1. Let $p$ be a prime number, let $k$ be a field of characteristic $p$, and let $G$ be a finite group of order divisible by $p$. Let $V$ be the one-dimensional $k$-linear subspace of $k[G]$ spanned by $\sum_{g \in G} g$.

(a) Show that $V$ is a left $k[G]$-submodule of $k[G]$.

(b) Let $f : k[G] \to V$ be a $k[G]$-linear map. Show that the kernel of $f$ contains $V$.

(c) Deduce that the ring $k[G]$ is not semi-simple.

2. Let $D$ be a division ring, and let $n$ be a positive integer. Show that the ring homomorphism $D \to \text{Mat}_n(D)$ sending each $\lambda \in D$ to $\lambda I$ (where $I$ is the identity matrix) induces a ring isomorphism $Z(D) \isom Z(\text{Mat}_n(D))$.

3. Let $R$ be a commutative ring. Show that $R$ is semi-simple if and only if $R$ is a finite product of fields.

4. Let $R$ be a ring. We say that $R$ is right semi-simple if every right $R$-module is semi-simple. Show that $R$ is semi-simple if and only if $R$ is right semi-simple.

5. Let $k$ be a field, and let $D$ be a division algebra over $k$ such that $[D : k] = \dim_k D$ is finite. Prove that for every $\alpha \in D$, the subalgebra $k[\alpha] = \sum_{i \geq 0} k\alpha^i$ of $D$ is a field and is a finite extension of $k$.

6. Let $R$ be a ring, let $M_1, \ldots, M_n$ be left $R$-modules, let $M$ be the left $R$-module $\bigoplus_{i=1}^n M_i$, and let $E$ be the Abelian group $\bigoplus_{i,j=1}^n R\text{Hom}(M_j, M_i)$.

(a) Show that there is a canonical isomorphism

$$\phi : R\text{End}(M) \isom E$$

of Abelian groups.

(b) Describe the unique ring structure on $E$ for which $\phi$ is a ring isomorphism. (Hint: think of matrix multiplication).

(c) Suppose $M_1 = \ldots = M_n$. Show that there is a canonical ring isomorphism

$$R\text{End}(M) \isom \text{Mat}_n(R\text{End}(M_1)).$$

(d) Suppose that the $R$-modules $M_1, \ldots, M_n$ are simple and pairwise non-isomorphic. Show that there is a canonical ring isomorphism

$$R\text{End}(M) \isom \prod_{i=1}^n R\text{End}(M_i).$$
7. Let $A_4$ be the alternating group on 4 elements, and let $k$ be an algebraically closed field of characteristic not 2 or 3.
   (a) Show that up to isomorphism, $A_4$ has exactly four irreducible $k$-linear representations.
   (b) Show that up to isomorphism, $A_4$ has exactly three $k$-linear representations of dimension 1 and exactly one irreducible $k$-linear representation of dimension 3.

8. Let $S_4$ be the symmetric group on 4 elements, and let $k$ be an algebraically closed field of characteristic not 2 or 3.
   (a) Show that up to isomorphism, $S_4$ has exactly five irreducible $k$-linear representations.
   (b) Show that up to isomorphism, $S_4$ has exactly two $k$-linear representations of dimension 1, exactly one irreducible $k$-linear representation of dimension 2 and exactly two irreducible $k$-linear representations of dimension 3.

(Hint for Exercises 7 and 8: it is not necessary to give any representation explicitly.)

9. Let $S_3$ be the symmetric group of order 6, and let $k$ be a field of characteristic not 2 or 3. Give an explicit $k$-algebra isomorphism
   $$k[S_3] \overset{\sim}{\to} k \times k \times \text{Mat}_2(k).$$

10. Let $D_4$ be the dihedral group of order 8, and let $k$ be a field of characteristic different from 2. Determine positive integers $n_1, \ldots, n_m$ and an explicit $k$-algebra isomorphism
    $$k[D_4] \overset{\sim}{\to} \prod_{i=1}^m \text{Mat}_{n_i}(k).$$

11. Let $Q$ be the quaternion group of order 8. Determine division algebras $D_1, \ldots, D_m$ over $\mathbb{R}$, positive integers $n_1, \ldots, n_m$ and an explicit $\mathbb{R}$-algebra isomorphism
    $$\mathbb{R}[Q] \overset{\sim}{\to} \prod_{i=1}^m \text{Mat}_{n_i}(D_i).$$

(Note that in Exercises 9, 10 and 11 the base field is not (necessarily) algebraically closed.)