Global class field theory

1. The idèle group of a global field

Let K be a global field. For every place v of K we write K_v for the completion of K at v and U_v for the group of units of K_v (which is by definition the group of positive elements of K_v if v is a real place and the whole group $K_v^\times$ if v is a complex place).

We have seen the definition of the idèle group of K:

$$J_K = \varinjlim_{S} \left(\prod_{v\in S} K_v^\times \times \prod_{v\notin S} U_v \right),$$

where S runs over all finite sets of places of K. This is a locally compact topological group containing $K^\times$ as a discrete subgroup.

2. The reciprocity map

Let L/K be an extension of global fields and $(L/K)^{\rm ab}$ the maximal Abelian extension of K inside L. For any place v of K, we choose a place $v'|v$ in $(L/K)^{\rm ab}$, and we denote the corresponding Abelian extension by $((L/K)^{\rm ab})_v$. The Galois group of this extension can be idenfied with the decomposition group of v in $\mathrm{Gal}((L/K)^{\rm ab})$.

For every place v of K we have the local norm residue symbol

$$(\ \ ,((L/K)^{\rm ab})_v)\colon K_v^\times \to \mathrm{Gal}(((L/K)^{\rm ab})_v).$$

This local norm residue symbol gives rise to group homomorphisms

$$\left(\frac{\ \ ,(L/K)^{\rm ab}}{v}\right)\colon K_v^\times \to \mathrm{Gal}((L/K)^{\rm ab})$$

by identifying $\mathrm{Gal}(((L/K)^{\rm ab})_v)$ with the decomposition group of v in $\mathrm{Gal}((L/K)^{\rm ab})$. Since $(L/K)^{\rm ab}$ is Abelian, this map does not depend on the choice of a place of L above v.

We now *define* the global reciprocity map by taking the product of the local norm residue maps:

$$\begin{aligned}(\ \ ,L/K)\colon J_K &\longrightarrow \mathrm{Gal}((L/K)^{\rm ab})\\ (x_v)_v &\longmapsto \prod_v \left(\frac{x_v,(L/K)^{\rm ab}}{v}\right).\end{aligned}$$

This is a surjective homomorphism which only depends on $(L/K)^{\rm ab}$ and not on L/K itself; however, it is still useful to define it for arbitrary L/K, as parts (iv) and (v) of the following theorem show.

Theorem 2.1. *Let L/K be an extension of global fields, and let $M/L/K$ be a tower of global fields for (iii)–(v).*

(i) *The map $(\ \ ,(L/K)^{\rm ab})$ is surjective and has kernel $K^\times \mathrm{N}_{L/K}(J_L)$, i.e. we have a short exact sequence*

$$1 \longrightarrow K^\times \mathrm{N}_{L/K}(J_L) \longrightarrow J_K \longrightarrow \mathrm{Gal}((L/K)^{\rm ab})$$

(ii) *For every v, the image of $K_v^\times$ (viewed as a subgroup of J_K) under $(\ \ ,L/K)$ is the decomposition group of v in $\mathrm{Gal}((L/K)^{\rm ab})$.*

(iii) *We have a commutative diagram*

$$\begin{array}{ccc} J_K & \xrightarrow{(\ \ ,L/K)} & \mathrm{Gal}((M/K)^{\rm ab}) \\ & {\scriptstyle (\ \ ,M/K)}\searrow & \downarrow \\ & & \mathrm{Gal}((L/K)^{\rm ab}), \end{array}$$

where the vertical map is the canonical one (note that the maximal Abelian extension of K in M contains the maximal Abelian extension of K in L).

(iv) *We have a commutative diagram*

$$\begin{array}{ccc} J_L & \xrightarrow{(\ ,M/L)} & \mathrm{Gal}((M/L)^{\mathrm{ab}}) \\ {\scriptstyle \mathrm{N}_{L/K}}\downarrow & & \downarrow \\ J_K & \xrightarrow{(\ ,M/K)} & \mathrm{Gal}((M/K)^{\mathrm{ab}}) \end{array}$$

where the vertical map on the right is the canonical one (note that the maximal Abelian extension of K in M is contained in the maximal Abelian extension of L in M).

(v) *We have a commutative diagram*

$$\begin{array}{ccc} J_K & \xrightarrow{(\ ,M/K)} & \mathrm{Gal}((M/K)^{\mathrm{ab}}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Ver}} \\ J_L & \xrightarrow{(\ ,M/L)} & \mathrm{Gal}((M/L)^{\mathrm{ab}}), \end{array}$$

where the map on the left is the canonical inclusion and the map on the right is the transfer (or Verlagerung).

The most remarkable property of the global reciprocity map is the fact that $K^\times$ is in its kernel. We define the *Hasse symbols* on $K^\times$ by

$$\begin{aligned} \left(\frac{\ ,(L/K)^{\mathrm{ab}}}{v}\right): K^\times &\longrightarrow \mathrm{Gal}(L/K), \\ x &\longmapsto \left(\frac{i_v(x),(L/K)^{\mathrm{ab}}}{v}\right), \end{aligned}$$

where $i_v\colon K \to K_v$ is the inclusion. Then the fact that the global reciprocity map vanishes on $K^\times$ is equivalent to the *product formula*

$$\prod_v \left(\frac{x,(L/K)^{\mathrm{ab}}}{v}\right) = 1 \quad \text{for all } x \in K^\times.$$

Example (quadratic reciprocity law). Let p be a (positive) odd prime number and

$$K = \mathbf{Q}, \quad L = \mathbf{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}}p}\right)$$

(note that $(-1)^{\frac{p-1}{2}}p$ is the generator of $p\mathbf{Z}$ which is congruent to 1 modulo 4). Then the extension $L/\mathbf{Q}$ is only ramified at p (we consider the extension $\mathbf{C}/\mathbf{R}$ as unramified, following for example Neukirch and Gras but in opposition to many other books). Let a be an integer coprime to $2p$. To compute $\left(\frac{a,L/\mathbf{Q}}{v}\right)$ for a place v of $\mathbf{Q}$, we use that by local class field theory this symbol is trivial precisely when a is a norm in the extension L_v/K_v. Furthermore, by multiplicativity we reduce to the case where a equals -1 or a positive odd prime number $q \neq p$. We distinguish the cases $v = \infty$, $v = p$ and $v = r$ with r a prime number different from p This means that we have to compute

$$\left(\frac{-1,L/\mathbf{Q}}{\infty}\right), \quad \left(\frac{-1,L/\mathbf{Q}}{p}\right), \quad \left(\frac{-1,L/\mathbf{Q}}{r}\right) \text{ for } r \neq p,$$
$$\left(\frac{q,L/\mathbf{Q}}{\infty}\right), \quad \left(\frac{q,L/\mathbf{Q}}{p}\right), \quad \left(\frac{q,L/\mathbf{Q}}{r}\right) \text{ for } r \neq p.$$

For convenience we will identify $\mathrm{Gal}(L/\mathbf{Q})$ with $\{\pm 1\}$. We start with $v = \infty$. Since

$$L_\infty/\mathbf{Q}_\infty = \begin{cases} \mathbf{R}/\mathbf{R} & \text{if } p \equiv 1 \pmod 4, \\ \mathbf{C}/\mathbf{R} & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

we see that q is always a norm in $L_\infty/\mathbf{Q}_\infty$ and that -1 is a norm in $L_\infty/\mathbf{Q}_\infty$ if and only if $p \equiv 1 \pmod 4$, i.e.

$$\left(\frac{-1, L/\mathbf{Q}}{\infty}\right) = (-1)^{\frac{p-1}{2}} \quad \text{and} \quad \left(\frac{q, L/\mathbf{Q}}{\infty}\right) = 1.$$

Next we compute $\left(\frac{-1,L/\mathbf{Q}}{r}\right)$ and $\left(\frac{-1,L/\mathbf{Q}}{r}\right)$. Since r is unramified in $L/\mathbf{Q}$ we have

$$\left(\frac{a, L/\mathbf{Q}}{r}\right) = \sigma_r^{\mathrm{ord}_r(a)}$$

where $\sigma_r \in \mathrm{Gal}(L/\mathbf{Q})$ is the Frobenius element at r, which gives

$$\left(\frac{-1, L/\mathbf{Q}}{r}\right) = 1 \quad \text{for all } r,$$
$$\left(\frac{q, L/\mathbf{Q}}{r}\right) = 1 \quad \text{for all } r \neq q,$$

and

$$\left(\frac{q, L/\mathbf{Q}}{q}\right) = \sigma_q.$$

Next we use that σ_q is trivial if and only if q is split in $L/\mathbf{Q}$, which is the case if and only if $(-1)^{\frac{p-1}{2}}p$ is a square in $\mathbf{Q}_q$. This means that

$$\left(\frac{q, L/\mathbf{Q}}{q}\right) = \left(\frac{(-1)^{\frac{p-1}{2}}p}{q}\right),$$

where the right-hand side denotes the usual Kronecker symbol. Finally, to compute $\left(\frac{-1,L/\mathbf{Q}}{p}\right)$ and $\left(\frac{q,L/\mathbf{Q}}{p}\right)$, we note that an element of $U_p = \mathbf{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbf{Z}_p)$ is a norm if and only if it is in the unique subgroup $U_p^2 = \mu_{p-1}^2 \times (1 + p\mathbf{Z}_p)$ of index 2 in U_p, i.e. if and only if it is a square modulo p. Thus we see that

$$\begin{aligned}\left(\frac{-1, L/\mathbf{Q}}{p}\right) &= \left(\frac{-1}{p}\right) \\ &= (-1)^{\frac{p-1}{2}}\end{aligned}$$

and that

$$\left(\frac{q, L/\mathbf{Q}}{p}\right) = \left(\frac{q}{p}\right),$$

where the right-hand side is the usual quadratic residue symbol.

Now the product formula applied to the prime number $q \nmid 2p$ reads

$$\left(\frac{q}{p}\right)\left(\frac{(-1)^{\frac{p-1}{2}}p}{q}\right) = 1$$

or equivalently

$$\begin{aligned}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) &= \left(\frac{-1}{q}\right)^{\frac{p-1}{2}} \\ &= (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.\end{aligned}$$

The other main theorem of global class field theory is the *global existence theorem*.

Theorem 2.2. *For any closed subgroup N of finite index in $J_K/K^\times$, there is a unique Abelian extension L of K such that the image of $J_L/L^\times$ under the norm map $J_L/L^\times \to J_K/K^\times$ (or equivalently, by the previous theorem, the kernel of the reciprocity map $J_K/K^\times \to \mathrm{Gal}(L/K)$) equals N. This gives a Galois correspondence between the closed subgroups of finite index in J_K containing $K^\times$ and finite Abelian extensions of K (up to isomorphism, or viewed inside the maximal Abelian extension of K).*

Remark. Given the existence of the correspondence stated in the theorem, the fact that it has the Galois properties follows from parts (i) and (iii) of Theorem 2.1.

3. The class group formulation of class field theory

The closed subgroups of finite index in J_K containing $K^\times$ are precisely the open subgroups containing $K^\times$, and it is known that these are exactly the subgroups of J_K which contain $K^\times$ and a subgroup of the form

$$U_\mathfrak{m} = \prod_v U_v^{n_v}$$

for some integral ideal $\mathfrak{m} = \prod_v (\mathfrak{m}_v^{n_v} \cap K)$ of K. This implies the existence of a *conductor* for every Abelian extension L/K.

Definition. The *conductor* of a finite extension L/K is the least integral ideal $\mathfrak{f}$ of K (least with respect to divisibility) such that the open subgroup $K^\times \mathrm{N}_{L/K}(J_L)$ of J_K contains $U_\mathfrak{f}$.

The group $J_K/K^\times U_\mathfrak{m}$ is called the *ray class group* modulo $\mathfrak{m}$. It is isomorphic to the group

$$\mathrm{Cl}_\mathfrak{m}(K) = \{\text{fractional ideals of } K \text{ coprime to } \mathfrak{m}\}/\{(a) \mid a \text{ totally positive and } 1 \pmod{\mathfrak{m}}\}.$$

For any finite Abelian extension L/K whose conductor divides $\mathfrak{m}$, the global reciprocity map induces a surjective homomorphism (the *Artin map*)

$$\alpha_{L/K}\colon \mathrm{Cl}_\mathfrak{m}(K) \to \mathrm{Gal}(L/K).$$

The existence theorem can be rephrased that for every subgroup $H \subseteq \mathrm{Cl}_\mathfrak{m}(K)$ there exists an Abelian extension K_H of $\mathrm{Cl}_\mathfrak{m}(K)$ such that the Artin map $\alpha_{K_H/K} \to \mathrm{Gal}(K_H/K)$ has kernel H.

The Abelian extension of K which corresponds to $K^\times U_\mathfrak{m}$ under the correspondence of global class field theory is called the *ray class field* modulo $\mathfrak{m}$ and is denoted by $K_{(\mathfrak{m})}$. In the class group formulation of class field theory, $K_{(\mathfrak{m})}$ corresponds to the trivial subgroup of $\mathrm{Cl}_\mathfrak{m}(K)$. In particular, the Artin map is an isomorphism:

$$\mathrm{Cl}_\mathfrak{m}(K) \xrightarrow{\sim} \mathrm{Gal}(K_{(\mathfrak{m})}/K)$$

The conductor of $K_{(\mathfrak{m})}$ divides $\mathfrak{m}$ but is not necessarily equal to it.

A special example of a ray class field is the *(narrow) Hilbert class field* H_K of a global field K, which is defined as the ray class field of K modulo 1. The extension H_K/K is the largest unramified extension of K; note that according to our conventions the extension $\mathbf{C}/\mathbf{R}$ is unramified. The so-called *principal ideal theorem* states that every ideal of K becomes principal when extended to an ideal of H_K.

References

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[2] J. Neukirch, *Algebraische Zahlentheorie*. Springer-Verlag, Berlin/Heidelberg, 1992.