of A on $\mathcal{E}^*$ is given by

$$\lambda_\xi \, b := \lambda_\xi \circ \phi(b) = \lambda_{\phi(b)\xi} \, ,$$

the second equality being easily established. The A-valued Hermitian product on $\mathcal{E}^*$ uses the left $\mathcal{K}_A(\mathcal{E})$-valued Hermitian product on $\mathcal{E}$:

$$\langle \lambda_\xi, \lambda_\eta \rangle := \phi^{-1}(\theta_{\xi,\eta}) \, ,$$

and $\mathcal{E}^*$ is full as well. Next, define a $*$-homomorphism $\phi^* : A \to \mathcal{L}_A(\mathcal{E}^*)$ by

$$\phi^*(b)(\lambda_\xi) := \lambda_{\xi \cdot b^*},$$

which is in fact an isomorphism $\phi^* : A \to \mathcal{K}_A(\mathcal{E}^*)$. Thus, the pair $(\mathcal{E}^*, \phi^*)$ is a self-Morita equivalence bimodule over A, according to Definition 2.1. It is the *inverse* to the self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ with respect to the operation given by the interior tensor product.

For a C*-algebra A, the collection of unitary equivalence classes of self-Morita equivalence bimodules over A has a natural group structure with respect to the interior tensor product, with identity element the class of the self-Morita equivalence bimodule (A, Id). Thinking of self-Morita equivalence bimodules as *noncommutative line bundles*, this group is named the *Picard group* of A, denoted by $\mathrm{Pic}(A)$, in analogy with the classical situation for which the Picard group of a space X is the group of isomorphism classes of line bundles over X.

2.0.1. *Automorphism.*

Definition 2.2. *Let $\mathcal{A}$ be a *-algebra. A *-automorphism is a linear invertible map $\alpha : \mathcal{A} \to \mathcal{A}$ that satisfies*

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(a^*) = \alpha(a)^*.$$

The group of automorphisms of $\mathcal{A}$ is denoted by $\mathrm{Aut}(\mathcal{A})$.

As a consequence of Gelfand duality, for a commutative C*-algebra $A \simeq C(X)$ we have an isomorphism between the group $Aut(A)$ and the group of one-to-one homeomorphisms of X.

Let now A be a C*-algebra. Recall that every automorphism $\alpha \in \mathrm{Aut}(A)$ yields a self-Morita equivalence bimodule (A, α) for A. If β is another automorphism, then by Proposition 1.5 their product is equivalent to $(A, \beta\alpha)$. Thus one has an anti-homomorphism from the group of automorphisms of A to the Picard group, that is denoted by

$$\Phi_A : \mathrm{Aut}(A) \to \mathrm{Pic}(A).$$

2.0.2. *The Picard group of a commutative C*-algebra.* Recall that for a commutative C*-algebra A, the classical Picard group $\mathrm{CPic}(A)$ is the group of Hilbert line bundles over the spectrum $\sigma(A)$ of A (*cf.* [**10**]), and one can prove that it agrees with the group $\mathrm{Pic}(\sigma(X))$. The following result relates the noncommutative Picard group with its classical counterpart:

Theorem 2.1 ([**1**, Theorem 1.12]). *Let A a commutative C*-algebra. Then* $\mathrm{CPic}(A)$ *is a normal subgroup in* $\mathrm{Pic}(A)$ *and*

$$\mathrm{Pic}(A) \simeq \mathrm{CPic}(A) \rtimes \mathrm{Aut}(A),$$

where the action of $\mathrm{Aut}(A)$ *is given by conjugation.*

CHAPTER 4

Cuntz–Pimsner algebras and circle bundles

1. Pimsner algebras

In his breakthrough paper [**17**], starting from a full C*-correspondence $(\mathcal{E}, \phi)$ such that the left action $\phi : A \to \mathcal{L}_A(\mathcal{E})$ is an isometric $*$-homomorphism, Pimsner constructed two C*-algebras: these are now referred to as the *Toeplitz algebra* and the *Cuntz-Pimsner algebra* of the C*-correspondence $(\mathcal{E}, \phi)$, denoted by $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{O}_E$ respectively. The former is actually an extension of the second, and can be thought of as a generalization of the Toeplitz algebra, while the latter encompasses a large class of examples, like Cuntz-Krieger algebras and crossed products by the integers. Both algebras are characterized by universal properties and depend only on the isomorphism class of the C*-correspondence.

We will conclude with six term exact sequences in KK-theory, naturally associated to any Pimsner algebra.

1.1. The Toeplitz algebra of a full C*-correpsondence. In this section we will work under the following assumption:

Assumption 1.1. *The image of ϕ is contained in $\mathcal{K}_A(\mathcal{E})$.*

Iterating the construction of the interior tensor product module described in Subsection 1.4, one considers the the k-fold tensor products

$$\mathcal{E}^{(k)} := \mathcal{E}^{\widehat{\otimes}_\phi^k} \quad k > 0. \tag{28}$$

Then one builds the infinite direct sum module

$$\mathcal{E}_+ = A \oplus \bigoplus_{k=1}^{\infty} \mathcal{E}^{(k)}. \tag{29}$$

It is a C*-correspondence over A, with left action ϕ_+ given, for all $a \in A$, by

$$\phi_+(a)(\xi_1 \otimes \ldots \otimes \xi_k) = \phi(a)\xi_1 \otimes \ldots \otimes \xi_k;$$

for $k \geq 1$ and $\xi_1, \ldots, \xi_k \in \mathcal{E}$ and

$$\phi_+(a)(a') = aa'$$

for $a' \in B$. It is referred to as the *(positive) Fock correspondence* associated to the correspondence $\mathcal{E}_\phi$.

One can naturally associate to any element $\xi \in \mathcal{E}$ a creation and an annihilation operator in $\mathcal{L}_A(\mathcal{E}_+)$; the creation operator is given by

$$T_\xi(\xi_1 \otimes \ldots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \ldots \otimes \xi_k, \qquad T_\xi(b) = \xi b, \tag{30}$$

and its adjoint is the annihilation operator

$$T_\xi^*(\xi_1 \otimes \ldots \otimes \xi_k) = \phi(\langle \xi, \xi_1 \rangle)\xi_2 \otimes \ldots \otimes \xi_k, \qquad T_\xi^*(b) = 0. \tag{31}$$

Definition 1.1. *The Toeplitz algebra$\mathcal{T}_E$ of the C*-correspondence $\mathcal{E}_\phi$ is the smallest C*-subalgebra of $\mathcal{L}_A(\mathcal{E}_+)$ that contains all the T_ξ for $\xi \in \mathcal{E}$.*

The algebra is universal in the following sense:

THEOREM 1.2 ([**17**, Theorem 3.4]). *Let $(\mathcal{E},\phi)$ be a full C*-correspondence over A, C any C*-algebra and $\psi : A \to C$ a $*$-homomorphism with the property that there exist elements $t_\zeta \in C$ for all $\zeta \in \mathcal{E}$ such that*

(1) $\alpha t_\xi + \beta t_\eta = t_{\alpha\xi+\beta\eta}$ *for all* $\alpha,\beta \in \mathbb{C}$ *and* $\xi,\eta \in \mathcal{E}$*;*
(2) $t_\xi \psi(a) = t_{\xi a}$ *and* $\psi(a)t_\xi = t_{\phi(a)\xi}$ *for all* $\xi \in \mathcal{E}$ *and* $a \in A$ *;*
(3) $t_\xi^* t_\eta = \psi(\langle \xi,\eta\rangle) \in A$ *for all* $\xi,\eta \in \mathcal{E}$*;*

then there exists a unique extension $\tilde{\psi} : \mathcal{T}_\mathcal{E} \to C$ that maps T_ξ to t_ξ.

1.2. The Pimsner Algebra of a full C*-correspondence. The (Cuntz-)Pismner algebra $\mathcal{O}_E$ of a full C*-correspondence $(\mathcal{E},\phi)$ is a quotient of the Toeplitz algebra. Under Assumption 1.1 one has the following:

DEFINITION 1.2. *The Pimsner algebra $\mathcal{O}_\mathcal{E}$ of the C*-correspondence $(\mathcal{E},\phi)$ is the quotient algebra appearing in the exact sequence*

$$0 \longrightarrow \mathcal{K}_A(\mathcal{E}_+) \longrightarrow \mathcal{T}_\mathcal{E} \xrightarrow{\pi} \mathcal{O}_\mathcal{E} \longrightarrow 0. \tag{32}$$

It is easy to check that since $\mathcal{E}$ is full, then $\mathcal{E}_+$ is a full Hilbert module as well; hence $\mathcal{K}_A(\mathcal{E}_+)$ is by definition Morita equivalent to the algebra A.

The image of an element $T_\xi \in \mathcal{T}_\mathcal{E}$ under the quotient map π will be denoted by S_ξ.

EXAMPLE 1.1. *Let $A = \mathbb{C}$ and $\mathcal{E} = \mathbb{C}^n$ and ϕ the left action by multiplication. If one chooses a basis for $\mathbb{C}^n$, then the Toeplitz algebra of $(\mathcal{E},\phi)$ is generated by n isometries $V_1,\dots,V_n$ satisfying $\sum_i V_iV_i^* \le 1$. This is the Toeplitz extension for the Cuntz algebras $\mathcal{O}_n$:*

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(V_1,\dots,V_n) \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

In particular, for $n = 1$ one gets the classical Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{T}_\mathbb{C} \longrightarrow C(S^1) \longrightarrow 0. \tag{33}$$

EXAMPLE 1.2. *More generally, if the module $\mathcal{E}$ is finitely generated projective, the Pimsner algebra of $(\mathcal{E},\phi)$ can be realized explicitly in terms of generators and relations* [**14**]. *Since $\mathcal{E}$ is finitely generated projective, there exist $\{\eta_j\}_{j=1}^n$ such that for any $b \in A$:*

$$\phi(b)\eta_j = \sum_{j=1}^n \eta_i \langle \eta_i, \phi(b)\eta_j\rangle_A,$$

The C-algebra $\mathcal{O}_E$ is the universal C*-algebra generated by A together with n operators $S_1,\dots,S_n$, satisfying*

$$S_i^*S_j = \langle \eta_i,\eta_j\rangle_A, \quad \sum\nolimits_j S_jS_j^* = 1, \quad \text{and} \quad aS_j = \sum\nolimits_i S_i\langle \eta_i, \phi(a)\eta_j\rangle_A, \tag{34}$$

for $a \in A$, and $j = 1,\dots,n$. The generators S_i are partial isometries if and only if the frame satisfies $\langle \eta_i,\eta_j\rangle = 0$ for $i \neq j$. For $A = \mathbb{C}$ and $\mathcal{E}$ a Hilbert space of dimension n, one recovers the Cuntz algebra $\mathcal{O}_n$ of Example 1.1.

EXAMPLE 1.3. *Let A be a C*-algebra and $\alpha : A \to A$ an automorphism of A. Then $\mathcal{E} = B$ can be naturally made into a C*-correspondence.*
The right Hilbert A-module structure is the standard one, with right A-valued inner product $\langle a,b\rangle_A = a^*b$.
The automorphism α is used to define the left action via $a \cdot b = \alpha(a)b$ and left A-valued inner product given by ${}_A\langle a,b\rangle = \alpha(a^*b)$.

Each module $\mathcal{E}^{(k)}$ is isomorphic to A as a right-module, with left action

$$a \cdot (x_1 \otimes \cdots \otimes x_k) = \alpha^k(a)\alpha^{k-1}(x_1) \cdots \alpha(x_{k-1})x_k. \tag{35}$$

The corresponding Pimsner algebra $\mathcal{O}_E$ coincides then with the crossed product algebra $A \rtimes_\alpha \mathbb{Z}$, while the Toeplitz algebra $\mathcal{T}_\mathcal{E}$ agrees with the Toeplitz algebra $\mathcal{T}(A, \alpha)$ of [**16**, Section 2]*, that appears in the exact sequence*

$$0 \longrightarrow A \otimes \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{T}(A, \alpha) \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow 0. \tag{36}$$

Similarly to the Toeplitz algebra $\mathcal{T}_E$ (*cf.* Theorem 1.2), the Pimsner algebra $\mathcal{O}_E$ can be characterized in terms of its universal properties.

THEOREM 1.3 ([**17**, Theorem 3.12]). *Let $(\mathcal{E}, \phi)$ as above, C a C^*-algebra and $\psi : A \to C$ a $*$-homomorphism. Suppose that there exist elements $s_\zeta \in C$ for all $\zeta \in \mathcal{E}$ such that*

(1) $\alpha \cdot s_\xi + \beta \cdot s_\eta = s_{\alpha \cdot \xi + \beta \cdot \eta}$ *for all $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in \mathcal{E}$;*
(2) $s_\xi \cdot \psi(a) = s_{\xi \cdot a}$ *and* $\psi(a) \cdot s_\xi = s_{\phi(a)(\xi)}$ *for all $\xi \in \mathcal{E}$ and $a \in A$;*
(3) $s_\xi^* s_\eta = \psi(\langle \xi, \eta \rangle)$ *for all $\xi, \eta \in \mathcal{E}$;*
(4) $s_\xi s_\eta^* = \psi(\phi^{-1}(\theta_{\xi,\eta}))$ *for all $\xi, \eta \in \mathcal{E}$;*

then there exists a unique $*$-homomorphism $\widehat{\psi} : \mathcal{O}_E \to C$ with $\widehat{\psi}(S_\xi) = s_\xi$ for all $\xi \in \mathcal{E}$.

2. Circle actions

Pimsner algebras are endowed with a natural action of the circle $\gamma : S^1 \to \mathrm{Aut}(\mathcal{O}_E)$, known as the *gauge action*, a feature they have in common with ordinary crossed products by the integers and with Cuntz-Krieger algebras.

In this section we will describe the gauge action on a Pimsner algebra and its properties, and recall the theory of C*-algebras endowed with a circle action, focusing on their connection with the notion of generalized crossed products.

2.1. The gauge action. By the universal properties of Proposition 1.3 (with $C = A$, ψ the identity and $s_\xi := zS_\xi$), the map

$$S_\xi \to \sigma_w(S_\xi) := wS_\xi, \qquad w \in S^1,$$

extends to an automorphism of $\mathcal{O}_E$, that we will denote with γ. The action is strongly continuous.

The fixed point algebra $\mathcal{O}_E^\gamma$ is in general bigger than the algebra of scalars A. In the case of a self-Morita equivalence bimodule the two algebras agree, as shown by the following theorem.

THEOREM 2.1. *Let $\mathcal{O}_E$ be the Pimsner algebra of a full C*-correspondence over A. Then $\mathcal{E}$ is a self-Morita equivalence bimoduleif and only if the fixed point algebra $\mathcal{O}_E^\gamma$ agrees with the algebra of scalars A.*

2.2. Algebras and circle actions. Let A be a C^*-algebra endowed with a strongly continuous action $\sigma : S^1 \to \mathrm{Aut}(A)$. For each $k \in \mathbb{Z}$, one defines the *k-th spectral subspace* for the action σ to be

$$A_k := \left\{ \xi \in A \mid \sigma_w(\xi) = w^{-k}\, \xi \quad \text{for all } w \in S^1 \right\}.$$

Clearly, the invariant subspace $A_0 \subseteq A$ is a C*-subalgebra of A, with unit whenever A is unital; this is the *fixed-point subalgebra* A^σ for the action.

For every pair of integers $k, l \in \mathbb{Z}$, the subspace $A_k A_l$–meant as the *closed* linear span of the set of products xy with $x \in A_k$ and $y \in A_l$–is contained in A_{k+l}. Thus, the algebra A is $\mathbb{Z}$-graded and the grading is compatible with the involution, that is $A_k^* = A_{-k}$ for all $k \in \mathbb{Z}$.

In particular, for any $k \in \mathbb{Z}$ the space $A_k^* A_k$ is a closed two-sided ideal in A_0. Thus, each spectral subspace A_k has a natural structure of Hilbert A_0-bimodule (not necessarily full) with left and right Hermitian products:

$$ {}_{A_0}\langle x, y \rangle = xy^*, \qquad \langle x, y \rangle_{A_0} = x^* y, \qquad \text{for all } x, y \in A_k. \tag{37} $$

Definition 2.1. *The action is said to be* saturated *if*

$$ A_1^* A_1 = A_0 = A_1 A_1^*, \tag{38} $$

The condition above is equivalent to the condition that all bimodules A_k are full, that is $A_k^* A_k = A_0 = A_k A_k^*$ for all $k \in \mathbb{Z}$. When this happens, all bimodules A_k are self-Morita equivalence bimodules for A_0, with isomorphisms $\phi : A_0 \to \mathcal{K}_{A_0}(A_k)$ given by

$$ \phi(a)(\xi) := a\,\xi, \qquad \text{for all } a \in A_0,\ \xi \in A_k. \tag{39} $$

Theorem 2.2. [**2**, Theorem 3.5] *Let A be a C^*-algebra with a strongly continuous and saturated action of the circle. Then the Pimsner algebra $\mathcal{O}_{A_1}$ of the self-Morita equivalence (A_1, ϕ), with ϕ as in is* (39)*, is isomorphic to A. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_1$.*

Bibliography

[1] B. Abadie and R. Exel, Hilbert C^*-bimodules over commutative C^*-algebras and an isomorphism condition for quantum Heisenberg manifolds, Rev. Math. Phys. **9** (1997), no. 4, 411–423. MR 1456142

[2] F. Arici, J. Kaad and G. Landi, Pimsner algebras and Gysin sequences from principal circle actions, J. Noncommut. Geom. **10** (2016), no. 1, 29–64. MR3500816

[3] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York, 1968. MR0249491

[4] B. Blackadar, Shape theory for C^*-algebras, Math. Scand. **56** (1985), no. 2, 249–275. MR 0813640

[5] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, 122, Springer-Verlag, Berlin, 2006. MR 2188261

[6] L. G. Brown, P. Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, Pacific J. Math. **71** (1977), no. 2, 349–363. MR0463928

[7] J. Cuntz, Simple C^*-algebras generated by isometries, Comm. Math. Phys. **57** (1977), no. 2, 173–185. MR 0467330

[8] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, Invent. Math. **56** (1980), no. 3, 251–268. MR 0561974

[9] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, 1996. MR1402012

[10] M. J. Dupré and R. M. Gillette, *Banach bundles, Banach modules and automorphisms of C^*-algebras*, Research Notes in Mathematics, 92, Pitman (Advanced Publishing Program), Boston, MA, 1983. MR0721812

[11] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehrbücher., Birkhäuser Boston, Inc., Boston, MA, 2001. MR1789831

[12] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995. MR1325694

[13] G. Landi, *An introduction to noncommutative spaces and their geometries*, Lecture Notes in Physics. New Series m: Monographs, 51, Springer-Verlag, Berlin, 1997. MR1482228

[14] T. Kajiwara, C. Pinzari and Y. Watatani, Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules, J. Funct. Anal. **159** (1998), no. 2, 295–322. MR1658088

[15] M. Khalkhali, *Basic noncommutative geometry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2009. MR2567651

[16] M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras, J. Operator Theory **4** (1980), no. 1, 93–118. MR 0587369

[17] M. V. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by **Z**, in *Free probability theory (Waterloo, ON, 1995)*, 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI. MR1426840

[18] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, 60, American Mathematical Society, Providence, RI, 1998. MR1634408

[19] J.-P. Serre, Modules projectifs et espaces fibrés à fibre vectorielle, in *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23*, 18 pp, Secrétariat mathématique, Paris. MR0177011

[20] R. G. Swan, Vector bundles and projective modules, Trans. Amer. Math. Soc. **105** (1962), 264–277. MR0143225

[21] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994. MR1269324

[22] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR1222415