Representation Theory of Finite Groups, spring 2019

Problem Sheet 3

18 Februari

In the following exercises, "module" always means "left module".

1. Let k be a field, let $k[x]$ be the polynomial ring in one variable over k, let V be a k-vector space, and let $f\colon V \to V$ be a k-linear map.

(a) Show that the k-vector space structure on V can be extended to a $k[x]$-module structure (in other words, that there is a k-linear representation of $k[x]$ of V) in a unique way such that for all $v \in V$ we have $xv = f(v)$.

(b) Show that the ring $\mathrm{End}_{k[x]}(V)$ consists of all k-linear maps $g\colon V \to V$ satisfying $g \circ f = f \circ g$.

2. Let k be a field, let n be a non-negative integer, let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, and let V be a k-vector space. Show that giving a k-linear representation of R on V is equivalent to giving k-linear maps $f_1, \ldots, f_n\colon V \to V$ satisfying $f_i \circ f_j = f_j \circ f_i$ for all i, j.

3. Let k be a field, and let V be a k-vector space. Show that giving a k-linear representation of $k[x, 1/x]$ on V is equivalent to giving an invertible k-linear map $V \to V$.

Definition. A *division ring* is a ring D for which the unit group $D^\times$ equals $D \setminus \{0\}$. (In particular, the zero ring is not a division ring.)

4. Let R be a ring.

(a) Let M be a simple R-module. Show that the ring $\mathrm{End}_R(M)$ is a division ring.

(b) Let M and N be two simple R-modules. Show that the group $\mathrm{Hom}_R(M, N)$ of R-linear maps $M \to N$ is non-zero if and only if M and N are isomorphic.

5. Let R be a ring, and let $(M_i)_{i\in I}$ be a family of R-modules indexed by a set I.

(a) For each $i \in I$, let $p_i\colon \prod_{j\in I} M_j \to M_i$ be the projection onto the i-th factor, i.e. the R-linear map defined by $p_i((m_j)_{j\in I}) = m_i$. Let N be an R-module, and for every $i \in I$ let $f_i\colon N \to M_i$ be an R-linear map. Show that there exists a unique R-linear map $f\colon N \to \prod_{i\in I} M_i$ such that for every $i \in I$ we have $p_i \circ f = f_i$.

(b) For each $i \in I$, let $h_i\colon M_i \to \bigoplus_{j\in I} M_j$ be the inclusion into the i-th summand, i.e. the R-linear map defined by $h_i(m) = (m_j)_{j\in I}$, where $m_i = m$ and $m_j = 0 \in M_j$ for $j \ne i$. Let N be an R-module, and for every $i \in I$ let $g_i\colon M_i \to N$ be an R-linear map. Show that there exists a unique R-linear map $g\colon \bigoplus_{i\in I} M_i \to N$ such that for every $i \in I$ we have $g \circ h_i = g_i$.

(c) Conclude that for every R-module N, there are natural bijections

$$\mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, N\Big) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_R(M_i, N),$$

$$\mathrm{Hom}_R\Big(N, \prod_{i\in I} M_i\Big) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

6. Let R be a ring, and let M be an R-module. Show that M is semi-simple if and only if for every submodule $L \subset M$ there exists a submodule $N \subset M$ such that $L+N = M$ and $L \cap N = 0$.

7. Let R be a ring, and let M be a product of simple R-modules. Is M necessarily semi-simple? Give a proof or a counterexample.

8. Take $k = \mathbf{C}$, and let V and f be as in Exercise 1. Assume that V is finite-dimensional over $\mathbf{C}$.

(a) Show that V is simple as a $\mathbf{C}[x]$-module if and only if V is one-dimensional over $\mathbf{C}$.

(b) Show that V is semi-simple as a $\mathbf{C}[x]$-module if and only if f is diagonalisable.

9. Let k be a field, and let S_3 be the symmetric group on $\{1, 2, 3\}$.

(a) Show that there is a unique k-linear representation of S_3 on k^2 such that the permutations $(1\ 2)$ and $(1\ 3)$ act as the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, respectively.

(b) Show that the representation constructed in (a) makes k^2 into a simple $k[S_3]$-module. (Be careful to take into account all possible characteristics of k.)

10. Let n be a positive integer, and let C_n be a cyclic group of order n. Show that there are exactly n simple $\mathbf{C}[C_n]$-modules up to isomorphism, and that these are all one-dimensional as $\mathbf{C}$-vector spaces.

11. Let k be a field, let n be a positive integer, let R be the matrix algebra $\mathrm{Mat}_n(k)$, and let $V = k^n$ viewed as a left R-module in the usual way. Recall that V is simple (see problem 10 of problem sheet 2).

(a) Show that R, viewed as a left module over itself, is isomorphic to a direct sum of n copies of V.

(b) Show that every simple R-module is isomorphic to V.

12. Let G be a group, let H and H' be two subgroups of G, and let $N \triangleleft H$ and $N' \triangleleft H'$ be normal subgroups of H and H', respectively.

(a) Show that $N(H \cap N')$ is normal in $N(H \cap H')$, that $(N \cap H')N'$ is normal in $(H \cap H')N'$, and that $(H \cap N')(N \cap H')$ is normal in $H \cap H'$.

(b) Show that there are canonical isomorphisms

$$\frac{N(H \cap H')}{N(H \cap N')} \xleftarrow{\sim} \frac{H \cap H'}{(N \cap H')(H \cap N')} \xrightarrow{\sim} \frac{(H \cap H')N'}{(N \cap H')N'}$$

(This is Zassenhaus's butterfly lemma for groups.)