Representation Theory of Finite Groups, spring 2019

Problem Sheet 8

1 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

Let G be a finite group. The space of *class functions* of G is the $\mathbf{C}$-vector space

$$\mathbf{C}_{\text{class}}(G) = \{f\colon G \to \mathbf{C} \mid f(gxg^{-1}) = f(x) \text{ for all } x, g \in G\},$$

made into a $\mathbf{C}$-algebra by pointwise addition and multiplication. There is a Hermitean inner product on $\mathbf{C}_{\text{class}}(G)$ defined by

$$\langle f_1, f_2 \rangle = \frac{1}{\#G} \sum_{x \in G} \overline{f_1(x)} f_2(x).$$

For each irreducible representation V of G, the *character* of V is the class function

$$\begin{aligned} \chi_V\colon G &\longrightarrow \mathbf{C} \\ g &\longmapsto \operatorname{tr}_{\mathbf{C}}(g\colon V \to V), \end{aligned}$$

i.e. the trace of g viewed as a $\mathbf{C}$-linear endomorphism of V. Let $X(G) \subset \mathbf{C}_{\text{class}}(G)$ be the set of characters of irreducible representations of G. It has been shown in the lecture that $X(G)$ is an orthonormal basis of $\mathbf{C}_{\text{class}}(G)$ with respect to the inner product $\langle\ ,\ \rangle$.

1. Let G be a finite group, and let V be a finite-dimensional representation of G. By Maschke's theorem, V is isomorphic to a representation of the form $\bigoplus_{S \in \mathcal{S}} S^{n_S}$, where $\mathcal{S}$ is the set of irreducible representations of G up to isomorphism and the n_S are non-negative integers. Prove the identity

$$\langle \chi_V, \chi_V \rangle = \sum_{S \in \mathcal{S}} n_S^2.$$

2. Let G be a finite group, and let $f\colon G \to \mathbf{C}$ be a class function. Since $X(G)$ is a basis of $\mathbf{C}_{\text{class}}(G)$, we can write $f = \sum_{\chi \in X(G)} a_\chi \chi$ with $a_\chi \in \mathbf{C}$.

(a) Show that for each $\chi \in X(G)$, the coefficient a_χ equals $\langle \chi, f \rangle$.

(b) Show that f is the character of a finite-dimensional representation of G if and only if all the a_χ are non-negative integers.

3. Let G be a finite group, and consider the class function $\chi\colon G \to \mathbf{C}$ defined by

$$\chi(g) = \begin{cases} \#G & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

Show that χ is the character of a finite-dimensional representation of G. Which representation is this?

The *character table* of G is a matrix with rows labelled by the irreducible representations of G up to isomorphism and columns labelled by the conjugacy classes of G. The entry in the row labelled by an irreducible representation V and the column labelled by a conjugacy class $[g]$ is the complex number $\chi_V(g)$.

4. Determine the character tables of the dihedral group D_4 and of the quaternion group Q, both of order 8. Do you notice anything remarkable?

5. Determine the character table of the dihedral group D_5 of order 10.

6. Determine the character table of the alternating group A_4 of order 12.

7. Determine the character table of the symmetric group S_4 of order 24.

8. Determine the character table of the alternating group A_5 of order 60.

(*Hint for Exercises 4–8:* use explicit descriptions of low-dimensional representations and constraints on the inner products between rows of the character table. For Exercises 4, 6 and 7, you may also use results from problem sheet 7.)

9. Let G be the symmetric group S_3 of order 6. Let V be the unique two-dimensional irreducible representation of G, and let $\chi_2\colon G \to \mathbf{C}$ be its character.

(a) Express the class function $\chi_2^2 \in \mathbf{C}_{\mathrm{class}}(G)$ as a linear combination of characters of irreducible representations of G.

(b) From the result of (a), deduce how the 4-dimensional representation $V \otimes_{\mathbf{C}} V$ of G decomposes as a direct sum of irreducible representations.

10. As Exercise 9, but for $G = S_4$. (Note that S_4, like S_3, has a unique two-dimensional irreducible representation; see Exercise 8 of problem sheet 7).