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Gysin Sequences for Noncommutative Spaces

and their Applications

Francesca Arici (SISSA)

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Motivat	ion						

- T duality
- Chern Simons field theories

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Topological formulation.

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Topological formulation.

The Gysin exact sequence: a powerful tool at hand.

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Topological formulation.

The Gysin exact sequence: a powerful tool at hand.

Our question: look at the quantized version of these theories by studying noncommutative spaces.

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T-dualit	y						

T-duality is a symmetry in superstring theory, relating type IIA and IIB sigma models.

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The physics of a type IIA sigma model compactified along a circle of radius R is indistinguishable from a type IIB sigma model compactified on a circle of dual radius α'/R .

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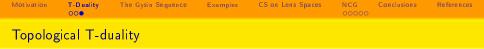
We consider strings in presence of a background field B with flux dB =: H.

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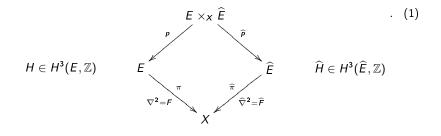
Ingredients for I-duality

Let E be our spacetime.

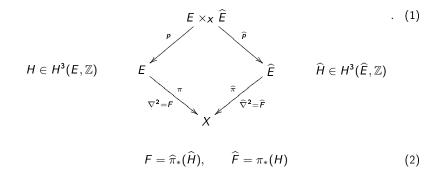
	Type IIA	Type IIB
Background <i>H</i> -flux	$H \in \Omega^{3}(E), dH = 0$	$H \in \Omega^{3}(E), dH = 0$
Ramond-Ramond (RR) field	$G\in \Omega^{even}(E)$	$G\in \Omega^{odd}(E)$
	$(d-H)\wedge G=0$	$(d-H)\wedge G=0$



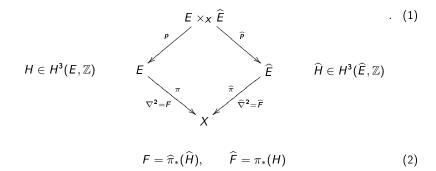












and on the correspondence space $E \times_X \widehat{E}$ we have $p^*(H) = \widehat{p}^*(\widehat{H})$.

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The Gy	sin Seque	ence and T-dua	ality				

Long exact sequence in cohomology, associated to any $U(1) \hookrightarrow E
ightarrow^{\pi} X.$



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 $\cdots \longrightarrow H^{k}(E) \xrightarrow{\pi_{*}} H^{k-1}(X) \xrightarrow{\cup c_{1}(E)} H^{k+1}(X) \xrightarrow{\pi^{*}} H^{k+1}(E) \longrightarrow \cdots$



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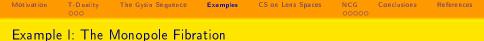
$$\cdots \longrightarrow H^{k}(E) \xrightarrow{\pi_{*}} H^{k-1}(X) \xrightarrow{\cup c_{1}(E)} H^{k+1}(X) \xrightarrow{\pi^{*}} H^{k+1}(E) \longrightarrow \cdots$$

The segment k = 3 is the statement of T-duality:

$$\cdots \longrightarrow H^3(E) \longrightarrow H^2(X) \longrightarrow 0$$

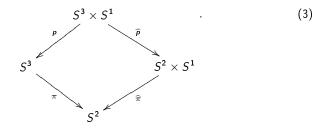
$$H \longmapsto \pi_*(H) =: \widehat{F}$$

the curvature of the T-dual bundle E'.



Let $E = S^3 \simeq SU(2)$. Let S^1 (maximal torus) act freely on E. The quotient $X = S^3/S^1$ is $\mathbb{CP}^1 \simeq S^2$.

H=0, and curvature $F=\omega$, with ω the generator of $H^2(S^2,\mathbb{Z})$.



Integrating along the fiber

$$c_1(E) = \widehat{\pi}^*(\widehat{H}) \quad c_1(\widehat{E}) = \pi^*(H) = 0.$$
(4)

Motivation	T-Duality 000	The Gysin Sequence	Examples	CS on Lens Spaces	NCG 00000	References
Example	e II: Lens	Spaces				

Orbit spaces of a free action of \mathbb{Z}_r on odd dimensional spheres. Particular case:

$$S^{3} = \{(z_{1}, z_{2}) \in \mathbb{C}^{2} \mid |z_{1}|^{2} + |z_{2}|^{2} = 1\}.$$
(5)

Action of \mathbb{Z}_r on S^3 by

$$k \cdot (z_1, z_2) = (z_1, \zeta^k z_2), \qquad \zeta = e^{\frac{2\pi i}{r}}, \ k = 0, \dots, r-1$$
 (6)

$$L(1;r) := \frac{S^3}{\mathbb{Z}_r}.$$
(7)

It is the total space of the circle bundle over S^2 with Chern class *r*-times the generator of $H^2(S^2, \mathbb{Z})$.

$$(L(1;j), H = k) \xleftarrow{\text{T-duality}} (L(1;k), H = j)$$





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This phenomenon lies again on the Gysin sequence.



The key idea in the generalization to noncommutative spaces is contained in the following diagram:



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geometry	algebra	
locally compact Hausdorff space	C*-algebra	
compact Hausdorff space	unital C*-algebra	
vector bundle	locally free projective module	
Lie Group	Hopf Algebra	
Action	Coaction	
Principal Bundle	Hopf-Galois Extension	
Singular cohomology	?	



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Principal Bundle	Hopf-Galois Extension		
Singular cohomology	?		
K-theory	Operator K-theory.		





Cyclic Six Term exact sequence with $K^1(\mathbb{CP}^1) = 0$.

$$0 \longrightarrow \mathcal{K}^{1}(\mathcal{L}(1,r)) \xrightarrow{\delta_{10}} \mathcal{K}^{0}(\mathbb{C}\mathrm{P}^{1}) \xrightarrow{\alpha} \mathcal{K}^{0}(\mathbb{C}\mathrm{P}^{1}) \xrightarrow{\pi^{*}} \mathcal{K}^{0}(\mathcal{L}(1,r)) \to 0, \quad (8)$$



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where α is the mutiliplication by the Euler class

$$\chi(\mathcal{L}_r) = 1 - [\mathcal{L}_r] \tag{9}$$

of the bundle $\mathcal{L}_r := \xi^{\otimes r}$, where ξ is the canonical line bundle on \mathbb{CP}^1 .



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of the bundle $\mathcal{L}_r := \xi^{\otimes r}$, where ξ is the canonical line bundle on \mathbb{CP}^1 Is there a **quantum** version?



Total space: $\mathcal{A}(SU_q(2))$, coordinate algebra of the quantum group $SU_q(2)$: *-algebra generated by *a*, *b* subject to the relations

$$ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c,$$

 $a^*a + c^*c = aa^* + q^2cc^* = 1,$ (10)



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The algebra of coinvariant elements,

$$\left(\mathcal{A}(SU_q(2))^{U(1)} \simeq \mathcal{A}(\mathbb{C}P),\right.$$
(12)

is the algebra of function on the noncommutative projective line.

Motivation	T-Duality 000	The Gysin Sequence	Examples	CS on Lens Spaces	NCG 00000	Conclusions	References
Line bui	ndles						

For the coordinate algebra of the quantum 3 sphere we have

$$\mathcal{A}(SU_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n \tag{13}$$

where the \mathcal{L}_n are finetely generated projective modules that can be thought of as line bundles:

$$\mathcal{L}_{n} := \{ \varphi \in \mathcal{A}(SU_{q}(2)) \mid \omega \cdot \varphi = \omega^{n} \varphi \}.$$
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We have

$$\mathcal{L}_0 = \mathcal{A}(\mathbb{C}\mathrm{P}^1_q) \tag{15}$$

$$\mathcal{L}_m \otimes_{\mathcal{A}(\mathbb{C}\mathrm{P}^1_q)} \mathcal{L}_n = \mathcal{L}_{n+m}.$$
(16)



We define the coordinate algebra of the quantum lens space

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This algebra agrees with the invariant algebra of $SU_q(2)$ under a \mathbb{Z}_r action defined as classically.

Moreover, we have a quantum principal bundle over the projective line $\mathcal{A}(\mathbb{CP}^1)$.



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- Moreover, we can define the pullback of line bundles:

$$j_*(\mathcal{L}_n) := \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}^1_q)} \mathcal{A}(L_q(1,r));$$
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• We have an index map constructed using KK-theory.

$$0 \longrightarrow \mathcal{K}_{1}(L_{q}(1, r)) \xrightarrow{\operatorname{Ind}} \mathcal{K}_{0}(\mathbb{C}\mathrm{P}_{q}^{1}) \xrightarrow{\alpha} \mathcal{K}_{0}(\mathbb{C}\mathrm{P}_{q}^{1}) \xrightarrow{\pi^{*}} \mathcal{K}_{0}(L_{q}(1, r)) \longrightarrow 0.$$
(19)



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Thank you very much for your attention!

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