Representation Theory of Finite Groups, spring 2019

Problem Sheet 11

6 May

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers unless otherwise mentioned.

1. Let V be a finite-dimensional $\mathbf{C}$-vector space, and let $g\colon V \to V$ be a $\mathbf{C}$-linear map such that $g^n = \mathrm{id}_V$ for some $n \geq 1$. Show that g is diagonalisable. (*Hint:* use the Jordan canonical form.)

2. Let $z = \sqrt{5} + 1 \in \mathbf{C}$. Show that z is an algebraic integer with $|z| > 2$ and that in $\bar{\mathbf{Z}}$ we have both $2 \mid z$ and $z \mid 2$.

(In particular, this shows that if z is an algebraic integer and n is a positive integer with $z \mid n$, it does not necessarily follow that $|z| \leq n$.)

3. Let G be a finite group, and let V be a $\mathbf{C}[G]$-module. We say that an element $g \in G$ *acts as a scalar on* V if there exists $\lambda \in \mathbf{C}$ such that $gv = \lambda v$ for all $v \in V$.

(a) Show that the set of elements of G that act as a scalar on V is a normal subgroup of G.

(b) Assume that V is irreducible. Show that all elements of G act as a scalar on V if and only if V is one-dimensional.

4. Determine all pairs (V, C) where V is an irreducible representation of S_4 (up to isomorphism) and $C \subset S_4$ is a conjugacy class such that the elements of C act as a scalar on V.

5. Let G be a finite group, and let $\rho\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ be a finite-dimensional representation of G.

(a) Show that there exists a $\mathbf{C}$-basis of V such that for every element $g \in G$, the matrix of g with respect to this basis has coefficients in the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. (*Hint:* consider the irreducible representations of G over $\overline{\mathbf{Q}}$.)

(b) Show that there exists a finite Galois extension K of $\mathbf{Q}$ contained in $\mathbf{C}$ such that for every element $g \in G$, the matrix of g with respect to a basis as in (a) has coefficients in K.

6. Let G be a finite group, let $\rho\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ be an irreducible representation of G with $\dim_{\mathbf{C}} V > 1$, and let $\chi\colon G \to \mathbf{C}$ be its character.

(a) Let $M = \frac{1}{\#G-1} \sum_{g \in G \setminus \{1\}} |\chi(g)|^2$. Show that $|M| < 1$.

(b) Let K be a number field as in Exercise 5(b), and let $P = \prod_{g \in G \setminus \{1\}} \chi(g) \in K$. Show that for every $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$, we have $|\sigma(P)| < 1$. (*Hint:* consider the "conjugated" representation of G obtained by applying σ to the entries of the matrices of the automorphisms $\rho(g)$ with respect to a basis as in Exercise 5(b).)

(c) Deduce that there exists $g \in G$ such that $\chi(g) = 0$.

7. Let G be the dihedral group D_n with $n \geq 3$ odd, and let X be the set of vertices of the regular n-gon with the standard action of G on X.

(a) Show that every element of $G \setminus \{1\}$ has at most one fixed point in X.

(b) Show (without using Frobenius's theorem) that the elements of G having no fixed points in X, together with the identity element, form a normal subgroup of G.

8. Let n be a positive integer. Suppose that there exists a transitive S_n-set X such that $1 < \#X < n!$ and every element of $S_n \setminus \{1\}$ has at most one fixed point in X. Prove that n equals 3. (*Hint:* use Frobenius's theorem and the fact that A_n is the only non-trivial normal subgroup of S_n if $n \geq 5$.)