Representation Theory of Finite Groups, spring 2019

Problem Sheet 10

29 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

1. Let $A \subset B$ be commutative rings such that B is finitely generated as an A-module, and let M be a finitely generated B-module. Show that M is finitely generated as an A-module.

2. Let $f\colon A \to B$ and $g\colon B \to C$ be homomorphisms of commutative rings. Let $b \in B$ be an element that is integral over $f(A)$. Show that $g(b)$ is integral over $g(f(A))$. (This shows that integrality is preserved under ring homomorphisms.)

Theorem (Cayley–Hamilton; Frobenius). Let A be a commutative ring, let n be a non-negative integer, and let M be an $n \times n$-matrix over A. Let $f = \det(tI - M) \in A[t]$ be the characteristic polynomial of A. Then we have $f(M) = 0$ in $\mathrm{Mat}_n(A)$.

3. The purpose of this exercise is to show that the Cayley–Hamilton theorem (CH) over an arbitrary commutative ring A follows from CH over $\mathbf{C}$ (where it is a well-known result, which can be proved for example using the Jordan normal form).

(a) Suppose that CH holds for $n \times n$-matrices over the polynomial ring $\mathbf{Z}[x_{i,j} \mid 1 \le i, j \le n]$ in n^2 variables over $\mathbf{Z}$. Show that CH holds for $n \times n$-matrices over any commutative ring A.

(b) Suppose that CH holds for $n \times n$-matrices over $\mathbf{C}$. Show that CH holds for $n \times n$-matrices over $\mathbf{Z}[x_{i,j} \mid 1 \le i, j \le n]$. (*Hint:* $\mathbf{C}$ contains infinitely many elements that are algebraically independent over $\mathbf{Q}$.)

4. Let $\alpha \in \mathbf{C}$ be algebraic over $\mathbf{Q}$. Show that α is integral over $\mathbf{Z}$ if and only if the minimal polynomial of α over $\mathbf{Q}$ has integral coefficients.

5. Let G be a finite group, and let $e = \frac{1}{\#G} \sum_{g \in G} g \in \mathbf{C}[G]$. Show that e lies in $Z(\mathbf{C}[G])$ and is integral over $\mathbf{Z}$.

6. Let $d \notin \{0, 1\}$ be a square-free integer. Determine the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. (*Hint:* the answer will depend on the residue class of d modulo 4.)

7. Let A be a commutative ring, let B and B' be two commutative rings containing A, let $\overline{A}$ be the integral closure of A in B, and let $\overline{A}'$ be the integral closure of A in B'. We view A as a subring of $B \times B'$ via the map $a \mapsto (a, a)$. Show that the integral closure of A in $B \times B'$ equals $\overline{A} \times \overline{A}'$.

8. (a) Give an explicit $\mathbf{C}$-algebra isomorphism $Z(\mathbf{C}[S_3]) \xrightarrow{\sim} \mathbf{C} \times \mathbf{C} \times \mathbf{C}$.

(b) Show that the integral closure of $\mathbf{Z}$ in $Z(\mathbf{C}[S_3])$ is isomorphic to $\overline{\mathbf{Z}} \times \overline{\mathbf{Z}} \times \overline{\mathbf{Z}}$ as a $\overline{\mathbf{Z}}$-algebra, and give a $\overline{\mathbf{Z}}$-basis for this integral closure as a $\overline{\mathbf{Z}}$-submodule of $Z(\mathbf{C}[S_3])$.

9. (a) Let V be a vector space over $\mathbf{C}$, and let $\phi\colon V \to V$ be an automorphism satisfying $\phi^k = \mathrm{id}_V$ for some $k \geq 1$. Show that all eigenvalues of ϕ are roots of unity of order dividing k.

(b) Let G be a finite group, and let χ be the character of a representation of G of finite dimension n. Show that for all $g \in G$, the complex number $\chi(g)$ is a sum of n roots of unity of order dividing $\#G$. (This fact was used without proof in the lecture.)

10. Let G be a finite group containing a conjugacy class C satisfying $\#C = p^k$ with p a prime number and $k \geq 1$. Is G necessarily solvable? Give a proof or a counterexample.

11. Show that the alternating group A_5 of order $5!/2 = 60 = 2^2 \cdot 3 \cdot 5$ is simple, i.e. has exactly two normal subgroups. (*Hint:* a subgroup H of a group G is normal if and only if H is a union of conjugacy classes of G.)