Noncommutative circle bundles, Pimsner Algebras and Gysin Sequences

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Advances in Noncommutative Geometry
University Paris Diderot
April 21, 2015

The Gysin Sequence for Quantum Lens Spaces
F. Arici, S. Brain, G. Landi
arXiv:1401.6788 [math.QA], *to appear in JNCG*;

Pimsner Algebras and Gysin Sequences from Principal Circle Actions
F. Arici, J. Kaad, G. Landi
arXiv:1409.5335 [math.QA], *to appear in JNCG*;

Principal Circle Bundles and Pimsner Algebras
F. Arici, F. D'Andrea, G. Landi
in preparation.

Principal circle bundles are a natural framework for many problems in mathematical physics:

- $\mathrm{U}(1)$-gauge theory;
- T-duality;
- Chern Simons field theories.

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- T-duality;
- Chern Simons field theories.

The Gysin sequence: long exact sequence in cohomology for any sphere bundle. In particular, for a principal circle bundle: $\mathrm{U}(1) \hookrightarrow P \xrightarrow{\pi} X$.

$$\cdots \longrightarrow H^k(P) \xrightarrow{\pi_*} H^{k-1}(X) \xrightarrow{e\cup} H^{k+1}(X) \xrightarrow{\pi^*} H^{k+1}(P) \longrightarrow \cdots$$

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where α is the mutiliplication by the Euler class

$$\chi(\mathcal{L}) = 1 - [\mathcal{L}] \tag{2}$$

of the line bundle $\mathcal{L} \to X$ with associated circle bundle $\pi : P \to X$.

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Let $\mathcal{A}$ be a complex unital algebra that it is a right comodule algebra over $\mathcal{O}(\mathrm{U}(1))$, i.e we have a homomorphism of unital algebras

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We will denote by

$$\mathcal{B} := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1\}$$

the unital subalgebra of coinvariant elements for the coaction.

Definition

One says that the datum $(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{B})$ is a *quantum principal* $\mathrm{U}(1)$*-bundle* when the *canonical map*

$$\chi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)) , \quad x \otimes y \mapsto x \cdot \Delta_R(y) ,$$

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Examples of quantum principal $\mathrm{U}(1)$-bundles: quantum spheres and lens spaces over quantum projective spaces (both θ and q-deformations).

Graded algebra structure: the coordinate algebra decomposes as a direct sum of line bundles over $\mathcal{B}$.

Let $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ be a $\mathbb{Z}$-graded unital algebra and let $\mathcal{O}(\mathrm{U}(1))$ as before.
The unital algebra homomorphism

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)) \quad x \mapsto x \otimes z^{-n}, \text{ for } x \in \mathcal{A}_n .$$

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Question: when is a graded algebra a principal circle bunlde?

Definition

Let $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ a $\mathbb{Z}$-graded algebra. $\mathcal{A}$ is *strongly graded* if and only if any of the following equivalent conditions is satisfied.

1. For all $n, m \in \mathbb{Z}$ we have $\mathcal{A}_n\mathcal{A}_m = \mathcal{A}_{n+m}$.
2. For all $n \in \mathbb{Z}$ we have $\mathcal{A}_n\mathcal{A}_{-n} = \mathcal{A}_0$.
3. $\mathcal{A}_1\mathcal{A}_{-1} = \mathcal{A}_0 = \mathcal{A}_{-1}\mathcal{A}_1$.

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strong grading $\longleftrightarrow$ principal action

To prove bijectivity of χ, one has to construct sequences

$$\{\xi_j\}_{j=1}^N\,,\ \{\beta_i\}_{i=1}^M \text{ in } \mathcal{A}_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^N\,,\ \{\alpha_i\}_{i=1}^M \text{ in } \mathcal{A}_{-1}$$

with the property that

$$\sum_{j=1}^N \xi_j \eta_j = 1_{\mathcal{A}} = \sum_{i=1}^M \alpha_i \beta_i .$$

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This means that the modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ are finetely generated projective. Indeed, we construct idempotents

$$\Phi_1 : \mathcal{A}_1 \to (\mathcal{A}_0)^N \qquad \Psi_1 : (\mathcal{A}_0)^N \to \mathcal{A}_1$$

$$\Phi_{-1} : \mathcal{A}_{-1} \to (\mathcal{A}_0)^N \qquad \Psi_{-1} : (\mathcal{A}_0)^N \to \mathcal{A}_1$$

with $\Psi_1 \Phi_1 = \mathrm{Id}_{\mathcal{A}_1}$ and $\Psi_{-1} \Phi_{-1} = \mathrm{Id}_{\mathcal{A}_{-1}}$.

The module $\mathcal{A}_1$ and its inverse $\mathcal{A}_{-1}$ play a crucial role.
They can be thought of as modules of sections of line bundles.

This phenomenon is related to a natural construction: Pimsner algebras.

Definition

A *self Morita equivalence bimodule (SMEB)* over B is a pair (E, ϕ) where E is a full right Hilbert C*-module over B and

$$\phi : B \to \mathcal{K}(E)$$

is an isomorphism.

Example: $A = C(X)$ and $E = \Gamma(\mathcal{L})$ the module of sections of a Hermitian line bundle $\mathcal{L} \to X$.

The C*-algebraic dual

$$E^* := \{\lambda_\xi, \xi \in E \mid \lambda_\xi(\eta) = \langle \xi, \eta \rangle\} \subseteq \mathrm{Hom}^*_B(E, B)$$

can be given the structure of a (right) Hilbert C^*-module over B using ϕ, with right action

$$\lambda_\xi \, b := \lambda_\xi \phi(b) \, ,$$

and inner product on E^* is given by

$$\langle \lambda_\xi, \lambda_\eta \rangle := \phi^{-1}(|\xi\rangle\langle\eta|) \, .$$

If we define ϕ^* as

$$\phi^*(a)(\lambda_\xi) := \lambda_{\xi \cdot a^*} \, ,$$

the pair (ϕ^*, E^*) gives an isomorphism $\phi^* : B \to \mathcal{K}(E^*)$.

We can take interior tensor product modules, that we will denote using with

$$E^{(n)} := \begin{cases} E^{\widehat{\otimes}_\phi n} & n > 0 \\ B & n = 0 \\ (E^*)^{\widehat{\otimes}_{\phi^*} n} & n < 0 \end{cases}.$$

Out of these we construct the Hilbert module

$$\mathcal{E}_\infty := \bigoplus_{n \in \mathbb{Z}} E^{(n)}$$

on which we will represent the Pimsner algebra.

We have natural creation and annihilation operators $S_\xi, S_\xi^* : \mathcal{E}_\infty \to \mathcal{E}_\infty$, defined at levels $1, 0, -1$ by

$$S_\xi(\eta) = \xi \otimes \eta \qquad S_\xi^*(\eta) = \langle \xi, \eta \rangle$$
$$S_\xi(b) = \xi b \qquad S_\xi^*(b) = \lambda_\xi b$$
$$S_\xi(\lambda_\eta) = \phi^{-1}(\theta_{\xi,\eta}) \qquad S_\xi^*(\lambda_\eta) = \lambda_\xi \otimes \lambda_\eta,$$

and extended on higher tensor powers.

Definition

The *Pimsner algebra* of the pair (ϕ, E), denoted $\mathcal{O}_E$, is the smallest C^*-subalgebra of $\mathrm{End}^*_B(\mathcal{E}_\infty)$ which contains the operators $S_\xi : \mathcal{E}_\infty \to \mathcal{E}_\infty$ for all $\xi \in E$.

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The representation of $\mathrm{U}(1)$ on $\mathcal{E}_\infty$ given by

$$t \circ x = t^n x \quad \forall t \in S^1, \; x \in E^{(n)}$$

induces an circle action on $\mathcal{O}_E$.

Let A be a C^*-algebra with an action $\{\sigma_z\}_{z \in S^1}$. When can we recover A as a Pimsner algebra?

For each $n \in \mathbb{Z}$, one can define the spectral subspaces

$$A_{(n)} := \left\{ \xi \in A \mid \sigma_z(\xi) = z^{-n} \xi \quad \text{for all } z \in S^1 \right\}.$$

It is easy to check that $A_{(n)}^* = A_{(-n)}$ and that $A_{(n)} A_{(m)} \subseteq A_{(n+m)}$.

Definition

The action σ has *large spectral subspaces* if $A_{(n)}^{*}A_{(n)} = A_{(0)}$ for all $n \in \mathbb{Z}$.

Note that σ has large spectral subspaces if and only if

$$A_{(1)}^{*}A_{(1)} = A_{(0)} = A_{(1)}A_{(1)}^{*}. \tag{3}$$

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Theorem

Let $\phi : A_{(0)} \to \mathrm{End}^*_{A_{(0)}}(A_{(1)})$ *simply defined by* $\phi(a)(\xi) := a\,\xi$. *Suppose that* $A_{(1)}$ *and* $A_{(-1)}$ *are full and countably generated over* $A_{(0)}$.
Then the circle action $\{\sigma_z\}$ *has large spectral subspaces.*
Moreover, the Pimsner algebra $\mathcal{O}_{A_{(1)}}$ *is isomorphic to* A.

Proposition (Gabriel-Grensing)

Let A be a unital, commutative C^*-algebra. Suppose that the first spectral subspace $E = A_{(1)}$ generates A as a C^*-algebra, and that it is finitely generated projective over $B = A_{(0)}$.
Then the following facts hold

1. $B = C(X)$ *for some compact space* X*;*
2. $E = \Gamma(\mathcal{L})$ *for some line bundle* $\mathcal{L} \to X$*;*
3. $A = C(P)$*, where* $P \to X$ *is the principal* S^1 *bundle over* X *associated to the line bundle* $\mathcal{L}$*.*

Since $\phi : B \to \mathcal{K}(E)$, we have a well defined class

$$\boxed{[E] := [(E, \phi, 0)] \in KK_0(B, B)} \tag{4}$$

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Since $\widehat{\phi} : \mathcal{O}_E \to \mathrm{End}^*_B(\mathcal{E}_\infty)$ is the inclusion, we have a class

$$\boxed{[\partial] := \left[(\mathcal{E}_\infty, \widehat{\phi}, F)\right] \in KK_1(\mathcal{O}_E, B)} \tag{5}$$

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To define the operator F, let $P : \mathcal{E}_\infty \to \mathcal{E}_\infty$ denotes the orthogonal projection with

$$\mathrm{Im}(P) = \bigoplus_{n=1}^{\infty} E^{(n)} \oplus B \subseteq \mathcal{E}_\infty \,,$$

and set $F := 2P - 1 \in \mathrm{End}_B^*(\mathcal{E}_\infty)$.

For any separable C^*-algebra C the Kasparov product induces the group homomorphisms

$$[E] : KK_*(B, C) \to KK_*(B, C)\,, \quad [E] : KK_*(C, B) \to KK_*(C, B)$$

and

$$[\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_E, C)\,, \quad [\partial] : KK_*(C, \mathcal{O}_E) \to KK_{*+1}(C, B)\,,$$

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$$[\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_E, C) , \quad [\partial] : KK_*(C, \mathcal{O}_E) \to KK_{*+1}(C, B) ,$$

The inclusion $j : B \hookrightarrow \mathcal{O}_E$ also induces maps in KK-theory.

$$j^* : KK_*(\mathcal{O}_E, C) \to KK_*(B, C) , \quad j_* : KK_*(C, B) \to KK_*(C, \mathcal{O}_E) ,$$

We get two six term exact sequences.

In particular, for $C = \mathbb{C}$ we get exact sequences in K-theory

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-[E]} & K_0(B) & \xrightarrow{j_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_E) & \xleftarrow[j_*]{} & K_1(B) & \xleftarrow[1-[E]]{} & K_1(B) \end{array} ,$$

and in K-homology

$$\begin{array}{ccccc} K^0(B) & \xleftarrow[1-[E]]{} & K^0(B) & \xleftarrow[j^*]{} & K^0(\mathcal{O}_E, C) \\ \downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\ K^1(\mathcal{O}_E) & \xrightarrow{j^*} & K^1(B) & \xrightarrow{1-[E]} & K^1(B) \end{array} .$$

- The previous sequences be interpreted as a *Gysin sequence* in K-theory and K-homology for the 'line bundle' E over the 'noncommutative base space' B.
- Multiplication by the Euler class is replaced with the Kasparov product with $1 - [E]$.

The coordinate algebra $\mathcal{A}(S_q^{2n+1})$ of the quantum S_q^{2n+1}:
$*$-algebra generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$ s.t.:

$$\begin{aligned} z_i z_j &= q^{-1} z_j z_i && 0 \leq i < j \leq n\,, \\ z_i^* z_j &= q z_j z_i^* && i \neq j\,, \\ [z_n^*, z_n] = 0\,, \quad [z_i^*, z_i] &= (1-q^2) \sum_{j=i+1}^{n} z_j z_j^* && i = 0, \dots, n-1\,, \\ 1 &= z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^*\,. \end{aligned}$$

(L. Vaksman, Ya. Soibelman)

$\mathrm{U}(1)$-action on the algebra $\mathcal{A}(\mathrm{S}_q^{2n+1})$:

$$(z_0, z_1, \dots, z_n) \mapsto (\lambda z_0, \lambda z_1, \dots, \lambda z_n), \qquad \lambda \in \mathrm{U}(1).$$

The coordinate algebra $\mathcal{A}(\mathbb{C}\mathrm{P}_q^n)$ of the quantum projective space $\mathbb{C}\mathrm{P}_q^n$ is the subalgebra of invariant elements.

We have a decomposition

$$\mathcal{A}(S_q^{2n+1}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k.$$

The $\mathrm{U}(1)$-action restricts to an action of the finite cyclic group $\mathbb{Z}_r$.

$$\boxed{\mathcal{A}(L_q^{(n,r)}) := \mathcal{A}(S_q^{2n+1})^{\mathbb{Z}_r}}$$

.

We have a decomposition

$$\mathcal{A}(L_q^{(n,r)}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_{rk}.$$

The C^*-algebras $C(\mathrm{S}_q^{2n+1})$, $C(L_q^{(n,r)})$ and $C(\mathbb{C}\mathrm{P}_q^n)$ of continuous functions: completions of $\mathcal{A}(\mathrm{S}_q^{2n+1})$, $\mathcal{A}(L_q^{(n,r)})$ and $\mathcal{A}(\mathbb{C}\mathrm{P}_q^n)$ in the universal C^*-norms

Let $r \geq 1$, then

$$\boxed{C(L_q^{(n,r)}) = \mathcal{O}_{E_{(r)}}}$$

with $E_{(r)}$ the r-th spectral subspaces for the circle action on $C(S_q^{2n+1})$.

Since $K_1(\mathbb{CP}^n_q) = 0$, we can compute $K_0(L_q^{(n,r)})$ as the kernel of a matrix representing the multiplication map $1 - [E] : K_0(\mathbb{CP}^n_q) \to K_0(\mathbb{CP}^n_q)$
This leads to

$$K_0(L_q^{(n,r)}) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_n\mathbb{Z} \qquad K_1(L_q^{(n,r)}) = \mathbb{Z},$$

where the α_i's depend on the divisibility properties of the integer r.
Explicit algebraic generators.

Joint work with S. Brain and G. Landi.

$\mathrm{U}(1)$-action on the algebra $\mathcal{A}(S_q^{2n+1})$: for a weight vector $\ell = (\ell_0, \ldots, \ell_n)$

$$(z_0, z_1, \ldots, z_n) \mapsto (\lambda^{\ell_0} z_0, \lambda^{\ell_1} z_1, \ldots, \lambda^{\ell_n} z_n), \qquad \lambda \in \mathrm{U}(1).$$

The coordinate algebra $\mathcal{A}(\mathbb{W}_q^n(\ell))$ of the quantum projective space $\mathbb{W}_q^n(\ell)$ is the subalgebra of invariant elements.
The C^*-algebras $C(\mathbb{W}_q^n(\ell))$ of continuous functions: completion in the universal C^*-norm.

We focus on n=1: weighted projective line.
$C(\mathbb{W}_q(k,l))$ is the universal C*-algebra generated by the elements

$$z_0^l (z_1^*)^k \quad \text{and} \quad z_1 z_1^* .$$

Notice that it does not depend on k and

$$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1}, \quad K_1(C(W_q(k,l))) = 0 .$$

Define the C*-algebra of the weighted quantum lens spaces $L_q(dkl,k,l)$ as a Pimsner algebra

$$\boxed{C(L_q(dkl,k,l)) := \mathcal{O}_{E_d}}$$

for the d-th spectral subspace $E_{(d)}$ for the weighted $\mathrm{U}(1)$-action on S_q^3

We have a Gysin sequence in K-theory

$$0 \longrightarrow K_1(C(L_q(dkl,k,l)) \longrightarrow \mathbb{Z}^{l+1} \xrightarrow{1-M^d} \mathbb{Z}^{l+1} \longrightarrow K_0(C(L_q(dkl,k,l)) \longrightarrow 0$$

Where $M = \{M_{sr}\} \in M_{l+1}(\mathbb{Z})$ is a matrix of pairings between the K-theory and K-homology of $\mathrm{C}(\mathrm{W}_q(k,l))$.

We compute the K-theory groups as

$$\boxed{K_1(C(L_q(dkl,k,l)) = \mathrm{Ker}(1-M^d) \quad K_0(C(L_q(dkl,k,l)) = \mathrm{Coker}(1-M^d)}$$

Joint work with J. Kaad and G.Landi.

- Quantum principal bundles are strongly graded algebras.
- Self Morita Equivalence are the C*-algebraic version of line bundles.
- The corresponding Pimsner algebra O_E is then the total space algebra of a principal circle bundle over B.
- Gysin-like sequences relates the KK-theories of O_E and of B.
- Explicit computations and representatives.
- Rich class of examples.
- Still open: understand the structure of other principal bundles.

Thank you very much for your attention!

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