Representation Theory of Finite Groups, spring 2019

Problem Sheet 5

4 March

1. Let $\mathcal{C}$ be a category equipped with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all objects X and Y of $\mathcal{C}$, such that composition of morphisms is bilinear. Let X be an object of $\mathcal{C}$.

(a) Show that the Abelian group $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X,X)$ has a natural ring structure with composition as multiplication.

(b) Show that X is a zero object in $\mathcal{C}$ if and only if $\mathrm{End}_{\mathcal{C}}(X)$ is the zero ring.

2. Let $\mathcal{C}$ be a category equipped with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all objects X and Y of $\mathcal{C}$, such that composition of morphisms is bilinear. Suppose that X and Y are objects of $\mathcal{C}$ and (S,i,j) is a sum of X and Y.

(a) Show that there are unique morphisms $p\colon S \to X$ and $q\colon S \to Y$ satisfying $p \circ i = \mathrm{id}_X$, $p \circ j = 0$, $q \circ i = 0$ and $q \circ j = \mathrm{id}_Y$.

(b) Show that the morphism $i \circ p + j \circ q \in \mathrm{End}_{\mathcal{C}}(S)$ equals id_S.

(c) Show that (S,p,q) is a product of X and Y in $\mathcal{C}$.

Definition. An *Abelian category* is a category $\mathcal{A}$, together with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ for all objects X and Y of $\mathcal{A}$, such that the following conditions are satisfied:

(1) Composition of morphisms is bilinear.

(2) There is a zero object in $\mathcal{A}$.

(3) For all objects X and Y of $\mathcal{A}$, there is an object S of $\mathcal{A}$ together with morphisms $i\colon X \to S$, $j\colon Y \to S$, $p\colon S \to X$ and $q\colon S \to Y$ such that (S,i,j) is a sum of X and Y and (S,p,q) is a product of X and Y.

(4) Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

(5) For every morphism $f\colon X \to Y$ in A, let $i\colon \ker f \to X$ and $p\colon Y \to \operatorname{coker} f$ be the kernel and cokernel of f. Then the unique morphism $\bar{f}\colon \operatorname{coker} i \to \ker p$ making the diagram

$$\begin{array}{ccccccc} \ker f & \xrightarrow{i} & X & \xrightarrow{f} & Y & \xrightarrow{p} & \operatorname{coker} f \\ & & q\downarrow & & \uparrow j & & \\ \operatorname{coim} f := & & \operatorname{coker} i & \xrightarrow{\bar{f}} & \ker p =: \operatorname{im} f & & \end{array}$$

commutative (the existence and uniqueness of $\bar{f}$ was proved in the lecture) is an isomorphism.

3. Let $\mathcal{A}$ be an Abelian category, and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Show that f is an isomorphism if and only if $0 \to X$ is a kernel of f and $Y \to 0$ is a cokernel of f.

4. Let $\mathcal{A}$ be an Abelian category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of two morphisms in $\mathcal{A}$ satisfying $g \circ f = 0$. Let $p\colon Y \to \operatorname{coker} f$ be the cokernel of f, let $i\colon \ker g \to Y$ be the kernel of g, and let $j\colon \operatorname{im} f = \ker p \to Y$ be the image of f, which is defined as the kernel of p. Show that there is a unique morphism $h\colon \operatorname{im} f \to \ker g$ satisfying $i \circ h = j$.

Definition. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an Abelian category is *exact at* Y if $g \circ f = 0$ and the morphism h defined in Exercise 4 is an isomorphism. A sequence of morphisms in $\mathcal{A}$ is *exact* if it is exact at every intermediate object.

5. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R-modules. Show that this sequence is exact according to the above definition if and only if the "usual" image of f equals the "usual" kernel of g (as submodules of M).

6. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R-modules. Show that this sequence is exact if and only if it fits into a commutative diagram of R-modules and R-linear maps

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & & \downarrow & & & & & & \\
0 & \longrightarrow & J & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 & & & & \\
 & & & & & \searrow^{f} & \downarrow & & & & & & \\
 & & & & & & M & & & & & & \\
 & & & & & & \downarrow & \searrow^{g} & & & & & \\
 & & & & 0 & \longrightarrow & P & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & 0 & & & & & &
\end{array}
$$

in which the two horizontal sequences and the vertical sequence are exact.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *additive* if for all objects X, Y of $\mathcal{A}$, the map $F\colon \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ is a group homomorphism. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is

- *exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ is exact.
- *left exact* if for every exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ is exact.
- *right exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ in $\mathcal{B}$ is exact.

7. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Show that the following statements are equivalent:

(1) The functor F is exact.

(2) The functor F is both left exact and right exact.

(3) For every short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ in $\mathcal{B}$ is exact.

(*Hint:* You may use without proof that the result of Exercise 6 holds in any Abelian category.)

8. Let R be a ring, and let M be a left R-module.

(a) Show that M is projective if and only if the functor ${}_R\mathrm{Hom}(M, \ \ ): {}_R\mathbf{Mod} \to \mathbf{Ab}$ is exact.

(b) Show that M is injective if and only if the functor ${}_R\mathrm{Hom}(\ \ , M): {}_R\mathbf{Mod}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

(See Problem Sheet 2 for projective and injective modules.)

Definition. Let R be a ring, let M be a right R-module, let N be a left R-module, and let A be an Abelian group. An *R-bilinear map* $M \times N \to A$ is a map $b: M \times N \to A$ satisfiying the following identities for all $r \in R$, $m, m' \in M$, and $n, n' \in N$:

$$\begin{aligned} b(m+m', n) &= b(m, n) + b(m', n) \\ b(m, n+n') &= b(m, n) + b(m, n') \\ b(mr, n) &= b(m, rn). \end{aligned}$$

The set of all R-bilinear maps $M \times N \to A$ is denoted by $\mathrm{Bil}_R(M, N, A)$. Note that this is an Abelian group under pointwise addition, i.e.

$$(b + b')(m, n) = b(m, n) + b'(m, n).$$

9. Let R be a ring, let M be a right R-module, and let N be a left R-module. Recall (as a special case of the generalities on bimodules treated in the lecture) that the Abelian group $\mathrm{Hom}(M, A)$ of all group homomorphisms $M \to A$ is a left R-module via $(rf)(m) = f(mr)$, and that $\mathrm{Hom}(N, A)$ is a right R-module via $(fr)(n) = f(rn)$.

(a) Show that there are canonical isomorphisms

$$\mathrm{Bil}_R(M, N, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}(N, A))$$

and

$$\mathrm{Bil}_R(M, N, A) \xrightarrow{\sim} {}_R\mathrm{Hom}(N, \mathrm{Hom}(M, A))$$

of Abelian groups.

(b) Let S and T be two further rings, and suppose in addition that that M is an (S, R)-bimodule and N is an (R, T)-bimodule. Show that $\mathrm{Bil}_R(M, N, A)$ has a natural (T, S)-bimodule structure.

10. Let R be a ring, and let $\iota: R \to R$ be an anti-automorphism of R, i.e. a ring isomorphism from R to itself except that the condition $\iota(xy) = \iota(x)\iota(y)$ that would have to hold for a ring homomorphism is replaced by $\iota(xy) = \iota(y)\iota(x)$. Let M be a right R-module. Show that the map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, M) &\longmapsto m\iota(r) \end{aligned}$$

makes M into a left R-module.

11. Let k be a field, and let G be a group. Define a map

$$\begin{aligned} \iota\colon k[G] &\longrightarrow k[G] \\ \sum_{g\in G} c_g g &\longmapsto \sum_{g\in G} c_g g^{-1}. \end{aligned}$$

(a) Show that ι is an anti-automorphism of $k[G]$ (see Exercise 10) that is compatible with the k-algebra structure.

(b) Let M be a left $k[G]$-module, and let $\operatorname{Hom}_k(M, k)$ be the k-vector space of k-linear maps $M \to k$. Show that the map

$$\begin{aligned} k[G] \times \operatorname{Hom}_k(M, k) &\longrightarrow \operatorname{Hom}_k(M, k) \\ (r, f) &\longrightarrow (m \mapsto f(\iota(r)m)) \end{aligned}$$

makes $\operatorname{Hom}_k(M, k)$ into a left $k[G]$-module.

(c) Let M and N be left $k[G]$-modules, and let $\operatorname{Hom}_k(M, N)$ be the k-vector space of k-linear maps $M \to N$. Show that the map

$$\begin{aligned} G \times \operatorname{Hom}_k(M, N) &\longrightarrow \operatorname{Hom}_k(M, N) \\ (g, f) &\longmapsto (m \mapsto g(f(g^{-1}m))) \end{aligned}$$

can be extended uniquely to a left $k[G]$-module structure on $\operatorname{Hom}_k(M, N)$ in such a way that the action of k is the "usual" scalar multiplication action of k on $\operatorname{Hom}_k(M, N)$.