Representation Theory of Finite Groups, spring 2019

Problem Sheet 7

25 March

1. Let p be a prime number, let k be a field of characteristic p, and let G be a finite group of order divisible by p. Let V be the one-dimensional k-linear subspace of $k[G]$ spanned by $\sum_{g\in G} g$.

(a) Show that V is a left $k[G]$-submodule of $k[G]$.

(b) Let $f\colon k[G] \to V$ be a $k[G]$-linear map. Show that the kernel of f contains V.

(c) Deduce that the ring $k[G]$ is not semi-simple.

2. Let D be a division ring, and let n be a positive integer. Show that the ring homomorphism $D \to \mathrm{Mat}_n(D)$ sending each $\lambda \in D$ to λI (where I is the identity matrix) induces a ring isomorphism $Z(D) \xrightarrow{\sim} Z(\mathrm{Mat}_n(D))$.

3. Let R be a *commutative* ring. Show that R is semi-simple if and only if R is a finite product of fields.

4. Let R be a ring. We say that R is *right semi-simple* if every right R-module is semi-simple. Show that R is semi-simple if and only if R is right semi-simple.

5. Let k be a field, and let D be a division algebra over k such that $[D : k] = \dim_k D$ is finite. Prove that for every $\alpha \in D$, the subalgebra $k[\alpha] = \sum_{i\ge 0} k\alpha^i$ of D is a field and is a finite extension of k.

6. Let R be a ring, let $M_1, \ldots, M_n$ be left R-modules, let M be the left R-module $\bigoplus_{i=1}^n M_i$, and let E be the Abelian group $\bigoplus_{i,j=1}^n {}_R\mathrm{Hom}(M_j, M_i)$.

(a) Show that there is a canonical isomorphism

$$\phi\colon {}_R\mathrm{End}(M) \xrightarrow{\sim} E$$

of Abelian groups.

(b) Describe the unique ring structure on E for which ϕ is a ring isomorphism. (*Hint:* think of matrix multiplication).

(c) Suppose $M_1 = \ldots = M_n$. Show that there is a canonical ring isomorphism

$${}_R\mathrm{End}(M) \xrightarrow{\sim} \mathrm{Mat}_n({}_R\mathrm{End}(M_1)).$$

(d) Suppose that the R-modules $M_1, \ldots, M_n$ are simple and pairwise non-isomorphic. Show that there is a canonical ring isomorphism

$${}_R\mathrm{End}(M) \xrightarrow{\sim} \prod_{i=1}^n {}_R\mathrm{End}(M_i).$$

7. Let A_4 be the alternating group on 4 elements, and let k be an algebraically closed field of characteristic not 2 or 3.

(a) Show that up to isomorphism, A_4 has exactly four irreducible k-linear representations.

(b) Show that up to isomorphism, A_4 has exactly three k-linear representations of dimension 1 and exactly one irreducible k-linear representation of dimension 3.

8. Let S_4 be the symmetric group on 4 elements, and let k be an algebraically closed field of characteristic not 2 or 3.

(a) Show that up to isomorphism, S_4 has exactly five irreducible k-linear representations.

(b) Show that up to isomorphism, S_4 has exactly two k-linear representations of dimension 1, exactly one irreducible k-linear representation of dimension 2 and exactly two irreducible k-linear representations of of dimension 3.

(*Hint for Exercises 7 and 8:* it is not necessary to give any representation explicitly.)

9. Let S_3 be the symmetric group of order 6, and let k be a field of characteristic not 2 or 3. Give an explicit k-algebra isomorphism

$$k[S_3] \xrightarrow{\sim} k \times k \times \mathrm{Mat}_2(k).$$

10. Let D_4 be the dihedral group of order 8, and let k be a field of characteristic different from 2. Determine positive integers $n_1, \ldots, n_m$ and an explicit k-algebra isomorphism

$$k[D_4] \xrightarrow{\sim} \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(k).$$

11. Let Q be the quaternion group of order 8. Determine division algebras $D_1, \ldots, D_m$ over $\mathbf{R}$, positive integers $n_1, \ldots, n_m$ and an explicit $\mathbf{R}$-algebra isomorphism

$$\mathbf{R}[Q] \xrightarrow{\sim} \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(D_i).$$

(Note that in Exercises 9, 10 and 11 the base field is not (necessarily) algebraically closed.)