Representation Theory of Finite Groups, spring 2019

Problem Sheet 1

4 Februari

Definition. A group G is *solvable* (Dutch: *oplosbaar*) if there exists a chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

of subgroups of G such that for $0 \le i < n$, the subgroup G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is Abelian.

1. Let G be a group. The *derived series* of G is the chain of subgroups of G defined by

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots$$

where $G_{i+1} = [G_i, G_i]$ for all $i \ge 0$. Show that G is solvable if and only if there exists $n \ge 0$ such that $G_n = \{1\}$.

2. Let G be a finite group. Show that G is solvable if and only if there exists a chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

of subgroups of G such that for $0 \le i < n$, the subgroup G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is cyclic of prime order.

3. (a) Show that every subgroup of a solvable group is solvable.

(b) Show that every quotient of a solvable group by a normal subgroup is solvable.

4. For every $n \ge 1$, the dihedral group D_n of order $2n$ is defined using generators and relations by

$$D_n = \langle \rho, \sigma \mid \rho^n, \sigma^2, (\sigma\rho)^2 \rangle.$$

Show that D_n is solvable.

5. Let G be the symmetry group (of order 48) of the 3-dimensional cube. Show that G is solvable by giving a chain of subgroups as in the definition of solvability. (*Hint:* use the action of G on the set of four lines passing through two opposite vertices.)

Definition. Let A be a commutative ring. An *A-algebra* is a (not necessarily commutative) ring R together with a ring homomorphism $i\colon A \to Z(R)$. Here $Z(R)$ is the centre of R, defined by $Z(R) = \{r \in R \mid \forall\, s \in R\colon rs = sr\}$.

Definition. Let R be a ring. A *(left) R-module* is an Abelian group M together with a map

$$R \times M \longrightarrow M$$
$$(r, m) \longmapsto r \cdot m$$

satisfying the following identities for all $r, s \in R$ and $m, n \in M$:

$$r \cdot (m + n) = r \cdot m + r \cdot n \qquad (rs) \cdot m = r \cdot (s \cdot m)$$
$$(r + s) \cdot m = r \cdot m + s \cdot m \qquad 1 \cdot m = m.$$

6. Let M be an Abelian group. Show that there is exactly one map $\mathbf{Z} \times M \to M$ with the property that it makes M into a $\mathbf{Z}$-module.

7. Let R be a ring. Show that the multiplication map $R \times R \to R$ makes R into a left R-module.

8. Let M be an Abelian group. Consider the set

$$\operatorname{End} M = \{f\colon M \to M \text{ group homomorphism}\}.$$

equipped with addition and multiplication maps defined by $(f+g)(m) = f(m)+g(m)$ and $fg = f \circ g$ for $f, g \in \operatorname{End} M$ and $m \in M$.

(a) Show that $\operatorname{End} M$ is a ring.

(b) Show that M is in a natural way a module over $\operatorname{End} M$.

9. Let R be a ring, and let M be an Abelian group. Show that giving an R-module structure on M is equivalent to giving a ring homomorphism $R \to \operatorname{End} M$.

10. Let k be a field, and let n be a non-negative integer. Show that k^n is in a natural way a module over the matrix algebra $\operatorname{Mat}_n(k)$.

11. Let R be a ring, and let M be an R-module. Consider the set

$$\operatorname{End}_R M = \{f \in \operatorname{End} M \mid f(r \cdot m) = r \cdot f(m) \text{ for all } r \in R\}.$$

Show that $\operatorname{End}_R M$ is a subring of $\operatorname{End} M$.

12. Let $\phi\colon R \to S$ be a ring homomorphism, and let N be an S-module. We write $\phi^* N$ for the Abelian group N equipped with the map

$$\begin{aligned} R \times N &\longrightarrow N \\ (r, m) &\longmapsto \phi(r) \cdot m. \end{aligned}$$

Show that $\phi^* N$ is an R-module.

13. Let A be a commutative ring, let R be an A-algebra, let $i\colon A \to R$ be the corresponding ring homomorphism (with image in $Z(R) \subset R$), and let M be an R-module. Let $i^* M$ be the A-module defined in Exercise 12. Show that the R-module structure on M gives a natural ring homomorphism

$$R \to \operatorname{End}_A(i^* M).$$

14. Let R and S be two rings, let M be an R-module, and let N be an S-module. Show that the map

$$\begin{aligned} (R \times S) \times (M \times N) &\longrightarrow M \times N \\ ((r, s), (m, n)) &\longmapsto (r \cdot m, s \cdot n) \end{aligned}$$

makes the product group $M \times N$ into a module over the product ring $R \times S$.

15. Let k be a field, and let G be a group, and consider the group algebra

$$k[G] = \left\{ \sum_{g \in G} c_g g \;\middle|\; c_g \in k, c_g = 0 \text{ for all but finitely many } g \right\}$$

with the multiplication as defined in the lecture. Show that $k[G]$ is commutative if and only if G is Abelian.