Representation Theory of Finite Groups, spring 2019

Problem Sheet 6

18 March

1. Let m and n be positive integers. Show that the tensor product $\mathbf{Z}/m\mathbf{Z} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ for some d, and determine d. Also describe the bilinear map $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z} \xrightarrow{\otimes} \mathbf{Z}/d\mathbf{Z}$.

2. Let M and N be $\mathbf{Z}$-modules (Abelian groups), and assume that M is a torsion group (every element has finite order) and N is a divisible group (multiplication by n on N is surjective for every positive integer n).

(a) Let A be an Abelian group, and let $b\colon M \times N \to A$ be a $\mathbf{Z}$-bilinear map. Show that b is the zero map.

(b) Deduce that $M \underset{\mathbf{Z}}{\otimes} N$ is the trivial group (and the universal bilinear map $M \times N \to M \underset{\mathbf{Z}}{\otimes} N$ is the zero map).

3. (a) Let R, S and T be three rings, let M be an (R, S)-bimodule, and let N be an (S, T)-bimodule. Show that the tensor product $M \underset{S}{\otimes} N$ has a natural (R, T)-bimodule structure.

(b) Let R and S be two rings, let L be a right R-module, let M be an (R, S)-bimodule, and let N be a left S-module. Show that there is a canonical isomorphism

$$(L \underset{R}{\otimes} M) \underset{S}{\otimes} N \xrightarrow{\sim} L \underset{R}{\otimes} (M \underset{S}{\otimes} N)$$

of Abelian groups.

4. Let A be a commutative ring, and let M and N be left A-modules. We also view M as a right A-module via $ma = am$ for $m \in M$ and $a \in A$, and similarly for N; this is possible because A is commutative. In particular, we have left A-modules $M \underset{A}{\otimes} N$ and $N \underset{A}{\otimes} M$. Show that there is a canonical isomorphism

$$M \underset{A}{\otimes} N \xrightarrow{\sim} N \underset{A}{\otimes} M$$

of left A-modules.

5. Let $\phi\colon R \to S$ be a ring homomorphism, and let M be a left R-module.

(a) Show that the Abelian group $S \underset{R}{\otimes} M$ (where S is viewed as a right R-module via $(s, r) \mapsto s\phi(r)$) has a natural left S-module structure.

(b) Let N be a left S-module, and let $\phi^* N$ be the Abelian group N viewed as a left R-module via $(r, n) \mapsto \phi(r)n$; cf. Exercise 12 of problem sheet 1. Show that there is a canonical isomorphism

$${}_S\mathrm{Hom}(S \underset{R}{\otimes} M, N) \xrightarrow{\sim} {}_R\mathrm{Hom}(M, \phi^* N)$$

of Abelian groups.

6. Let R and S be two rings, and let T be the Abelian group $T = R \underset{\mathbf{Z}}{\otimes} S$ (where R and S are viewed as $\mathbf{Z}$-modules).

(a) Show that the map

$$\begin{aligned} (R \times S) \times (R \times S) &\longrightarrow R \times S \\ ((r, s), (r', s')) &\longmapsto (rr', ss') \end{aligned}$$

induces a bilinear map $m: T \times T \to T$.

(b) Show that T has a natural ring structure, with the map m from (a) as the multiplication map.

(c) Show that there are canonical ring homomorphisms $i: R \to T$ and $j: S \to T$.

(d) Show that T, together with the maps i and j, is a sum of R and S in the category of rings.

7. Let A be a commutative ring. Formulate and prove an analogue of Exercise 6 for A-algebras.

8. Let $A \to B$ be a homomorphism of commutative rings, and let R be an A-algebra. Show that the A-algebra $B \underset{A}{\otimes} R$ has a natural B-algebra structure.

9. Let $k \to K$ be a field extension.

(a) Let n be a non-negative integer. Show that there is a canonical isomorphism

$$K \underset{k}{\otimes} \mathrm{Mat}_n(k) \xrightarrow{\sim} \mathrm{Mat}_n(K)$$

of K-algebras.

(b) Let G be a group. Show that there is a canonical isomorphism

$$K \underset{k}{\otimes} k[G] \xrightarrow{\sim} K[G]$$

of K-algebras.

10. Let $\mathbf{H}$ be the $\mathbf{R}$-algebra of Hamilton quaternions. We recall that this is the 4-dimensional $\mathbf{R}$-vector space with basis $(1, i, j, k)$, made into an $\mathbf{R}$-algebra with unit element 1 and multiplication defined on the other basis elements by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

and extended $\mathbf{R}$-bilinearly.

(a) Show that $\mathbf{H}$ is a division ring. (*Hint:* use the conjugation map $a + bi + cj + dk \mapsto a - bi - cj - dk$ for $a, b, c, d \in \mathbf{R}$.)

(b) Show that there is an isomorphism $\mathbf{C} \underset{\mathbf{R}}{\otimes} \mathbf{H} \xrightarrow{\sim} \mathrm{Mat}_2(\mathbf{C})$ of $\mathbf{C}$-algebras.

11. Let R be a ring that is semi-simple as a left module over itself, so there is a family $(M_i)_{i \in I}$ of simple R-modules such that R is isomorphic to $\bigoplus_{i \in I} M_i$ as an R-module.

(a) Show that the set I is finite. (*Hint:* write $1 \in R$ as a sum of elements of the M_i.)

(b) Show that every simple R-module is isomorphic to one of the M_i.

12. Let R and S be two semi-simple rings. Show, using the definition of semi-simple rings, that the product ring $R \times S$ is also semi-simple. (Do not use the classification of semi-simple rings; this has not yet been proved in the lecture.)