Exam: Representation Theory of Finite Groups

Monday 3 June 2019, 10:00–13:00

Note:

- You may consult books and lecture notes. The use of electronic devices is not allowed.
- You may use results proved in the lecture or in the exercises, unless this makes the question trivial. When doing so, clearly state the results that you use.
- This exam consists of five questions. The number of points that each question is worth is indicated in the margin. The grade for this exam is 1 + (number of points)/10.
- If you are unable to answer a subquestion, you may still use the result in the remainder of the question.
- Representations are taken to be over $\mathbf{C}$, unless mentioned otherwise.

(20 pt) **1.** Let G be a finite group, let $N \triangleleft G$ be a normal subgroup, let G/N be the quotient group, and let k be a field. Let V be a $k[G]$-module, and let W be the k-linear subspace of V spanned by the elements $v - nv$ for $v \in V$ and $n \in N$.

(a) Show that W is a sub-$k[G]$-module of V.

(b) Show that the quotient V/W has a natural $k[G/N]$-module structure.

(c) Suppose that the $k[G]$-module V is simple and that N acts trivially on V. Show that V/W is simple as a $k[G/N]$-module.

(16 pt) **2.** Let G be a finite group, let $H \subseteq G$ be a subgroup, and let G/H be the set of cosets gH for $g \in G$. Let $\mathbf{C}\langle G/H\rangle$ be the $\mathbf{C}$-vector space of formal linear combinations of elements of G/H. Consider $\mathbf{C}[G]$ as a right $\mathbf{C}[H]$-module in the natural way, and consider $\mathbf{C}$ as a left $\mathbf{C}[H]$-module with trivial H-action.

(a) Show that the map

$$t\colon \mathbf{C}[G] \times \mathbf{C} \longrightarrow \mathbf{C}\langle G/H\rangle$$
$$\left(\sum_{g\in G} c_g g, \lambda\right) \longmapsto \sum_{g\in G} (\lambda c_g) gH$$

is $\mathbf{C}[H]$-bilinear.

(b) Show (by verifying the universal property) that the $\mathbf{C}$-vector space $\mathbf{C}\langle G/H\rangle$ together with the $\mathbf{C}[H]$-bilinear map t is a tensor product of $\mathbf{C}[G]$ and $\mathbf{C}$ over $\mathbf{C}[H]$.

(16 pt) **3.** The character table of the alternating group A_4 looks as follows (here ζ satisfies $\zeta^2 + \zeta + 1 = 0$):

conj. class	[(1)]	[(12)(34)]	[(123)]	[(132)]
size	1	3	4	4
	1	1	1	1
	1	1	ζ	$-1-\zeta$
	1	1	$-1-\zeta$	ζ
	3	-1	0	0

Let X be the set of unordered pairs (two-element subsets) $\{i, j\}$ with $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, and let $\mathbf{C}\langle X\rangle$ be the $\mathbf{C}$-vector space of formal linear combinations of elements of X. The natural action of A_4 on X defines a permutation representation of A_4 on $\mathbf{C}\langle X\rangle$. Determine the decomposition of $\mathbf{C}\langle X\rangle$ as a direct sum of irreducible representations of A_4.

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(*18 pt*) **4.** The character table of the symmetric group S_4 looks as follows:

conj. class	[(1)]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
size	1	6	3	8	6
	1	1	1	1	1
	1	−1	1	1	−1
	2	0	2	−1	0
	3	1	−1	0	−1
	3	−1	−1	0	1

(a) Show that every finite-dimensional representation of S_4 is isomorphic to its dual.

(b) Consider the class function $f\colon S_4 \to \mathbf{C}$ defined by

$[g]$	[(1)]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
$f(g)$	7	−1	3	1	−3

Determine whether f is the character of a finite-dimensional representation of S_4.

(c) Let V be the unique 2-dimensional irreducible representation of S_4, and let W and W' be the two 3-dimensional irreducible representations of S_4. Show that the three representations $\mathrm{Hom}_{\mathbf{C}}(V, W)$, $V \otimes_{\mathbf{C}} W'$ and $W \oplus W'$ are pairwise isomorphic.

(*20 pt*) **5.** Let n be a positive integer, and let D_n be the dihedral group of order $2n$, generated by two elements r and s subject to the relations $r^n = 1$, $s^2 = 1$ and $srs^{-1} = r^{-1}$. We view the cyclic group C_n as the subgroup $\langle r \rangle$ of D_n.

(a) For all $g \in D_n$, let T_g be the set of all $t \in D_n$ satisfying $t^{-1}gt \in C_n$. Show that

$$T_g = \begin{cases} D_n & \text{if } g \in C_n, \\ \emptyset & \text{if } g \notin C_n. \end{cases}$$

Let V be a one-dimensional representation of C_n, and let $W = \mathrm{Ind}_{C_n}^{D_n} V$ be the induced representation. Let $\xi\colon C_n \to \mathbf{C}$ and $\chi\colon D_n \to \mathbf{C}$ be the characters of V and W, respectively.

(b) Show that for all $g \in D_n$, the character value $\chi(g)$ satisfies

$$\chi(g) = \begin{cases} \xi(g) + \xi(g^{-1}) & \text{if } g \in C_n, \\ 0 & \text{if } g \notin C_n. \end{cases}$$

(c) Let h be a generator of C_n, and let $\zeta = \xi(h)$. Suppose that $n \geq 3$ and that ζ is a primitive n-th root of unity. Show that W is an irreducible representation of D_n.