Representation Theory of Finite Groups, spring 2019

Problem Sheet 2

11 Februari

In the following exercises, "module" always means "left module".

1. Let A be a commutative ring, let R be an A-algebra, and let M be an Abelian group. Show that giving an R-module structure on M is equivalent to giving an A-module structure on M together with an A-algebra homomorphism $R \to \operatorname{End}_A(M)$.

2. Let k be a field, let G be a group, and let R be a k-algebra. Show that there is a natural bijection between the set of k-algebra homomorphisms $k[G] \to R$ and the set of group homomorphisms $G \to R^\times$.

3. Let R be a ring.

(a) Consider two exact sequences

$$\begin{aligned} L \longrightarrow M \longrightarrow N \longrightarrow 0, \\ 0 \longrightarrow N \longrightarrow P \longrightarrow Q \end{aligned} \tag{1}$$

of R-modules (note that N occurs twice). Show that there is a natural exact sequence

$$L \longrightarrow M \longrightarrow P \longrightarrow Q \tag{2}$$

of R-modules.

(b) Conversely, given an exact sequence of the form (2), give an R-module N and two exact sequences of the form (1).

4. Let R be a ring, and consider a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

Show that the following three statements are equivalent:

(1) there exists an R-linear map $r\colon M \to L$ satisfying $r \circ f = \mathrm{id}_L$;

(2) there exists an R-linear map $s\colon N \to M$ satisfying $g \circ s = \mathrm{id}_N$;

(3) there exists an isomorphism $h\colon M \xrightarrow{\sim} L \oplus N$ of R-modules such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_L}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

is commutative, where the R-linear maps i and p are defined by $i(l) = (l, 0)$ and $p(l, n) = n$.

Definition. A short exact sequence of R-modules is *split* if the equivalent conditions of Exercise 4 hold.

Definition. Let R be a ring. An R-module M is *simple* if M has exactly two R-submodules.

5. Show that simple modules over a field k are the same as 1-dimensional k-vector spaces.

Definition. Let R be a ring. A *left ideal* of R is an R-submodule of R, where R is viewed as left module over itself. A left ideal $I \subset R$ is *maximal* if there are exactly two left ideals $J \subset R$ with $I \subset J$.

6. Let R be a ring, and let M be an R-module. Show that M is simple if and only if M is isomorphic to an R-module of the form R/I with I a maximal left ideal of R.

7. Let R be a ring, and let M be a simple R-module. Show that every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of R-modules is split.

8. Let R be a ring. Show that R is simple as an R-module if and only if R is a division ring (i.e. $R \neq 0$ and every non-zero element of R is invertible).

9. Let k be a field, and let G be a finite group. Show that every simple $k[G]$-module is finite-dimensional as a k-vector space.

10. Let k be a field, let n be a positive integer, and let R be the k-algebra $\mathrm{Mat}_n(k)$. We view k^n as a module over R in the usual way; cf. Exercise 10 of problem sheet 1.

(a) Show that k^n is a simple R-module.

(b) Describe a maximal left ideal $I \subset R$ such that k^n is isomorphic to R/I as an R-module.

Definition. Let R be a ring. An R-module P is *projective* if for every R-module M and every surjective R-linear map $p\colon M \to P$, there exists an R-linear map $s\colon P \to M$ satisfying $p \circ s = \mathrm{id}_P$.

Definition. Let R be a ring. An R-module I is *injective* if for every R-module M and every injective R-linear map $i\colon I \to M$, there exists an R-linear map $r\colon M \to I$ satisfying $r \circ i = \mathrm{id}_I$.

11. Let R be a ring, and let P be an R-module. Show that P is projective if and only if for every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle h} & & \\ N' & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}$$

of R-modules and R-linear maps in which the bottom row is exact, there exists an R-linear map $h'\colon P \to N'$ satisfying $q \circ h' = h$.

12. Formulate and prove an analogue of Exercise 11 for injective modules.