Retake: Representation Theory of Finite Groups

Thursday 27 June 2019, 10:00–13:00

Note:

- You may consult books and lecture notes. The use of electronic devices is not allowed.
- You may use results proved in the lecture or in the exercises, unless this makes the question trivial. When doing so, clearly state the results that you use.
- This exam consists of five questions. The number of points that each question is worth is indicated in the margin. The grade for this exam is 1 + (number of points)/10.
- If you are unable to answer a subquestion, you may still use the result in the remainder of the question.
- Representations are taken to be over $\mathbf{C}$, unless mentioned otherwise.
- **Notation:** For any set S and any field k, we write $k\langle S\rangle$ for the k-vector space of formal finite k-linear combinations of elements of S.

(18 pt) **1.** Let G be a finite group, let $N \triangleleft G$ be a normal subgroup, let G/N be the quotient group, and let k be a field. Let V be a $k[G]$-module, and let

$$V^N = \{v \in V \mid nv = v \text{ for all } n \in N\}$$

be the set of N-invariant elements in V.

(a) Show that V^N is a sub-$k[G]$-module of V.

(b) Show that V^N has a natural $k[G/N]$-module structure.

(c) Consider k as a $k[N]$-module with trivial N-action. Show that the k-linear map

$$\begin{aligned} {}_{k[N]}\mathrm{Hom}(k, V) &\longrightarrow V \\ h &\longmapsto h(1) \end{aligned}$$

is injective with image equal to V^N.

(20 pt) **2.** Let D_5 be the dihedral group of order 10, generated by two elements r and s subject to the relations $r^5 = 1$, $s^2 = 1$ and $srs^{-1} = r^{-1}$.

In this question, you may only use general results about representations, as opposed to results specifically about representations of dihedral groups.

(a) Show that D_5 has exactly two irreducible representations of dimension 1 (up to isomorphism), and give these explicitly.

(b) Let ζ be a primitive fifth root of unity in $\mathbf{C}$. Show that there is a unique representation $\rho_\zeta\colon D_5 \to \mathrm{GL}_2(\mathbf{C})$ satisfying

$$\rho_\zeta(r) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad \rho_\zeta(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) Show that ρ_ζ is irreducible for every primitive fifth root of unity $\zeta \in \mathbf{C}$.

(d) Determine the character table of D_5.

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(*16 pt*) **3.** The character table of the symmetric group S_4 looks as follows:

conj. class	[(1)]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
size	1	6	3	8	6
	1	1	1	1	1
	1	-1	1	1	-1
	2	0	2	-1	0
	3	1	-1	0	-1
	3	-1	-1	0	1

(a) Let V be the unique two-dimensional irreducible representation of S_4. Determine the decomposition of $V \otimes_{\mathbf{C}} V \otimes_{\mathbf{C}} V$ as a direct sum of irreducible representations of S_4.

(b) Let T be a regular tetrahedron with a numbering of the four vertices by the set $\{1,2,3,4\}$. This gives an identification of S_4 with the group of isometries of T. Let E be the set of edges of T, so $\#E = 6$ and $\mathbf{C}\langle E\rangle$ is a (permutation) representation of S_4. Determine the decomposition of $\mathbf{C}\langle E\rangle$ as a direct sum of irreducible representations of S_4.

(*18 pt*) **4.** Let k be a field, let G be a finite group, let X be a finite right G-set, and let Y be a finite left G-set. Let Z be the quotient of the set $X \times Y$ by the left G-action defined by $g(x,y) = (xg^{-1}, gy)$. The image of an element (x,y) under the quotient map $X \times Y \to Z$ is denoted by $[x,y]$. Note that $k\langle X\rangle$ is a right $k[G]$-module and $k\langle Y\rangle$ is a left $k[G]$-module.

(a) Show that the map

$$t\colon k\langle X\rangle \times k\langle Y\rangle \longrightarrow k\langle Z\rangle$$
$$\left(\sum_{x\in X} c_x x, \sum_{y\in Y} d_y y\right) \longmapsto \sum_{(x,y)\in X\times Y} c_x d_y [x,y]$$

is $k[G]$-bilinear.

(b) Show (by verifying the universal property) that the k-vector space $k\langle Z\rangle$ together with the k-bilinear map t is a tensor product of $k\langle X\rangle$ and $k\langle Y\rangle$ over $k[G]$.

(*18 pt*) **5.** Let p be a prime number, and let G be the semidirect product $\mathbf{F}_p \rtimes \mathbf{F}_p^\times$, where $\mathbf{F}_p^\times$ acts on $\mathbf{F}_p$ by multiplication. (Thus G is the product set $\mathbf{F}_p \times \mathbf{F}_p^\times$ equipped with the group operation $(a,m)(a',m') = (a+ma', mm')$ for $(a,m),(a',m') \in G$.) We view the additive group $\mathbf{F}_p$ as a normal subgroup of G via the injection $a \mapsto (a,1)$. Let $\xi\colon \mathbf{F}_p \to \mathbf{C}^\times$ be the homomorphism defined by

$$\xi(a \bmod p) = \exp(2\pi i a/p).$$

Let $\mathrm{Ind}_{\mathbf{F}_p}^G \xi$ be the induced representation, and let $\chi\colon G \to \mathbf{C}^\times$ be the character of $\mathrm{Ind}_{\mathbf{F}_p}^G \xi$.

(a) Show that

$$\chi(a,m) = \begin{cases} p-1 & \text{if } a = 0 \text{ and } m = 1,\\ -1 & \text{if } a \neq 0 \text{ and } m = 1,\\ 0 & \text{if } m \neq 1.\end{cases}$$

(b) Show that the representation $\mathrm{Ind}_{\mathbf{F}_p}^G \xi$ is irreducible.