1. Let \( m \) and \( n \) be positive integers. Show that the tensor product \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \) for some \( d \), and determine \( d \). Also describe the bilinear map \( \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/d\mathbb{Z} \).

2. Let \( M \) and \( N \) be \( \mathbb{Z} \)-modules (Abelian groups), and assume that \( M \) is a torsion group (every element has finite order) and \( N \) is a divisible group (multiplication by \( n \) on \( N \) is surjective for every positive integer \( n \)).
   
   (a) Let \( A \) be an Abelian group, and let \( b: M \times N \rightarrow A \) be a \( \mathbb{Z} \)-bilinear map. Show that \( b \) is the zero map.
   
   (b) Deduce that \( M \otimes N \) is the trivial group (and the universal bilinear map \( M \times N \rightarrow M \otimes N \) is the zero map).

3. (a) Let \( R \), \( S \) and \( T \) be three rings, let \( M \) be an \( (R,S) \)-bimodule, and let \( N \) be an \( (S,T) \)-bimodule. Show that the tensor product \( M \otimes_{S} N \) has a natural \( (R,T) \)-bimodule structure.
   
   (b) Let \( R \) and \( S \) be two rings, let \( L \) be a right \( R \)-module, let \( M \) be an \( (R,S) \)-bimodule, and let \( N \) be a left \( S \)-module. Show that there is a canonical isomorphism
   \[
   (L \otimes_{R} M) \otimes_{S} N \sim L \otimes_{R} (M \otimes_{S} N)
   \]
   of Abelian groups.

4. Let \( A \) be a commutative ring, and let \( M \) and \( N \) be left \( A \)-modules. We also view \( M \) as a right \( A \)-module via \( ma = am \) for \( m \in M \) and \( a \in A \), and similarly for \( N \); this is possible because \( A \) is commutative. In particular, we have left \( A \)-modules \( M \otimes_{A} N \) and \( N \otimes_{A} M \). Show that there is a canonical isomorphism
   \[
   M \otimes_{A} N \sim N \otimes_{A} M
   \]
   of left \( A \)-modules.

5. Let \( \phi: R \rightarrow S \) be a ring homomorphism, and let \( M \) be a left \( R \)-module.
   
   (a) Show that the Abelian group \( S \otimes_{R} M \) (where \( S \) is viewed as a right \( R \)-module via \( (s,r) \mapsto s\phi(r) \)) has a natural left \( S \)-module structure.
   
   (b) Let \( N \) be a left \( S \)-module, and let \( \phi^{*}N \) be the Abelian group \( N \) viewed as a left \( R \)-module via \( (r,n) \mapsto \phi(r)n \); cf. Exercise 12 of problem sheet 1. Show that there is a canonical isomorphism
   \[
   s\text{Hom}(S \otimes_{R} M, N) \sim r\text{Hom}(M, \phi^{*}N)
   \]
   of Abelian groups.
6. Let $R$ and $S$ be two rings, and let $T$ be the Abelian group $T = R \otimes S$ (where $R$ and $S$ are viewed as $\mathbb{Z}$-modules).

(a) Show that the map
\[ ((r, s), (r', s')) \mapsto (rr', ss') \]
induces a bilinear map $m : T \times T \to T$.

(b) Show that $T$ has a natural ring structure, with the map $m$ from (a) as the multiplication map.

(c) Show that there are canonical ring homomorphisms $i : R \to T$ and $j : S \to T$.

(d) Show that $T$, together with the maps $i$ and $j$, is a sum of $R$ and $S$ in the category of rings.


8. Let $A \to B$ be a homomorphism of commutative rings, and let $R$ be an $A$-algebra. Show that the $A$-algebra $B \otimes_A R$ has a natural $B$-algebra structure.

9. Let $k \to K$ be a field extension.

(a) Let $n$ be a non-negative integer. Show that there is a canonical isomorphism
\[ K \otimes_k \text{Mat}_n(k) \cong \text{Mat}_n(K) \]
of $K$-algebras.

(b) Let $G$ be a group. Show that there is a canonical isomorphism
\[ K \otimes_k k[G] \cong K[G] \]
of $K$-algebras.

10. Let $H$ be the $\mathbb{R}$-algebra of Hamilton quaternions. We recall that this is the 4-dimensional $\mathbb{R}$-vector space with basis $(1, i, j, k)$, made into an $\mathbb{R}$-algebra with unit element 1 and multiplication defined on the other basis elements by
\[
i^2 = j^2 = k^2 = -1,
ij = -ji = k,
jk = -kj = i,
ki = -ik = j
\]
and extended $\mathbb{R}$-bilinearly.

(a) Show that $H$ is a division ring. (Hint: use the conjugation map $a + bi + cj + dk \mapsto a - bi - cj - dk$ for $a, b, c, d \in \mathbb{R}$.)

(b) Show that there is an isomorphism $\mathbb{C} \otimes_{\mathbb{R}} H \cong \text{Mat}_2(\mathbb{C})$ of $\mathbb{C}$-algebras.

11. Let $R$ be a ring that is semi-simple as a left module over itself, so there is a family $(M_i)_{i \in I}$ of simple $R$-modules such that $R$ is isomorphic to $\bigoplus_{i \in I} M_i$ as an $R$-module.

(a) Show that the set $I$ is finite. (Hint: write $1 \in R$ as a sum of elements of the $M_i$.)

(b) Show that every simple $R$-module is isomorphic to one of the $M_i$.

12. Let $R$ and $S$ be two semi-simple rings. Show, using the definition of semi-simple rings, that the product ring $R \times S$ is also semi-simple. (Do not use the classification of semi-simple rings; this has not yet been proved in the lecture.)