Throughout this problem sheet, representations and characters are taken to be over the field $\mathbb{C}$ of complex numbers unless otherwise mentioned.

1. Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, and let $g: V \to V$ be a $\mathbb{C}$-linear map such that $g^n = \text{id}_V$ for some $n \geq 1$. Show that $g$ is diagonalisable. (Hint: use the Jordan canonical form.)

2. Let $z = \sqrt{5} + 1 \in \mathbb{C}$. Show that $z$ is an algebraic integer with $|z| > 2$ and that in $\mathbb{Z}$ we have both $2 | z$ and $z | 2$.
   (In particular, this shows that if $z$ is an algebraic integer and $n$ is a positive integer with $z | n$, it does not necessarily follow that $|z| \leq n$.)

3. Let $G$ be a finite group, and let $V$ be a $\mathbb{C}[G]$-module. We say that an element $g \in G$ acts as a scalar on $V$ if there exists $\lambda \in \mathbb{C}$ such that $gv = \lambda v$ for all $v \in V$.
   (a) Show that the set of elements of $G$ that act as a scalar on $V$ is a normal subgroup of $G$.
   (b) Assume that $V$ is irreducible. Show that all elements of $G$ act as a scalar on $V$ if and only if $V$ is one-dimensional.

4. Determine all pairs $(V, C)$ where $V$ is an irreducible representation of $S_4$ (up to isomorphism) and $C \subset S_4$ is a conjugacy class such that the elements of $C$ act as a scalar on $V$.

5. Let $G$ be a finite group, and let $\rho: G \to \text{Aut}_\mathbb{C} V$ be a finite-dimensional representation of $G$.
   (a) Show that there exists a $\mathbb{C}$-basis of $V$ such that for every element $g \in G$, the matrix of $g$ with respect to this basis has coefficients in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. (Hint: consider the irreducible representations of $G$ over $\overline{\mathbb{Q}}$.)
   (b) Show that there exists a finite Galois extension $\mathbb{K}$ of $\mathbb{Q}$ contained in $\mathbb{C}$ such that for every element $g \in G$, the matrix of $g$ with respect to a basis as in (a) has coefficients in $\mathbb{K}$.

6. Let $G$ be a finite group, let $\rho: G \to \text{Aut}_\mathbb{C} V$ be an irreducible representation of $G$ with $\dim_{\mathbb{C}} V > 1$, and let $\chi: G \to \mathbb{C}$ be its character.
   (a) Let $M = \frac{1}{\#G} \sum_{g \in G \setminus \{1\}} |\chi(g)|^2$. Show that $|M| < 1$.
   (b) Let $K$ be a number field as in Exercise 5(b), and let $P = \prod_{g \in G \setminus \{1\}} \chi(g) \in K$. Show that for every $\sigma \in \text{Gal}(K/\mathbb{Q})$, we have $|\sigma(P)| < 1$. (Hint: consider the “conjugated” representation of $G$ obtained by applying $\sigma$ to the entries of the matrices of the automorphisms $\rho(g)$ with respect to a basis as in Exercise 5(b).)
   (c) Deduce that there exists $g \in G$ such that $\chi(g) = 0$. 

\[1\]
7. Let $G$ be the dihedral group $D_n$ with $n \geq 3$ odd, and let $X$ be the set of vertices of the regular $n$-gon with the standard action of $G$ on $X$.

(a) Show that every element of $G \setminus \{1\}$ has at most one fixed point in $X$.

(b) Show (without using Frobenius’s theorem) that the elements of $G$ having no fixed points in $X$, together with the identity element, form a normal subgroup of $G$.

8. Let $n$ be a positive integer. Suppose that there exists a transitive $S_n$-set $X$ such that $1 < \#X < n!$ and every element of $S_n \setminus \{1\}$ has at most one fixed point in $X$. Prove that $n$ equals 3. (Hint: use Frobenius’s theorem and the fact that $A_n$ is the only non-trivial normal subgroup of $S_n$ if $n \geq 5$.)