The K-Theory and K-homology of Quantum Lens Spaces

Francesca Arici (SISSA)

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The Gysin Sequence for Quantum Lens Spaces

F. Arici, S. Brain, G. Landi

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Pimsner Algebras and Gysin Sequences from Principal Circle Actions

F. Arici, J. Kaad, G. Landi

in preparation.

1. Topology:
   - Quotient of odd dimensional spheres by an action of a finite cyclic group.

$$\mathrm{L}^{(n,r)} := \mathrm{S}^{2n+1}/\mathbb{Z}_r \tag{1}$$

   - Torsion phenomena, e.g. $\pi_1\left(\mathrm{L}^{(n,r)}\right) = \mathbb{Z}_r$.
   - Total spaces of $U(1)$ bundles over complex projective spaces.

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   - Torsion phenomena, e.g. $\pi_1\left(\mathrm{L}^{(n,r)}\right) = \mathbb{Z}_r$.
   - Total spaces of $U(1)$ bundles over complex projective spaces.
2. Problems in high energy physics:
   - T duality
   - Chern Simons field theories

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Long exact sequence in cohomology, associated to any sphere bundle.

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where α is the mutiliplication by the Euler class

$$\chi(\mathcal{L}_r) = 1 - [\mathcal{L}_r] \tag{3}$$

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... Is there a **quantum** version?

L. Vaksman, Ya. Soibelman, 1991 M. Welk, 2000

The coordinate algebra $\mathcal{A}(\mathrm{S}_q^{2n+1})$ quantum sphere S_q^{2n+1} is the $*$-algebra generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$ s.t. $z_i z_j = q^{-1} z_j z_i$ etc...

Sphere relation:

$$1 = z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* \ .$$

The coordinate algebra $\mathcal{A}(\mathbb{C}\mathrm{P}_q^n)$ of the quantum projective space $\mathbb{C}\mathrm{P}_q^n$ is made of invariant elements for the $\mathrm{U}(1)$-action on the algebra $\mathcal{A}(\mathrm{S}_q^{2n+1})$ given by

$$(z_0, z_1, \ldots, z_n) \mapsto (\lambda z_0, \lambda z_1, \ldots, \lambda z_n), \qquad \lambda \in \mathrm{U}(1).$$

The C^*-algebras $C(\mathrm{S}_q^{2n+1})$ and $C(\mathbb{C}\mathrm{P}_q^n)$ of continuous functions: completions of $\mathcal{A}(\mathrm{S}_q^{2n+1})$ and $\mathcal{A}(\mathbb{C}\mathrm{P}_q^n)$ in the universal C^*-norms

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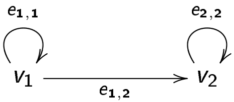
These are graph algebras J.H. Hong, W. Szymański 2002.



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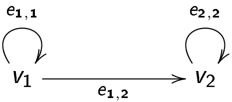

Their K-theory can be computed out of the *incidence matrix*.

F. D'Andrea, G. Landi 2010 Generators of the homology group $K^0(C(\mathbb{C}\mathrm{P}^n_q))$ given explicitly as (classes of) even Fredholm modules

$$\mu_k = (\mathcal{A}(\mathbb{C}\mathrm{P}^n_q), \mathcal{H}_{(k)}, \pi^{(k)}, \gamma_{(k)}, F_{(k)}), \quad \text{for} \quad 0 \leq k \leq n.$$

Generators of the K-theory $K_0(\mathbb{C}\mathrm{P}^n_q)$ also given explicitly as projections whose are polynomial functions:

For $N \in \mathbb{Z}$, let $\Psi_N := (\psi^N_{j_0,\ldots,j_n})$ be the vector-valued function with entries in $\mathcal{A}(\mathbb{C}\mathrm{P}^n_q)$

Such that $\Psi_N^* \Psi_N = 1$;

$\Rightarrow \qquad P_N := \Psi_N \Psi_N^*$ is a projection

The inclusion $\mathcal{A}(\mathbb{CP}^n_q) \hookrightarrow \mathcal{A}(\mathrm{S}^{2n+1}_q)$ is a $\mathrm{U}(1)$ q.p.b.
$\mathcal{L}_N$ made of elements of $\mathcal{A}(\mathrm{S}^{2n+1}_q)$ transforming under $\mathrm{U}(1)$ as

$$\varphi_N \mapsto \varphi_N \lambda^{-N}$$

$\mathcal{L}_0 = \mathcal{A}(\mathbb{CP}^n_q)$; each $\mathcal{L}_N$ is an $\mathcal{L}_0$-bimodule – the bimodule of equivariant maps for the IRREP of $\mathrm{U}(1)$ with weight N.

$$\mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{CP}^n_q)} \mathcal{L}_{N'} \simeq \mathcal{L}_{N+N'} \quad \mathcal{L}_N^{\otimes k} = \mathcal{L}_{kN}. \tag{4}$$

We denote $[P_N] = [\mathcal{L}_N]$ in the group $K_0(\mathbb{CP}^n_q)$.

The module $\mathcal{L}_N$ is a line bundle, in the sense that its 'rank' (as computed by pairing with $[\mu_0]$) is equal to 1

Completely characterized by its 'first Chern number' (as computed by pairing with the class $[\mu_1]$):

Proposition (D'Andrea - Landi 2010)

For all $N \in \mathbb{Z}$ it holds that

$$\langle [\mu_0], [\mathcal{L}_N] \rangle = 1 \quad \textit{and} \quad \langle [\mu_1], [\mathcal{L}_N] \rangle = -N \,.$$

The line bundle $\mathcal{L}_{-1}$ emerges as a central character:

$$\langle [\mu_0], [\mathcal{L}_{-1}] \rangle = 1 \qquad\qquad \langle [\mu_1], [\mathcal{L}_{-1}] \rangle = 1$$

$\mathcal{L}_{-1}$ is the *tautological line bundle* for the QPS $\mathbb{C}\mathrm{P}^n_q$.
Consider $u = \chi([\mathcal{L}_{-1}]) := 1 - [\mathcal{L}_{-1}]$, the Euler class of $\mathcal{L}_{-1}$.

$$u^j = (1 - [\mathcal{L}_{-1}])^j \simeq \sum_{N=0}^{j} (-1)^N \binom{j}{N} [\mathcal{L}_{-N}] \, .$$

Proposition (D'Andrea - Landi 2010)

$$K_0(\mathbb{C}\mathrm{P}^n_q) \simeq \mathbb{Z}[\mathcal{L}_{-1}]/(1 - [\mathcal{L}_{-1}])^{n+1} \simeq \mathbb{Z}[u]/u^{n+1} \, .$$

Fix an integer $r \geq 2$ and define

$$\mathcal{A}(\mathrm{L}_q^{(n,r)}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_{rN} \,.$$

Proposition

$\mathcal{A}(\mathrm{L}_q^{(n,r)})$ is a $*$-algebra; all elements of $\mathcal{A}(\mathrm{S}_q^{2n+1})$ invariant under the action $\alpha_r : \mathbb{Z}_r \to \mathrm{Aut}(\mathcal{A}(\mathrm{S}_q^{2n+1}))$ of the cyclic group $\mathbb{Z}_r$:

$$(z_0, z_1, \ldots, z_n) \mapsto (e^{2\pi \mathrm{i}/r} z_0, e^{2\pi \mathrm{i}/r} z_1, \ldots, e^{2\pi \mathrm{i}/r} z_n) \,.$$

It can be interpreted as a deformation of the classical lens space $L^{(n,r)} = \mathrm{S}^{2n+1}/\mathbb{Z}_r$

Proposition

The algebra inclusion $\mathcal{A}(\mathrm{L}_q^{(n,r)}) \hookrightarrow \mathcal{A}(\mathrm{S}_q^{2n+1})$ *is a quantum principal bundle with structure group* $\mathbb{Z}_r$.

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More structure:

Proposition

The algebra inclusion $j : \mathcal{A}(\mathbb{C}\mathrm{P}_q^n) \hookrightarrow \mathcal{A}(\mathrm{L}_q^{(n,r)})$ *is a quantum principal bundle with structure group* $\widetilde{\mathrm{U}}(1) := \mathrm{U}(1)/\mathbb{Z}_r$*:*

$$\mathcal{A}(\mathbb{C}\mathrm{P}_q^n) = \mathcal{A}(\mathrm{L}_q^{(n,r)})^{\widetilde{\mathrm{U}}(1)},$$

in analogy with the identification $\mathcal{A}(\mathbb{C}\mathrm{P}_q^n) = \mathcal{A}(\mathrm{S}_q^{2n+1})^{\mathrm{U}(1)}$

A way to 'pull-back' line bundles from $\mathbb{CP}_q^n$ to $\mathrm{L}_q^{(n,r)}$:

$$\begin{array}{ccc} \widetilde{\mathcal{L}}_N & \xleftarrow{\;j_*\;} & \mathcal{L}_N \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathcal{A}(\mathrm{L}^{(n,r)_q}) & \xleftarrow[\;j\;]{} & \mathcal{A}(\mathbb{CP}_q^n)\,. \end{array}$$

i.e, the algebra inclusion $j : \mathcal{A}(\mathbb{CP}_q^n) \to \mathcal{A}(\mathrm{L}_q^{(n,r)})$ induces a map

$$j_* : K_0(\mathbb{CP}_q^n) \to K_0(\mathrm{L}_q^{(n,r)})$$

They are obtained as

$$j_*(\mathcal{L}_N) = \mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{CP}_q^n} \mathcal{A}\mathrm{L}_q^{(n,r)} =: \widetilde{\mathcal{L}}_N$$

Proposition

There are left $\mathcal{A}(\mathrm{L}_q^{(n,r)})$-module isomorphisms

$$\widetilde{\mathcal{L}}_N \simeq (\mathcal{A}(\mathrm{L}_q^{(n,r)}))^{d_N} P_N$$

and right $\mathcal{A}(\mathrm{L}_q^{(n,r)})$-module isomorphisms

$$\widetilde{\mathcal{L}}_N \simeq P_{-N} (\mathcal{A}(\mathrm{L}_q^{(n,r)}))^{d_N} .$$

Projections P_N here are as before; now:
$[\widetilde{\mathcal{L}}_N] \simeq [P_N]$ as an element in $K_0(\mathrm{L}_q^{(n,r)})$.

$\mathcal{L}_N$ versus its pull-back $\widetilde{\mathcal{L}}_N$

- marked difference: each $\mathcal{L}_N$ is not free when $N \neq 0$;
  The pull-back $\widetilde{\mathcal{L}}_{-r}$ of the line bundle $\mathcal{L}_{-r}$ is free:
- The condition $\Psi_{-r}^{*}\Psi_{-r} = 1$ implies that P_{-r} is equivalent to 1, that is the class of the module $\widetilde{\mathcal{L}}_{-r}$ is trivial in $K_0(\mathrm{L}_q^{(n,r)})$.
  Such pulled-back line bundles $\widetilde{\mathcal{L}}_{-N}$ thus define *torsion classes*; furthermore, they generate the group $K_0(\mathrm{L}_q^{(n,r)})$.

A second crucial ingredient

$$\alpha : K_0(\mathbb{C}\mathrm{P}^n_q) \to K_0(\mathbb{C}\mathrm{P}^n_q),$$

α is multiplication by $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$
the Euler class of the line bundle $\mathcal{L}_{-r}$

Assembly these into an exact sequence, the *Gysin sequence*

$$0 \to K_1(\mathrm{L}_q^{(n,r)}) \longrightarrow K_0(\mathbb{C}\mathrm{P}_q^n) \xrightarrow{\alpha} K_0(\mathbb{C}\mathrm{P}_q^n) \xrightarrow{j_*} K_0(\mathrm{L}_q^{(n,r)}) \longrightarrow 0$$

Some practical and important applications, notably, the computation of the K-theory of the quantum lens spaces $\mathrm{L}_q^{(n,r)}$.

Thus

$$K_1(\mathrm{L}_q^{(n,r)}) \simeq \ker(\alpha), \qquad K_0(\mathrm{L}_q^{(n,r)}) \simeq \operatorname{coker}(\alpha)\,.$$

$$K_1(\mathrm{L}_q^{(n,r)}) \qquad\qquad K_0(\mathrm{L}_q^{(n,r)})$$

for the latter as pulled-back line bundles from $\mathbb{C}\mathrm{P}_q^n$ to $\mathrm{L}_q^{(n,r)}$

Some Notation: from now on we will be writing

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The infinitesimal generator of the circle action determines an unbounded self-adjoint regular operator $\mathfrak{D} : \mathrm{Dom}(\mathfrak{D}) \to X$ The pair $(X, \mathfrak{D})$ yields a class in the bivariant K-theory $KK_1(A, F)$ and the Kasparov product with the class $[(X, \mathfrak{D})]$ thus furnishes

$$\mathrm{Ind}_{\mathfrak{D}} : K_*(A) \to K_{*+1}(F), \qquad \mathrm{Ind}_{\mathfrak{D}}(-) := - \widehat{\otimes}_A [(X, \mathfrak{D})].$$

Then the sequence becomes

$$0 \to K_1(A) \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} K_0(F) \xrightarrow{\alpha} K_0(F) \xrightarrow{j_*} K_0(A) \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} 0$$

At this point we are saying nothing about exactness of the sequence.

More on $\mathrm{Ind}_{\mathfrak{D}}$

A.L. Carey, S. Neshveyev, R. Nest, A. Rennie 2011

F sits inside A as the fixed point subalgebra,

$$F = \{a \in A : \sigma_t(a) = a \text{ for all } t \in \widetilde{\mathrm{U}}(1)\}$$

and one has a faithful conditional expectation

$$\tau : A \to F, \qquad \tau(a) := \int_0^{2\pi} \sigma_t(a)\mathrm{d}t\,,$$

leading to an F-valued inner product on A by defining

$$\langle \cdot, \cdot \rangle_F : A \times A \to F, \qquad \langle a, b \rangle_F := \tau(a^* b).$$

A is a right pre-Hilbert F-module, with Hilbert module X say.

The mapping cone of the pair (F, A) is the C^*-algebra

$$M(F, A) := \{f \in C([0,1], A) \mid f(0) = 0,\ f(1) \in F\}.$$

$$0 \to S(A) \xrightarrow{i} M(F, A) \xrightarrow{\mathrm{ev}} F \to 0,$$

$S(A) := C_0((0,1)) \otimes A$ the suspension;
with $i(f \otimes a)(t) := f(t)a$; $\mathrm{ev}(f) := f(1)$
Using the vanishing of $K_1(F)$, and of $K_1(M(F, A))$, the corresponding six term exact sequence is

$$0 \to K_1(A) \xrightarrow{i_*} K_0(M(F, A)) \xrightarrow{\mathrm{ev}_*} K_0(F) \xrightarrow{j_*} K_0(A) \to 0.$$

The above is an equivalent variant of the Gysin sequence

Theorem

There is a diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_1(A) & \xrightarrow{i_*} & K_0(M(F,A)) & \xrightarrow{\mathrm{ev}_*} & K_0(F) & \xrightarrow{j_*} & K_0(A) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{Ind}_{\widehat{\mathfrak{D}}}} & & \big\downarrow{\scriptstyle B_{\boldsymbol{F}}} & & \big\downarrow{\scriptstyle B_{\boldsymbol{A}}} & & \\
0 & \longrightarrow & K_1(A) & \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} & K_0(F) & \xrightarrow{\alpha} & K_0(F) & \xrightarrow{j_*} & K_0(A) & \longrightarrow & 0
\end{array}$$

where squares commute and vertical arrows are isomorphisms

The merit of our construction is not only in computing the K-theory groups: this could be done by means of graph algebras.

Explicit generators as classes of 'line bundles', torsion ones.

Since the map j_* in the sequence is surjective, the group $K_0(\mathrm{L}_q^{(n,r)})$ can be obtained by 'pulling back' classes from $K_0(\mathbb{C}\mathrm{P}_q^n)$.

The matrix A of the map α with respect to the $\mathbb{Z}$-module basis $\{1, u, \ldots, u^n\}$. Using the condition $u^{n+1} = 0$ one has

$$\chi(\mathcal{L}_{-r}) = 1 - (1-u)^r = \sum_{j=1}^{\min(r,n)} (-1)^{j+1} \binom{r}{j} u^j \,.$$

Thus A is an $(n+1)\times(n+1)$ strictly lower triangular matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots\cdots & & 0 \\ r & 0 & 0 & \cdots\cdots & & 0 \\ -\binom{r}{2} & r & 0 & \cdots\cdots & & 0 \\ \binom{r}{3} & -\binom{r}{2} & r & & & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ & & & & & \\ 0 & 0 & 0 & \cdots\cdots & r & 0 \end{pmatrix}.$$

Proposition

The $(n+1)\times(n+1)$ matrix A has rank n:

$$K_1(C(\mathrm{L}_q^{(n,r)})) \simeq \mathbb{Z}\,.$$

On the other hand, the structure of the cokernel of the matrix A depends on the divisibility properties of the integer r.

The Smith normal form for matrices over a principal ideal domain, such as $\mathbb{Z}$: there exist invertible matrices P and Q having integer entries which transform A to a diagonal matrix

$$\mathrm{Sm}(A) := PAQ = \mathrm{diag}(\alpha_1, \cdots, \alpha_n, 0)\,.$$

Integer entries $\alpha_i \geq 1$, given by

$$\alpha_1 = d_1(A) \qquad \alpha_i = d_i(A)/d_{i-1}(A)$$

$d_i(A)$ is the greatest common divisor of the non-zero determinants of the minors of order i of the matrix A.

This leads to

$$K_0(\mathrm{L}_q^{(n,r)}) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_n\mathbb{Z}\,.$$

Construction of explicit generators.

The module $\mathcal{L}_{-r}$ over the fixed point algebra $F = C(\mathbb{C}\mathrm{P}^n_q)$ plays a crucial role in our construction.
Related construction: Cuntz-Pimsner Algebras
Ingredients:

- A C*-algebra F;
- A C*-correspondence E over F.

One constructs a C*-algebra $\mathcal{O}_E$ that generalizes Cuntz-Krieger algebras and crossed products.
All the information about $\mathcal{O}_E$ is encoded in (F, E).

If F is a C*-algebra, a right Hilbert C*-module is a Banach space E together with a right action of F on E and an F-valued inner product $\langle \cdot, \cdot \rangle$ such that:

1. $\langle \xi, \eta f \rangle_F = \langle \xi, \eta \rangle_F f$,
2. $\langle \xi, \eta \rangle_F = \langle \eta, \xi \rangle_F^*$,
3. $\langle \xi, \xi \rangle_F \geq 0$ and $\langle \xi, \xi \rangle_F = 0$ if and only if $x = 0$,

for all $\xi, \eta \in E,\ f \in F$.

- $\mathcal{L}(E)$ denotes the C^*-algebra of bounded adjointable operators on E;
- $\mathcal{K}(E) \subseteq \mathcal{L}(E)$ denotes the C^*-algebra of compact operators on E, i.e. the closed two sided ideal given by

$$\overline{\mathrm{span}}\{\theta_{\xi,\eta} \mid \xi, \eta \in E\},$$

where $\theta_{\xi,\eta}\zeta = \xi\langle\eta, \zeta\rangle$.

A C^*-correspondence E over F is a countably generated (right) Hilbert C^*-module over the separable C^*-algebra F toghether with a $*$-homomorphism

$$\phi : B \to \mathcal{L}(E).$$

We make the following assumptions:

1. E is taken to be full, i.e $\langle E, E \rangle := \mathrm{span}_{\mathbb{C}}\big\{\langle \xi, \eta \rangle \,|\, \xi, \eta \in E\big\}$ is dense in F.
2. ϕ induces an isomorphism $\phi : B \to \mathcal{K}(E)$.

Let E^* denote the dual of E, thus as a vector space

$$E^* := \left\{ \lambda \in \mathrm{Hom}_B(E, B) \,|\, \exists \xi \in E \text{with} \lambda(\eta) = \langle \xi, \eta \rangle \ \forall \eta \in E \right\}.$$

Then E^* is a C*-correspondence over F, w.r.t. the $*$-homomorphism $\phi^* : E^* \to \mathcal{L}(E)$ given by

$$\phi^*(b)\lambda_\xi := \lambda_{\xi b^*}.$$

Moreover, the pair (E^*, ϕ^*) satisfies the assumptions.

Next, for each $n \in$, let $E^{\widehat{\otimes}_\phi n}$ and $(E^*)^{\widehat{\otimes}_{\phi^*} n}$ be the n-fold interior tensor product of E over B and of E^* over B, respectively.
Define the Hilbert C^*-module over B,

$$E_\infty := \left(\oplus_{n=1}^{\infty} E^{\widehat{\otimes}_\phi n} \right) \oplus B \oplus \left(\oplus_{n=1}^{\infty} (E^*)^{\widehat{\otimes}_\phi n} \right) .$$

Then, for each $\xi \in E$ we have a bounded adjointable operator $S_\xi : E_\infty \to E_\infty$ defined compontent-wise by

$$\begin{aligned} B \ni b &\longmapsto \xi b \\ E^{\widehat{\otimes}_\phi n} \ni \xi_1 \otimes \cdots \otimes \xi_n &\longmapsto \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \ , \ \ \xi_1 \otimes \cdots \otimes \xi_n \in , \\ (E^*)^{\widehat{\otimes}_{\phi^*} n} \ni \lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_n} &\longmapsto \lambda_{\xi_2 \phi^{-1}(\theta_{\xi_1, \xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_n} \ . \end{aligned}$$

In particular, $S_\xi(\lambda_{\xi_1}) = \phi^{-1}(\theta_{\xi, \xi_1}) \in B$.
The adjoint of S_ξ is easily found to be given by $S_{\lambda_\xi} := S_\xi^* : E_\infty \to E_\infty$:

1. $\phi : F \to \mathcal{L}(E)$ factorizes through the compacts $\mathcal{K}(E) \subseteq \mathcal{L}(E)$.
   The class in $KK_0(F, F)$ defined by the even Kasparov module $(E, \phi, 0)$ (with trivial grading) will be denoted by $[E]$.
2. Next, let $P : E_\infty \to E_\infty$ denote the orthogonal projection with image

$$\mathrm{Im}(P) = \big(\oplus_{n=1}^{\infty} E^{\widehat{\otimes}_\varphi n} \big) \oplus F \subseteq E_\infty \, .$$

   Then, let $Q := 2P - 1 \in \mathcal{L}(E_\infty)$ and recall the inclusion $\widetilde{\phi} : \mathcal{O}_E \to \mathcal{L}(E_\infty)$.
   The class in $KK_1(\mathcal{O}_E, F)$ defined by the odd Kasparov module $(E_\infty, \widetilde{\phi}, Q)$ will be denoted by $[\partial]$.

For any separable C^*-algebra B we then have the group homomorphisms

$$[E] : KK_*(F,B) \to KK_*(F,B)\,, \quad [E] : KK_*(B,F) \to KK_*(B,F)$$

and

$$[\partial] : KK_*(B,\mathcal{O}_E) \to KK_{*+1}(B,F)\,, \quad [\partial] : KK_*(F,B) \to KK_{*+1}(\mathcal{O}_E,B)\,,$$

which are induced by the Kasparov product.

We get two exact sequences:

$$\begin{array}{ccccc} KK_0(B,F) & \xrightarrow{1-[E]} & KK_0(B,F) & \xrightarrow{j_*} & KK_0(B,\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ KK_1(B,\mathcal{O}_E) & \xleftarrow[j_*]{} & KK_1(B,F) & \xleftarrow[1-[E]]{} & KK_1(B,F) \end{array}$$

and

$$\begin{array}{ccccc} KK_0(F,B) & \xleftarrow[1-[E]]{} & KK_0(F,B) & \xleftarrow[j^*]{} & KK_0(\mathcal{O}_E,C) \\ {\scriptstyle [\partial]}\downarrow & & & & \uparrow{\scriptstyle [\partial]} \\ KK_1(\mathcal{O}_E,B) & \xrightarrow{j^*} & KK_1(F,B) & \xrightarrow{1-[E]} & KK_1(F,B) \end{array}$$

where j^* and j_* are induced by $j : F \hookrightarrow \mathcal{O}_E$.

For $B = \mathbb{C}$, the first sequence above reduces to

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{1-[E]} & K_0(F) & \xrightarrow{j_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_E) & \xleftarrow[j_*]{} & K_1(F) & \xleftarrow[1-[E]]{} & K_1(F) \end{array} .$$

Can be interpreted as a *Gysin sequence* in K-theory. for the 'line bundle' E over the 'noncommutative space' F and with the map $1 - [E]$ having the role of the *Euler class* $\chi(E) := 1 - [E]$ of the line bundle E.

The second sequence above reduces to

$$\begin{array}{ccccc} K^0(F) & \xleftarrow[1-[E]]{} & K^0(F) & \xleftarrow[j^*]{} & K^0(\mathcal{O}_E) \\ \big\downarrow{\scriptstyle [\partial]} & & & & \big\uparrow{\scriptstyle [\partial]} \\ K^1(\mathcal{O}_E) & \xrightarrow{j^*} & K^1(F) & \xrightarrow{1-[E]} & K^1(F) \end{array}$$

Can be interpreted as a *Gysin sequence* in K-homology. for the 'line bundle' E over the 'noncommutative space' F and with the map $1-[E]$ having the role of the *Euler class* $\chi(E) := 1-[E]$ of the line bundle E.

Example of this construction.

F:= quantum weighted proective space;

$\mathcal{O}_E$:= quantum weighted lens space

Fixed point algebra under a weighted circle action $\{\sigma_w^{(k,l)}\}_{w \in S^1}$ on $\mathcal{A}(S_q^3)$ defined on generators by

$$\sigma_w^L : z_0 \mapsto w^k z_0 \quad z_1 \mapsto w^l z_1 \,.$$

The algebraic quantum projective line $\mathcal{A}(W_q(k,l))$ agrees with the unital $*$-subalgebra of $\mathcal{A}(S_q^3)$ generated by the elements $z_0^l (z_1^*)^k$ and $z_1 z_1^*$.
The C^*-algebra $C(W_q(k,l))$ is defined as the completion in the universal C*-norm. Notice that it does not depend on k.

As a consequence one has the folowing corollary due to Brzeziński and Fairfax.

Corollary

The K-groups of $C(W_q(k,l))$ are:

$$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1}, \quad K_1(C(W_q(k,l))) = 0 .$$

We construct the coordinate algebra of the quantum weighted lens spaces out of a finetely generated projective modules $A_{(dn)}(k,l)$ over $\mathcal{A}(W_q(k,l))$.

$$\mathcal{A}(L_q(dlk;k,l)) \cong \oplus_{n\in\mathbb{Z}} A_{(dn)}(k,l)\,.$$

The C*-algebra is obtained O_E for the corresponding C*-module E over $C(W_q(k,l))$.
We can compute the K-groups using the Gysin-Pimsner sequence.

- We constructed a Gysin exact sequence for quantum lens spaces using operator algebraic tecniques.
- The key role is played by a line bundle.
- Look at C^*-correspondences.
- The corresponding Pimsner algebra O_E is then the total space algebra of a principal circle bundle over A.
- Gysin-like sequences relates the KK-theories of O_E and of A.
- More examples.

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in preparation