# Sketch of a construction of the Néron model of an Abelian variety

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## Introduction

Throughout these notes, $R$ denotes a discrete valuation ring, $K$ its field of fractions and $k$ its residue field.

Let $X_K$ be a smooth scheme of finite type over $K$. Recall that a *Néron model* for $X_K$ over $R$ is a smooth $R$-scheme $X$ with generic fibre $X_K$, having the following universal property: for any smooth $R$-scheme $Y$ and any morphism $f: Y_K \to X_K$, there is a unique morphism $Y \to X$ extending $f$. In other words, the canonical map

$$\mathrm{Hom}_R(Y, X) \to \mathrm{Hom}_K(Y_K, X_K)$$

is bijective.

We are going to sketch the construction of Néron models of Abelian varieties. We start by defining two technical concepts which are essential for this construction: rational and birational maps over a base scheme, and Henselian local rings.

## Rational and birational maps

The concept of a rational map of schemes is analogous to that of a rational function in (for example) complex analysis. In certain situations one encounters functions which are defined on a dense open subset of a variety, but cannot be extended to the whole variety. We will now define a relative version of density and of rational maps between schemes over a base scheme $S$ (which in our situation is the spectrum of a discrete valuation ring)

**Definition.** Let $X \to S$ be a morphism of schemes, with $X$ reduced. An open subset $U \subseteq X$ is called *$S$-dense* if for every point $s \in S$, the intersection of $U$ with the fibre $X_s$ is Zariski dense in $X_s$.

*Remark.* It is easy to check that every $S$-dense open subset of $X$ is Zariski dense in $X$. The reason we have assumed $X$ to be reduced is that a more useful definition of density in the general case is that of *schematic density*; see [BLR, § 2.5], or [EGA IV$_4$, définition 11.10.2].

**Definition.** Let $S$ be a Noetherian scheme, and let $X, Y$ be schemes of finite type over $S$, with $X$ reduced and $Y$ separated over $S$. An *$S$-rational map* $f: X \dashrightarrow Y$ from $X$ to $Y$ is an equivalence class of pairs $(U, f_U)$, with $U$ an $S$-dense open subset of $X$ and $f_U: U \to Y$ an $S$-morphism, and where two pairs $(U, f_U)$ and $(V, f_V)$ are equivalent if $f_U|_{U\cap V} = f_V|_{U\cap V}$. The $S$-rational map $f$ is said to be *defined (by the morphism $f_U$)* on an open subset $U \subseteq X$ if $(U, f_U)$ occurs in the equivalence class. The largest open subset $U \subseteq X$ on which $f$ is defined is called the *domain of definition* of $f$.

*Remark.* The assumption that $X$ is reduced and $Y \to S$ separated is necessary to ensure that this is an equivalence relation.

**Definition.** Let $S$ be a Noetherian scheme, and let $X, Y$ be reduced $S$-schemes, separated and of finite type over $S$. A *$S$-birational map* from $X$ to $Y$ is an $S$-rational map which can be represented by $(U, f)$ with $U$ an $S$-dense open subset of $X$ and $f$ an isomorphism from $U$ to an $S$-dense open subset of $Y$.

*Remark.* There are more general notions of rational maps; see [BLR, § 2.5] for a definition of $S$-rational maps for smooth schemes over any base scheme $S$, or see [EGA IV$_4$, § 20.2] for the definition of *pseudo-morphisms*, or *strict rational maps*, between arbitrary schemes.

## Henselian local rings

In this section we define a special class of local rings, namely those with the so-called Henselian property. They are characterised by the following fact [BLR, § 2.3, Proposition 4]: Let $R$ be a Henselian local ring with residue field $k$. Then every étale morphism from a scheme $X$ to $\operatorname{Spec} R$ is a local isomorphism at each $k$-rational point of $X$ lying above the closed point of $\operatorname{Spec} R$. In particular, if $R$ is strictly Henselian, then every étale morphism to $\operatorname{Spec} R$ is a local isomorphism at all points above the closed point of $\operatorname{Spec} R$. In fact, it is sufficient to require the following algebraic property, which is at first sight weaker (it implies the Henselian properties for open subsets, étale over $\operatorname{Spec} R$, of $R$-schemes of the form $\operatorname{Spec}(R[x]/(f))$ with $f \in R[x]$ a monic polynomial).

**Definition.** A local ring $R$ with maximal ideal $\mathfrak{m}$ and residue field $k$ is called a *Henselian local ring* if the following condition (known as Hensel's lemma) holds:

> For every monic polynomial $f \in R[x]$ and every simple zero of $f$ modulo $\mathfrak{m}$ (i.e. every $\alpha \in k$ such that $\bar{f}(\alpha) = 0$ and $\bar{f}'(\alpha) \neq 0$, where $\bar{f} \in k[x]$ is the reduction of $f$ modulo $\mathfrak{m}$), there is a unique $\tilde{\alpha} \in R$ such that $(\tilde{\alpha} \bmod \mathfrak{m}) = \alpha$ and $f(\tilde{\alpha}) = 0$.

A local ring $R$ is called *strictly Henselian* if it is Henselian with separably closed residue field.

Given any local ring $R$, it is possible to construct a 'smallest' Henselian local ring containing $R$, called the *Henselisation* of $R$, as well as a 'smallest' strictly Henselian local ring containing $R$, the *strict Henselisation* of $R$. The precise definition is as follows:

**Definition.** Let $R$ be a local ring. A *Henselisation* of $R$ is a Henselian local ring $R^{\mathrm{h}}$ together with a local homomorphism $i\colon R \to R^{\mathrm{h}}$ such that for every Henselian local ring $A$ together with a local homomorphism $f\colon R \to A$, there is a unique local homomorphism $f^{\mathrm{h}}\colon R^{\mathrm{h}} \to A$ such that $f = f^{\mathrm{h}} \circ i$.

**Definition.** Let $R$ be a local ring with residue field $k$. Fix a separable closure $k^{\mathrm{s}}$ of $k$. A *strict Henselisation* of $R$ (with respect to $k^{\mathrm{s}}$) is a Henselian local ring $R^{\mathrm{sh}}$, together with a local homomorphism $j\colon R \to R^{\mathrm{sh}}$ and an isomorphism from $k^{\mathrm{s}}$ to the residue field of $R^{\mathrm{sh}}$, such that for any strictly Henselian local ring $A$ together with a local homomorphism $f\colon R \to A$ and a $k$-embedding of $k^{\mathrm{s}}$ into the residue field of $A$, there is a unique local homomorphism $f^{\mathrm{sh}}\colon R^{\mathrm{sh}} \to A$ such that $f = f^{\mathrm{sh}} \circ j$ and such that $f^{\mathrm{sh}}$ induces the given embedding of residue fields. In other words, the diagram

$$\begin{array}{ccc} R & \xrightarrow{j} & R^{\mathrm{sh}} \\ \downarrow & & \downarrow \\ k & \hookrightarrow & k^{\mathrm{s}} \end{array}$$

is universal in the "category of local morphisms from $R$ to a strictly Henselian local ring $A$ together with an embedding of $k^{\mathrm{s}}$ into the residue field of $A$".

The (strict) Henselisation of a local ring $R$ can be constructed as as a direct limit of local rings of the form $\mathcal{O}_{X,x}$, where $X \to \operatorname{Spec} R$ is an étale morphism of schemes and $x$ is a point of $X$ lying above the closed point of $\operatorname{Spec} R$. This looks like the way in which the local ring of a scheme is constructed; in fact, if $S$ is a scheme and $s$ a point of $S$, then the strict Henselisation of $\mathcal{O}_{S,s}$ can be viewed as a local ring for the *étale topology* on $S$.

It is not hard to show that for any local ring $R$ the morphisms $R \to R^{\mathrm{h}} \to R^{\mathrm{sh}}$ are injective, and that the maximal ideal of $R$ generates the maximal ideals of $R^{\mathrm{h}}$ and $R^{\mathrm{sh}}$; these facts follow from the construction via direct limits mentioned above. Furthermore, it can be shown that if $R$ is reduced (resp. normal, resp. regular, resp. Noetherian), then its (strict) Henselisation has the same property. In particular, we have the following fact which will be of importance for us:

**Proposition.** *Let $R$ be a discrete valuation ring. Then $R^{\mathrm{h}}$ and $R^{\mathrm{sh}}$ are discrete valuation rings, and a uniformising element of $R$ is also a uniformising element of $R^{\mathrm{h}}$ and of $R^{\mathrm{sh}}$.*

*Proof*. This follows from the fact that the normal Noetherian local rings with principal maximal ideal are precisely the fields and discrete valuation rings, and from the properties of the (strict) Henselisation mentioned above. □

**Overview of the construction**

We will sketch a construction of the Néron model over $R$ of an Abelian variety $A_K$ over $K$. This goes in several steps:

(0) Construct a proper model $A_0$ for $A_K$ over $R$. This step is easy: embed $A_K$ in a projective space over $K$ and take the Zariski closure in the corresponding projective space over $R$.

(1) Apply the *smoothening process*; blowing up $A_0$ according to certain rules gives a proper model $A_1$ of $A_K$ over $R$ which possesses the following properties:

(a) For every $K^{\mathrm{sh}}$-valued point of $A_K$, the properness of $A_1$ gives a unique $R^{\mathrm{sh}}$-valued point of $A_1$ extending it; the image of this point is contained in the smooth locus of $A_1$.

(b) Let $Z$ be a smooth $R$-scheme, and let $u_K\colon Z_K \dashrightarrow A_K$ be a $K$-rational map. Then there exists an $R$-rational map $u$ from $Z$ into the smooth locus of $A_1$ which extends $u_K$.

(2) Construct a so-called *weak Néron model* $A_2$ out of $A_1$. This is again easy: we leave out the non-smooth locus of the special fibre of $A_1$. The model $A_2$ is smooth, separated and of finite type, but not necessarily proper. It is also not unique in general. It follows immediately from the properties a) and b) of $A_1$ that $A_2$ has the following two properties:

(a′) The natural map $A_2(R^{\mathrm{sh}}) \to A_K(K^{\mathrm{sh}})$ is bijective.

(b′) Let $Z$ be a smooth $R$-scheme, and let $u_K\colon Z_K \dashrightarrow A_K$ be a $K$-rational map. Then there exists an $R$-rational map $u\colon Z \dashrightarrow A_2$ extending $u_K$.

(3) The special fibre of $A_2$ is the disjoint union of its irreducible components, which are smooth, separated and of finite type over $k$; in particular they are integral. We leave out the components which are not $\omega$-minimal (see below for the definition). Then the model $A_3$ with which we are left is no longer a weak Néron model, but instead has the following two properties:

(c) Let $Z$ be a smooth $R$-scheme and $\zeta$ a generic point of its special fibre. Let $R'$ be the discrete valuation ring $\mathcal{O}_{Z,\zeta}$ and $K'$ its field of fractions. Then each translation of $A_{K'}$ by one of its $K'$-valued points extends to an $R'$-birational morphism of $A_3 \otimes_R R'$ which is an open immersion on its domain of definition.

(d) The group law on $A_K$ extends to an $R$-birational group law on $A_3$, i.e. an $R$-rational map

$$m\colon A_3 \times_R A_3 \dashrightarrow A_3$$

such that the universal translations

$$\Phi, \Psi\colon A_3 \times_R A_3 \dashrightarrow A_3 \times_R A_3$$

defined by $\Phi(x,y) = (x, m(x,y))$ and $\Psi(x,y) = (m(x,y), y)$ are $R$-birational. Furthermore, $m$ is associative (in an obvious sense).

(4) There is a unique embedding $A_3 \hookrightarrow A$ into a group scheme $A$ over $R$ (smooth, separated and of finite type) which is compatible with this birational group law. This $A$ will then be the Néron model of $A_K$ over $R$. The construction of $A$ is rather involved: its existence is proved first after the base change $R \to R^{\mathrm{sh}}$, and then *descent* is used to construct $A$ over $R$.

**The defect of smoothness**

We recall a 'differential' criterion for smoothness:

**Proposition.** *Let $f: X \to Y$ be morphism of schemes which is flat and locally of finite presentation, and let $x$ be a point of $X$. Then $f$ is smooth at $x$ if and only if the $\mathcal{O}_X$-module $\Omega_{X/Y}$ is locally free in a neighbourhood of $x$, of rank equal to the relative dimension of $f$ at $x$.*

*Proof*. See [EGA IV$_4$, corollaire 17.5.2 and proposition 17.15.15].

In the next lemma we use the following notation: let $X$ be an $R$-scheme which is locally of finite type, and let $x$ be a point of $X$. Then we write $\kappa(x)$ for the residue field of the local ring $\mathcal{O}_{X,x}$, and we put $\Omega_{X/R}(x) = \Omega_{X/R,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

**Lemma.** *Let $X$ be an $R$-scheme which is locally of finite type, and let $x \in X_k$, $\xi \in X_K$ be points of $X$ lying in the special and generic fibre of $X$, respectively, and such that $x \in \overline{\{\xi\}}$. Suppose that $X_K$ is smooth of relative dimension $d$ at $\xi$, and that the $\kappa(x)$-vector space $\Omega_{X/R}(x)$ has dimension $d$. Then $X$ is smooth of relative dimension $d$ at $x$.*

*Proof*. First we prove that the special fibre $X_k$ over $k$ is smooth of relative dimension $d$ at $x$. By a theorem of Chevalley [EGA IV$_3$, théorème 13.1.3], the dimension of the fibres of a morphism of finite type $f: X \to Y$ is an *upper semi-continuous function* on $X$, which is to say that for all $n \geq 0$ the set

$$F_n(X) = \{x \in X \mid \dim_x(X_{f(x)}) \geq n\}$$

is closed in $X$. Since in our case (with $Y = \operatorname{Spec} R$) the point $x$ is in the closure of $\{\xi\}$, this means that the dimension of $X_k$ at $x$ is at least $d$. On the other hand, $\Omega_{X_k/k,x} \otimes \kappa(x) = \Omega_{X/R}(x)$ is a $\kappa(x)$-vector space of dimension $d$ by assumption; this implies that the dimension of $X_k$ at $x$ is equal to $d$, and that the fibre $X_k$ is smooth over $k$ at $x$.

It remains to show that $X$ is flat at $x$. Since the problem is local on $X$, we may assume that there is a closed immersion $i: X \to Z$ with $Z$ an affine $R$-scheme which is smooth at $x$ (e.g. $Z = \mathbf{A}^n_R$ for some $n \geq 0$). We use induction on the relative dimension $n$ of $Z$ at $x$ to prove that in this situation $X$ is flat at $x$. If $n = d$, then $X_k$ and $X_K$ are identical to $Z_k$ and $Z_K$ in some open neighbourhood of $x$, so $X = Z$ in an open neighbourhood of $x$ and we are done. Now suppose the claim is true with $n-1$ in place of $n$. Write $I$ for the ideal of $\Gamma(Z, \mathcal{O}_Z)$ defining $X$, i.e. the kernel of $\Gamma(Z, \mathcal{O}_Z) \to \Gamma(X, \mathcal{O}_X)$; then because of the exact sequence

$$I/I^2 \xrightarrow{d} \Gamma(X, i^*\Omega_{Z/R}) \longrightarrow \Gamma(X, \Omega_{X/R}) \longrightarrow 0$$

and the fact that $\dim_{\kappa(x)}(i^*\Omega_{Z/R})(x) > \dim_{\kappa(x)} \Omega_{X/R}(x)$, there exists an element $g \in I$ such that the image of $dg$ in $(i^*\Omega_{Z/R})(x) = \Omega_{Z/R}(x)$ is non-zero. Let $j: Y \to Z$ be the closed immersion defined by $g$; then $Y$ is a closed subscheme of $Z$ which contains $X$ and is of relative dimension $n-1$ at $x$. We are done if we can show that $Y$ is smooth over $R$ at $x$, since then by the induction hypothesis we can apply the lemma to the immersion $X \to Y$. The choice of $g$ implies that $\dim_{\kappa(x)} \Omega_{Y/R}(x) = n-1$, and the same argument as above shows that the fibre $Y_k$ is smooth over $k$ at $x$. To prove that $Y$ is flat at $x$, we consider the exact sequence

$$0 \longrightarrow g\mathcal{O}_{Z,x} \longrightarrow \mathcal{O}_{Z,x} \longrightarrow \mathcal{O}_{Y,x} \longrightarrow 0.$$

Changing the base to $k$ and using the fact that $\mathcal{O}_{Z,z}$ is flat over $R$ gives the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(\mathcal{O}_{Y,x}, k) \longrightarrow g\mathcal{O}_{Z,x} \otimes_R k \longrightarrow \mathcal{O}_{Z_k,x} \longrightarrow \mathcal{O}_{Y_k,x} \longrightarrow 0.$$

Since $Z_k$ is smooth over $k$ at $x$, the local ring $\mathcal{O}_{Z_k,x}$ is a domain; furthermore, $\bar{g} \neq 0$ because of the choice of $g$ such that the image of $d\bar{g}$ in $(j^*\Omega_{Z/R})(x) = \Omega_{Z/R}(x)$ is non-zero. This implies that the composed morphism

$$\mathcal{O}_{Z_k,x} = \mathcal{O}_{Z,x} \otimes_R k \xrightarrow{g \otimes 1} g\mathcal{O}_{Z,x} \otimes_R k \longrightarrow \mathcal{O}_{Z_k,x},$$

which equals multiplication by $\bar{g}$, is injective. On the other hand, the map $g \otimes 1$ is surjective; therefore $g \otimes 1$ is an isomorphism and the map $g\mathcal{O}_{Z,x} \otimes_R k \to \mathcal{O}_{Z_k,x}$ is injective. This amounts to saying that $\operatorname{Tor}_1^R(\mathcal{O}_{Y,x}, k) = 0$. The local criterion of flatness [AK, Theorem 3.2] now implies that $\mathcal{O}_{Y,x}$ is flat over $R$, as we had to show. □

Now let $X$ be a scheme of finite type over $R$, and assume that its generic fibre $X_K$ is smooth over $K$. As before, let $R^{\mathrm{sh}}$ be a strict Henselisation of $R$ (with respect to a separable closure $k^{\mathrm{s}}$ of $k$) and $K^{\mathrm{sh}}$ its field of fractions. For any point $a \in X(R^{\mathrm{sh}})$, we write $a^*\Omega_{X/R}$ for the pull-back to $R^{\mathrm{sh}}$ of the sheaf of relative differentials of $X$ over $R$; this is a finitely generated $R^{\mathrm{sh}}$-module. Let $(a^*\Omega_{X/R})_{\mathrm{tor}}$ denote the torsion submodule of $a^*\Omega_{X/R}$; then $a^*\Omega_{X/R}/(a^*\Omega_{X/R})_{\mathrm{tor}}$ is torsion-free, hence free (by the structure theorem for finitely generated modules over a principal ideal domain).

**Lemma.** *The module $a^*\Omega_{X/R}$ over $R^{\mathrm{sh}}$ is free (equivalently, $(a^*\Omega_{X/R})_{\mathrm{tor}} = 0$) if and only if the image of $a$ lies in the smooth locus of $X \to \operatorname{Spec} R$.*

*Proof*. One implication follows easily from the above characterisation of smoothness: if $X \to S$ is smooth on an open subset containing the image of $a$, then $\Omega_{X/R}$ is locally free on an open subset containing this image, hence $a^*\Omega_{X/R}$ is free.

Conversely, suppose $a^*\Omega_{X/R}$ is free; then its rank must be the relative dimension of $X$ over $K$ at $x_K$. Denote by $x$ and $\xi$ the (topological) images of the special and generic points of $\operatorname{Spec} R^{\mathrm{sh}}$ under $a$, respectively. By continuity, $x$ is in the closure of $\{\xi\}$. The claim now follows from the previous lemma. □

The torsion submodule of $a^*\Omega_{X/R}$ turns out to be a useful measure for the non-smoothness of $X$ at $a$.

**Definition.** For any point $a \in X(R^{\mathrm{sh}})$, we define the *defect of smoothness* of $X$ at $a$, denoted by $\delta(a)$, as the length of the $R^{\mathrm{sh}}$-module $(a^*\Omega_{X/R})_{\mathrm{tor}}$ (which is a finitely generated torsion module, hence of finite length).

### The smoothening process

Let $X$ be an $R$-scheme of finite type such that the generic fibre $X_K$ is smooth over $K$. Let $k^{\mathrm{s}}$, $R^{\mathrm{sh}}$, $K^{\mathrm{sh}}$ be as in the preceding sections. For every point $a \in X(R^{\mathrm{sh}})$ we write $a_k\colon \operatorname{Spec} k^{\mathrm{s}} \to X_k$ for the specialisation of $a$ and $a_K\colon \operatorname{Spec} K^{\mathrm{sh}} \to X_K$ for the $K^{\mathrm{sh}}$-valued point defined by $a$. We say that $a$

**Definition.** Let $E$ be a subset of $X(R^{\mathrm{sh}})$, and let $Y$ be a geometrically reduced closed subscheme of $X_k$. Write $U$ for the largest open subscheme of $Y$ such that $U$ is smooth over $k$ and $\Omega_{X/R}|_U$ is locally free; this is a dense open subscheme of $Y$. Finally, let $E_Y$ be the subset of $E$ consisting of those points which specialise into points of $Y$. Then the subscheme $Y$ of $X_k$ is called *$E$-permissible* if it is geometrically reduced and if the images of the specialisations of the points in $E_Y$ form a Zariski dense subset of $Y$ which is contained in $U$.

If $X' \to X$ is obtained by blowing up $X$ in a closed subscheme of its special fibre, then $X' \to X$ is proper, and $X(K^{\mathrm{sh}}) \cong X'(K^{\mathrm{sh}})$ (since $X_K = X_{K'}$). Applying the valuative criterion of properness to $X' \to X$ shows that every point $a \in X(R^{\mathrm{sh}})$ lifts uniquely to a point $a' \in X'(R^{\mathrm{sh}})$. For any subset $E \subseteq X(R^{\mathrm{sh}})$, we denote by $E'$ the image of $E$ under the bijection $X(R^{\mathrm{sh}}) \xrightarrow{\sim} X'(R^{\mathrm{sh}})$.

The fundamental tool in the smoothening process is the following lemma, which says that the defect of smoothness is reduced by blowing up $X$ in suitable closed subschemes.

**Lemma.** *Let $E$ be a subset of $X(R^{\mathrm{sh}})$, and let let $Y$ be an $E$-permissible closed subscheme of $X_k$. Let $X' \to X$ be the blowing-up of $X$ in $Y$. Let $a$ be a point in $E$, and let $a' \in E'$ be the point corresponding to $a$ under the bijection $X(R^{\mathrm{sh}}) \xrightarrow{\sim} X'(K^{\mathrm{sh}})$.*
*(a) If $a$ specialises into a point of $X_k \setminus Y$, then $\delta(a') = \delta(a)$.*
*(b) If $a$ specialises into a point of $Y$, then $\delta(a') \le \max\{0, \delta(a) - 1\}$.*

*Proof*. See [BLR, § 3.4, Lemma 1]. (Actually, the proof there is based on the schematic dominance of the family of morphisms $E_Y$. The union of the images of the specialisations $a_k\colon \operatorname{Spec} k^{\mathrm{s}} \to X$, for $a \in E_Y$, is Zariski dense in $Y$. Since $Y$ is reduced, the family of morphisms $\{a_k \mid a \in E_Y\}$ is schematically dominant [EGA IV$_3$, définition 11.10.2 and proposition 11.10.4].) □

**Theorem.** *Let $X$ be an $R$-scheme of finite type with smooth generic fibre $X_K$. Then there exists a proper morphism $X' \to X$ of $R$-schemes, which can be obtained by a finite sequence of blowing-ups* *with centres contained in the non-smooth loci of the corresponding schemes, such that the image of each point in $X'(R^{\mathrm{sh}})$ lies in the smooth locus of $X'$.*

*Proof*. (This is a bit sketchy; see [BLR, § 3.4, Theorem 2] for a more formal proof.) Let $E$ be the subset of $X(R^{\mathrm{sh}})$ consisting of the points which specialise into the non-smooth locus of $X$. Consider the filtration $E = E_1 \supset E_2 \supset \ldots$ (strict inclusions) constructed as follows:

$Y_1$ = Zariski closure of the set $\{\operatorname{im} a_k \mid a \in E_1\}$;
$U_1$ = largest open subscheme of $Y_1$ such that $U_1$ is smooth over $k$ and $\Omega_{X/R}|_{U_1}$ is locally free;
$E_2$ = points in $E_1$ which specialise into $Y_1 \setminus U_1$;
$Y_2$ = Zariski closure of the set $\{\operatorname{im} a_k \mid a \in E_2\}$;
$U_2$ = largest open subscheme of $Y_2$ such that $U_2$ is smooth over $k$ and $\Omega_{X/R}|_{U_2}$ is locally free;
...

By construction, each $Y_i$ is $(E \setminus E_{i+1})$-permissible, though the fact that $Y_i$ is geometrically reduced is not entirely trivial to prove; see [BLR, § 3.3, Lemma 4]. Since $Y_1 \supset Y_2 \supset \ldots$ is a strictly decreasing chain of closed subsets of the Noetherian scheme $X_k$, we get $E_{t+1} = \emptyset$ for some least natural number $t$. If $t = 0$, we are done. Otherwise, let $\delta(E_t) = \max\{\delta(a) \mid a \in E_t\}$; this number is finite [BLR, § 3.3, Proposition 3]. Now $Y_t$ is by construction an $E$-permissible subscheme of $X$. The preceding lemma implies that the blowing-up $X' \to X$ of $X$ in $Y_t$ has the property that $\delta(a') < \delta(E_t)$ for all $a'$ in the subset $E'_t \subseteq X'(R^{\mathrm{sh}})$ corresponding to $E_t$. We may throw away all the points of $E'_t$ with image in the smooth locus of $X'$. Next we construct a filtration $E'_t = E'_{t,1} \supset E'_{t,2} \supset \ldots \supset E'_{t,u} \supset E'_{t,u+1} = \emptyset$ in the same way as for $E$; then either $u = 0$ (i.e. $E'_t = \emptyset$), in which case we continue by blowing up in $Y'_{t-1}$, or we blow up in $Y'_{t,u}$. We go on recursively like this; after finitely many steps, we get a morphism $X'' \to X$, obtained by blowing-ups in the non-smooth loci of the special fibres, such that all points in the lift of $E$ to $X''(R^{\mathrm{sh}})$ land in the smooth locus of $X''$. Since $X'' \to X$ is an isomorphism above the smooth locus of $X$, it follows that all points of $X''(R^{\mathrm{sh}})$ land in the smooth locus of $X''$, and we are finished. □

**Weak Néron models**

In this section we denote by $R^{\mathrm{sh}}$ be a strict Henselisation of $R$ with respect to a separable closure $k^{\mathrm{s}}$ of $k$, and we write $K^{\mathrm{sh}}$ for its field of fractions.

**Definition.** Let $X_K$ be a smooth projective $K$-scheme. A *weak Néron model* of $X_K$ is a model $X'$ of $X_K$ over $R$ which is smooth, separated and of finite type, such that the natural map

$$X'(R^{\mathrm{sh}}) \to X'(K^{\mathrm{sh}}) \cong X_K(K^{\mathrm{sh}})$$

(which is injective because of the separatedness of $X'$) is bijective. In words: every $K^{\mathrm{sh}}$-point of $X_K$ extends to a $R^{\mathrm{sh}}$-point of $X'$.

The reason for the name *weak Néron model* for the scheme $X'$ is that it satisfies a variant of the Néron property for rational maps, the so-called *weak Néron property.*

**Proposition.** *Let $X_K$ be a smooth projective $K$-scheme, let $X_0$ be a proper model of $X_K$ (e.g. the Zariski closure of $X_K$ embedded in some projective space), and let $X_1$ be the model obtained from $X_0$ by the smoothening process. Let $X_2$ be the weak Néron model of $X_K$ over $R$ obtained by removing the non-smooth locus of $X_1$. For every smooth $R$-scheme $Z$ and every $K$-rational map $u_K\colon Z_K \dashrightarrow X_K$, there exists an $R$-rational map $u\colon Z \dashrightarrow X_2$ extending $u_K$.*

*Proof*. We may assume that the special fibre $Z_k$ is irreducible. The local ring $\mathcal{O}_{Z,\zeta}$ of $Z$ at the generic point $\zeta$ of $Z_k$ is a discrete valuation ring whose field of fractions $L$ is the function field of the connected component of $Z$ containing $\zeta$. The $K$-rational map $u_K\colon Z_K \dashrightarrow X_K$ induces an $R$-morphism $\operatorname{Spec} L \to X_2$. By the valuative criterion of properness, this extends uniquely to an $R$-morphism $\operatorname{Spec} \mathcal{O}_{Z,\zeta} \to X_1$. Since $X_2$ is locally of finite type over $R$, there exists a $R$-dense open neighbourhood $U$ of $\zeta$ such that $u_K$ is defined by a morphism $u\colon U \to X_1$. Since $U$ is smooth over $R$, the set of $k^{\mathrm{s}}$-rational points of $U_k$ is Zariski dense in $U_k$, and all these points are specialisations of $R^{\mathrm{sh}}$-valued points of $Z$ [BLR, § 2.2, Corollary 13 and Proposition 14]. By property (a) of the model $X_1$, the images of all $k^{\mathrm{s}}$-rational points of $U$ under the morphism $u$ lie in the smooth locus $X_2$ of $X_1$. By continuity and the fact that $X_2$ is open in $X_1$, the special fibre of $u^{-1}X_2$ is a dense open subset of $U_k$. This implies that there is an $R$-dense open subset $U' \subseteq U$ such that $u_K$ is defined by an $R$-morphism $u\colon U' \to X_2$. □

*Remark.* One can prove the same result by using only the definition of weak Néron models, without knowing how they can be constructed; see [BLR, § 3.5, Proposition 3].

### The $\omega$-minimal model

**Notation.** Let $G$ be a group scheme over a scheme $S$. For any $S$-scheme $T$, we write $G_T$ for the fibred $G \times_S T$, viewed as a $T$-scheme, and $p_T$ for the canonical map $G_T \to G$. Furthermore, for any differential form $\omega \in \Gamma(G, \Omega^i_{G/S})$ we write $\omega_T$ for the pull-back $p_T^*\omega \in \Gamma(G_T, \Omega^i_{G_T/T})$.

**Definition.** Let $T$ be an $S$-scheme and $g \in G(T) = \mathrm{Hom}_S(T, G)$ a $T$-valued point of $G$. For any $T$-scheme $f\colon T' \to T$, there is a map

$$\begin{aligned} t_g(T')\colon G_T(T') &\to G_T(T') \\ g' &\mapsto m(g \circ f, g'), \end{aligned}$$

where $m$ is the group law on $T'$-valued points. This map is functorial in $T'$, and hence induces a morphism

$$t_g\colon G_T \to G_T$$

of $T$-schemes. This morphism is called *left translation by the point $g$*.

**Definition.** An *left-invariant differential form* of degree $i \geq 0$ is a global section $\omega$ of the sheaf $\Omega^i_{G/K}$ such that for every $S$-scheme $T$ and every $T$-valued point $g \in G(T)$ the pull-back $t_g^*\omega_T$ of $\omega_T$ under the left translation map $t_g\colon G_T \to G_T$ satisfies $t_g^*\omega_T = \omega_T$.

**Proposition.** *Let $G$ be a group scheme over a scheme $S$. Then for all $i \geq 0$, the map*

$$\begin{aligned} \{\textit{left-invariant differentials of degree } i \textit{ on } G\} &\longrightarrow \Gamma(S, e^*\Omega^i_{G/S}) \\ \omega &\longmapsto e^*\omega, \end{aligned}$$

*where $e\colon S \to G$ is the neutral section, is bijective.*

Let $A_K$ be an Abelian variety of dimension $d$ over the field $K$. The left-invariant differential forms on $A_K$ are simply called *invariant differential forms*. Since $\Omega^d_{A_K/K}$ is a line bundle on $A_K$, the $K$-vector space $e^*\Omega^d_{A_K/K}$ is of dimension 1. The previous proposition implies that there exists a non-zero invariant differential form $\omega$ of degree $d$ on $A_K$; it is unique up to multiplication by an element of $K^\times$. We fix one such form from now on.

Let $A'$ be a weak Néron model of $A_K$ over $R$. Let $C_1, \ldots, C_r$ be the irreducible components of the special fibre $A'_k$ of $A'$. Each local ring $\mathcal{O}_{A',\zeta_i}$, where $\zeta_i$ is the generic point of $C_i$, is a discrete valuation ring. Therefore, the invariant $d$-form $\omega$, viewed as a rational section of $\Omega_{A'/R}$, has a well-defined order of vanishing $n_i$ along each $C_i$ (if $n_i < 0$, then $\omega$ has a pole of order $-n_i$ along $C_i$). We put $n_0 = \min\{n_1, \ldots, n_r\}$, so that $\pi^{-n_0}\omega$ vanishes exactly on the components $C_i$ for which $n_i > n_0$. Let $A''$ be the $R$-model of $A_K$ obtained by removing all these $C_i$; then we have the following result.

**Proposition.** *The group law on $A_K$ extends to a birational group law on $A''$, i.e. an $R$-birational map*

$$m\colon A'' \times_R A'' \dashrightarrow A''$$

*such that the universal translations*

$$\begin{aligned} \Phi\colon A_3 \times_R A_3 &\dashrightarrow A_3 \times_R A_3 \\ (x, y) &\longmapsto (x, m(x, y)) \end{aligned}$$

*and*

$$\begin{aligned}\Psi\colon A_3 \times_R A_3 &\dashrightarrow A_3 \times_R A_3\\ (x,y) &\longmapsto (m(x,y),y)\end{aligned}$$

*are $R$-birational, and such that $m$ is associative in the sense that $m \circ (m \times 1)$ and $m \circ (1 \times m)$ coincide wherever they are defined.*

*Proof*. See [BLR, § 4.3, Proposition 5].

### From birational group laws to group schemes

**Theorem.** *Let $X$ be an $R$-scheme which is smooth, separated, of finite type and surjective. Suppose that $X_K$ is a group scheme such that the group law on $X_K$ extends to an $R$-birational group law*

$$m\colon X \times_S X \dashrightarrow X$$

*on $X$. Then there exists a group scheme $\bar{X}$ over $R$ and an open immersion $X \to \bar{X}$ onto an $R$-dense open subscheme of $\bar{X}$ which is an isomorphism on the generic fibres and such that the group law on $\bar{X}$ restricts to $m$ on $X$.*

*Proof*. See [BLR, § 5.1, Theorem 5].

**Theorem.** *Let $A_K$ be an Abelian variety over $K$, and let $A'$ be the $\omega$-minimal model of $A$ over $R$. Then the group scheme $\overline{A'}$ from the previous theorem is a Néron model for $X$ over $R$.*

*Proof*. See [BLR, § 4.4, Corollary 4].

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