Gysin Sequences for Noncommutative Spaces and their Applications

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Motivation

Problems in high energy physics:

- T duality
- Chern Simons field theories

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Our question: look at the quantized version of these theories by studying noncommutative spaces.

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We consider strings in presence of a background field B with flux $dB =: H$.

Ingredients for T-duality

Let E be our spacetime.

	Type IIA	Type IIB
Background H-flux	$H \in \Omega^3(E)$, $dH = 0$	$H \in \Omega^3(E)$, $dH = 0$
Ramond-Ramond (RR) field	$G \in \Omega^{even}(E)$	$G \in \Omega^{odd}(E)$
	$(d - H) \wedge G = 0$	$(d - H) \wedge G = 0$

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$$
\begin{array}{ccccccc}
 & & & E \times_X \widehat{E} & & & \\
 & & \overset{p}{\swarrow} & & \overset{\widehat{p}}{\searrow} & & \\
H \in H^3(E,\mathbb{Z}) & E & & & & \widehat{E} & \widehat{H} \in H^3(\widehat{E},\mathbb{Z}) \\
 & & \underset{\nabla^2 = F}{\overset{\pi}{\searrow}} & & \underset{\widehat{\nabla}^2 = \widehat{F}}{\overset{\widehat{\pi}}{\swarrow}} & & \\
 & & & X & & &
\end{array} \quad . \qquad (1)
$$

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$$F = \widehat{\pi}_*(\widehat{H}), \qquad \widehat{F} = \pi_*(H) \qquad (2)$$

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$$
F = \widehat{\pi}_*(\widehat{H}), \qquad \widehat{F} = \pi_*(H) \tag{2}
$$

and on the correspondence space $E \times_X \widehat{E}$ we have $p^*(H) = \widehat{p}^*(\widehat{H})$.

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Long exact sequence in cohomology, associated to any $U(1) \hookrightarrow E \rightarrow^{\pi} X$.

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The segment $k = 3$ is the statement of T-duality:

$$\cdots \longrightarrow H^3(E) \longrightarrow H^2(X) \longrightarrow 0$$

$$H \longmapsto \pi_*(H) =: \widehat{F}$$

the curvature of the T-dual bundle E'.

Example I: The Monopole Fibration

Let $E = S^3 \simeq SU(2)$. Let S^1 (maximal torus) act freely on E. The quotient $X = S^3/S^1$ is $\mathbb{CP}^1 \simeq S^2$.

$H = 0$, and curvature $F = \omega$, with ω the generator of $H^2(S^2, \mathbb{Z})$.

$$
\begin{array}{ccccc}
 & & S^3 \times S^1 & & \\
 & \swarrow^{p} & & \searrow^{\widehat{p}} & \\
S^3 & & & & S^2 \times S^1 \\
 & \searrow_{\pi} & & \swarrow_{\widehat{\pi}} & \\
 & & S^2 & &
\end{array}
\quad . \tag{3}
$$

Integrating along the fiber

$$
c_1(E) = \widehat{\pi}^*(\widehat{H}) \quad c_1(\widehat{E}) = \pi^*(H) = 0. \tag{4}
$$

Example II: Lens Spaces

Orbit spaces of a free action of $\mathbb{Z}_r$ on odd dimensional spheres.

Particular case:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{5}$$

Action of $\mathbb{Z}_r$ on S^3 by

$$k \cdot (z_1, z_2) = (z_1, \zeta^k z_2), \qquad \zeta = \mathrm{e}^{\frac{2\pi i}{r}}, \; k = 0, \dots, r-1 \tag{6}$$

$$L(1; r) := \frac{S^3}{\mathbb{Z}_r}. \tag{7}$$

It is the total space of the circle bundle over S^2 with Chern class r-times the generator of $H^2(S^2, \mathbb{Z})$.

$$(L(1;j), H = k) \xleftrightarrow{\text{T-duality}} (L(1;k), H = j)$$

Chern Simons on Lens Spaces

The study of the CS action functional and its partition function simplify slightly when one does not consider general 3 manifolds, but rather total spaces of U(1) bundles (Blau, Thompson 2006) and Seifert fibrations (Blau, Thompson 2013).

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This phenomenon lies again on the Gysin sequence.

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vector bundle	locally free projective module
Lie Group	Hopf Algebra
Action	Coaction
Principal Bundle	Hopf-Galois Extension
Singular cohomology	?

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K-theory	**Operator K-theory**.

The Gysin Sequence in K-Theory

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Cyclic Six Term exact sequence with $K^1(\mathbb{CP}^1) = 0$.

$$0 \longrightarrow K^1(L(1,r)) \xrightarrow{\delta_{10}} K^0(\mathbb{CP}^1) \xrightarrow{\alpha} K^0(\mathbb{CP}^1) \xrightarrow{\pi^*} K^0(L(1,r)) \to 0\,, \quad (8)$$

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where α is the mutiliplication by the Euler class

$$\chi(\mathcal{L}_r) = 1 - [\mathcal{L}_r] \quad (9)$$

of the bundle $\mathcal{L}_r := \xi^{\otimes r}$, where ξ is the canonical line bundle on $\mathbb{CP}^1$.

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... Is there a **quantum** version?

Algebraic Ingerdients: the Hopf q-Monopole Fibration

Total space: $\mathcal{A}(SU_q(2))$, coordinate algebra of the quantum group $SU_q(2)$:

*-algebra generated by a, b subject to the relations

$$\begin{aligned} ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c, \\ a^*a + c^*c = aa^* + q^2cc^* = 1, \end{aligned} \tag{10}$$

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The algebra of coinvariant elements,

$$(\mathcal{A}(SU_q(2))^{U(1)} \simeq \mathcal{A}(\mathbb{C}\mathrm{P}), \tag{12}$$

is the algebra of function on the noncommutative projective line.

Line bundles

For the coordinate algebra of the quantum 3 sphere we have

$$\mathcal{A}(SU_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n \tag{13}$$

where the $\mathcal{L}_n$ are finetely generated projective modules that can be thought of as line bundles:

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We have

$$\mathcal{L}_0 = \mathcal{A}(\mathbb{C}\mathrm{P}^1_q) \tag{15}$$

$$\mathcal{L}_m \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \mathcal{L}_n = \mathcal{L}_{n+m}. \tag{16}$$

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This algebra agrees with the invariant algebra of $SU_q(2)$ under a $\mathbb{Z}_r$ action defined as classically.

Moreover, we have a quantum principal bundle over the projective line $\mathcal{A}(\mathbb{C}\mathrm{P}^1)$.

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- Moreover, we can define the pullback of line bundles:

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$$0 \longrightarrow K_1(L_q(1,r)) \xrightarrow{Ind} K_0(\mathbb{C}\mathrm{P}^1_q) \xrightarrow{\alpha} K_0(\mathbb{C}\mathrm{P}^1_q) \xrightarrow{\pi^*} K_0(L_q(1,r)) \to 0\,. \tag{19}$$

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Thank you very much for your attention!

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