Problem Sheet 3

18 February

In the following exercises, “module” always means “left module”.

1. Let $k$ be a field, let $k[x]$ be the polynomial ring in one variable over $k$, let $V$ be a $k$-vector space, and let $f : V \to V$ be a $k$-linear map.
   
   (a) Show that the $k$-vector space structure on $V$ can be extended to a $k[x]$-module structure (in other words, that there is a $k$-linear representation of $k[x]$ of $V$) in a unique way such that for all $v \in V$ we have $xv = f(v)$.
   
   (b) Show that the ring $\text{End}_{k[x]}(V)$ consists of all $k$-linear maps $g : V \to V$ satisfying $g \circ f = f \circ g$.

2. Let $k$ be a field, let $n$ be a non-negative integer, let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $k$, and let $V$ be a $k$-vector space. Show that giving a $k$-linear representation of $R$ on $V$ is equivalent to giving $k$-linear maps $f_1, \ldots, f_n : V \to V$ satisfying $f_i \circ f_j = f_j \circ f_i$ for all $i, j$.

3. Let $k$ be a field, and let $V$ be a $k$-vector space. Show that giving a $k$-linear representation of $k[x, 1/x]$ on $V$ is equivalent to giving an invertible $k$-linear map $V \to V$.

**Definition.** A division ring is a ring $D$ for which the unit group $D^\times$ equals $D \setminus \{0\}$. (In particular, the zero ring is not a division ring.)

4. Let $R$ be a ring.
   
   (a) Let $M$ be a simple $R$-module. Show that the ring $\text{End}_R(M)$ is a division ring.
   
   (b) Let $M$ and $N$ be two simple $R$-modules. Show that the group $\text{Hom}_R(M, N)$ of $R$-linear maps $M \to N$ is non-zero if and only if $M$ and $N$ are isomorphic.

5. Let $R$ be a ring, and let $(M_i)_{i \in I}$ be a family of $R$-modules indexed by a set $I$.
   
   (a) For each $i \in I$, let $p_i : \prod_{j \in I} M_j \to M_i$ be the projection onto the $i$-th factor, i.e. the $R$-linear map defined by $p_i((m_j)_{j \in I}) = m_i$. Let $N$ be an $R$-module, and for every $i \in I$ let $f_i : N \to M_i$ be an $R$-linear map. Show that there exists a unique $R$-linear map $f : N \to \prod_{i \in I} M_i$ such that for every $i \in I$ we have $p_i \circ f = f_i$.
   
   (b) For each $i \in I$, let $h_i : M_i \to \bigoplus_{j \in I} M_j$ be the inclusion into the $i$-th summand, i.e. the $R$-linear map defined by $h_i(m) = (m_j)_{j \in I}$, where $m_i = m$ and $m_j = 0 \in M_j$ for $j \neq i$. Let $N$ be an $R$-module, and for every $i \in I$ let $g_i : M_i \to N$ be an $R$-linear map. Show that there exists a unique $R$-linear map $g : \bigoplus_{i \in I} M_i \to N$ such that for every $i \in I$ we have $g \circ h_i = g_i$.
   
   (c) Conclude that for every $R$-module $N$, there are natural bijections
   
   $\text{Hom}_R\left(\bigoplus_{i \in I} M_i, N\right) \xrightarrow{\sim} \prod_{i \in I} \text{Hom}_R(M_i, N),$
   
   $\text{Hom}_R\left(N, \prod_{i \in I} M_i\right) \xrightarrow{\sim} \prod_{i \in I} \text{Hom}_R(N, M_i).$
6. Let $R$ be a ring, and let $M$ be an $R$-module. Show that $M$ is semi-simple if and only if for every submodule $L \subset M$ there exists a submodule $N \subset M$ such that $L + N = M$ and $L \cap N = 0$.

7. Let $R$ be a ring, and let $M$ be a product of simple $R$-modules. Is $M$ necessarily semi-simple? Give a proof or a counterexample.

8. Take $k = \mathbb{C}$, and let $V$ and $f$ be as in Exercise 1. Assume that $V$ is finite-dimensional over $\mathbb{C}$.
   (a) Show that $V$ is simple as a $\mathbb{C}[x]$-module if and only if $V$ is one-dimensional over $\mathbb{C}$.
   (b) Show that $V$ is semi-simple as a $\mathbb{C}[x]$-module if and only if $f$ is diagonalisable.

9. Let $k$ be a field, and let $S_3$ be the symmetric group on $\{1, 2, 3\}$.
   (a) Show that there is a unique $k$-linear representation of $S_3$ on $k^2$ such that the permutations $(1 2)$ and $(1 3)$ act as the matrices $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, respectively.
   (b) Show that the representation constructed in (a) makes $k^2$ into a simple $k[S_3]$-module. (Be careful to take into account all possible characteristics of $k$.)

10. Let $n$ be a positive integer, and let $C_n$ be a cyclic group of order $n$. Show that there are exactly $n$ simple $\mathbb{C}[C_n]$-modules up to isomorphism, and that these are all one-dimensional as $\mathbb{C}$-vector spaces.

11. Let $k$ be a field, let $n$ be a positive integer, let $R$ be the matrix algebra $\text{Mat}_n(k)$, and let $V = k^n$ viewed as a left $R$-module in the usual way. Recall that $V$ is simple (see problem 10 of problem sheet 2).
   (a) Show that $R$, viewed as a left module over itself, is isomorphic to a direct sum of $n$ copies of $V$.
   (b) Show that every simple $R$-module is isomorphic to $V$.

12. Let $G$ be a group, let $H$ and $H'$ be two subgroups of $G$, and let $N \triangleleft H$ and $N' \triangleleft H'$ be normal subgroups of $H$ and $H'$, respectively.
   (a) Show that $N(H \cap N')$ is normal in $N(H \cap H')$, that $(N \cap H')N'$ is normal in $(H \cap H')N'$, and that $(H \cap N')(N \cap H')$ is normal in $H \cap H'$.
   (b) Show that there are canonical isomorphisms
   \[
   \frac{N(H \cap H')}{N(H \cap N')} \cong \frac{H \cap H'}{(N \cap H')(H \cap N')} \cong \frac{(H \cap H')N'}{(N \cap H')N'}
   \]
   (This is Zassenhaus’s butterfly lemma for groups.)