Definition. A group $G$ is solvable (Dutch: oplosbaar) if there exists a chain
\[ G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\} \]
of subgroups of $G$ such that for $0 \leq i < n$, the subgroup $G_{i+1}$ is normal in $G_i$ and the quotient group $G_i/G_{i+1}$ is Abelian.

1. Let $G$ be a group. The derived series of $G$ is the chain of subgroups of $G$ defined by
\[ G = G_0 \supset G_1 \supset G_2 \supset \cdots \]
where $G_{i+1} = [G_i, G_i]$ for all $i \geq 0$. Show that $G$ is solvable if and only if there exists $n \geq 0$ such that $G_n = \{1\}$.

2. Let $G$ be a finite group. Show that $G$ is solvable if and only if there exists a chain
\[ G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\} \]
of subgroups of $G$ such that for $0 \leq i < n$, the subgroup $G_{i+1}$ is normal in $G_i$ and the quotient group $G_i/G_{i+1}$ is cyclic of prime order.

3. (a) Show that every subgroup of a solvable group is solvable.
(b) Show that every quotient of a solvable group by a normal subgroup is solvable.

4. For every $n \geq 1$, the dihedral group $D_n$ of order $2n$ is defined using generators and relations by
\[ D_n = \langle \rho, \sigma \mid \rho^n, \sigma^2, (\sigma \rho)^2 \rangle. \]
Show that $D_n$ is solvable.

5. Let $G$ be the symmetry group (of order 48) of the 3-dimensional cube. Show that $G$ is solvable by giving a chain of subgroups as in the definition of solvability. (Hint: use the action of $G$ on the set of four lines passing through two opposite vertices.)

Definition. Let $A$ be a commutative ring. An $A$-algebra is a (not necessarily commutative) ring $R$ together with a ring homomorphism $i: A \to Z(R)$. Here $Z(R)$ is the centre of $R$, defined by $Z(R) = \{r \in R \mid \forall s \in R: rs = sr\}$.

Definition. Let $R$ be a ring. A (left) $R$-module is an Abelian group $M$ together with a map
\[ R \times M \longrightarrow M \]
\[ (r, m) \longmapsto r \cdot m \]
satisfying the following identities for all $r, s \in R$ and $m, n \in M$:
\[ r \cdot (m + n) = r \cdot m + r \cdot n \quad (rs) \cdot m = r \cdot (s \cdot m) \]
\[ (r + s) \cdot m = r \cdot m + s \cdot m \quad 1 \cdot m = m. \]
6. Let $M$ be an Abelian group. Show that there is exactly one map $\mathbb{Z} \times M \to M$ with the property that it makes $M$ into a $\mathbb{Z}$-module.

7. Let $R$ be a ring. Show that the multiplication map $R \times R \to R$ makes $R$ into a left $R$-module.

8. Let $M$ be an Abelian group. Consider the set
   
   $\text{End} M = \{ f: M \to M \text{ group homomorphism} \}$

   equipped with addition and multiplication maps defined by $(f + g)(m) = f(m) + g(m)$ and $fg = f \circ g$ for $f, g \in \text{End} M$ and $m \in M$.

   (a) Show that $\text{End} M$ is a ring.

   (b) Show that $M$ is in a natural way a module over $\text{End} M$.

9. Let $R$ be a ring, and let $M$ be an Abelian group. Show that giving an $R$-module structure on $M$ is equivalent to giving a ring homomorphism $R \to \text{End} M$.

10. Let $k$ be a field, and let $n$ be a non-negative integer. Show that $k^n$ is in a natural way a module over the matrix algebra $\text{Mat}_n(k)$.

11. Let $R$ be a ring, and let $M$ be an $R$-module. Consider the set

   $\text{End}_R M = \{ f \in \text{End} M \mid f(r \cdot m) = r \cdot f(m) \text{ for all } r \in R \}$

   Show that $\text{End}_R M$ is a subring of $\text{End} M$.

12. Let $\phi: R \to S$ be a ring homomorphism, and let $N$ be an $S$-module. We write $\phi^* N$ for the Abelian group $N$ equipped with the map

   $R \times N \to N$

   $(r, m) \mapsto \phi(r) \cdot m$.

   Show that $\phi^* N$ is an $R$-module.

13. Let $A$ be a commutative ring, let $R$ be an $A$-algebra, let $i: A \to R$ be the corresponding ring homomorphism (with image in $\mathbb{Z}(R) \subseteq R$), and let $M$ be an $R$-module. Let $i^* M$ be the $A$-module defined in Exercise 12. Show that the $R$-module structure on $M$ gives a natural ring homomorphism

   $R \to \text{End}_A(i^* M)$.

14. Let $R$ and $S$ be two rings, let $M$ be an $R$-module, and let $N$ be an $S$-module. Show that the map

   $(R \times S) \times (M \times N) \to M \times N$

   $((r, s), (m, n)) \mapsto (r \cdot m, s \cdot n)$

   makes the product group $M \times N$ into a module over the product ring $R \times S$.

15. Let $k$ be a field, and let $G$ be a group, and consider the group algebra

   $k[G] = \left\{ \sum_{g \in G} c_g g \mid c_g \in k, c_g = 0 \text{ for all but finitely many } g \right\}$

   with the multiplication as defined in the lecture. Show that $k[G]$ is commutative if and only if $G$ is Abelian.