Throughout this problem sheet, representations and characters are taken to be over the field \( \mathbb{C} \) of complex numbers.

Let \( G \) be a finite group. The space of class functions of \( G \) is the \( \mathbb{C} \)-vector space \( \mathbb{C}_{\text{class}}(G) = \{ f: G \to \mathbb{C} \mid f(gxg^{-1}) = f(x) \text{ for all } x, g \in G \} \), made into a \( \mathbb{C} \)-algebra by pointwise addition and multiplication. There is a Hermitean inner product on \( \mathbb{C}_{\text{class}}(G) \) defined by
\[
\langle f_1, f_2 \rangle = \frac{1}{\#G} \sum_{x \in G} f_1(x)f_2(x).
\]

For each irreducible representation \( V \) of \( G \), the character of \( V \) is the class function \( \chi_V: G \to \mathbb{C} \),
\[
\chi_V(g) = \text{tr}_\mathbb{C}(g: V \to V),
\]
i.e. the trace of \( g \) viewed as a \( \mathbb{C} \)-linear endomorphism of \( V \). Let \( X(G) \subset \mathbb{C}_{\text{class}}(G) \) be the set of characters of irreducible representations of \( G \). It has been shown in the lecture that \( X(G) \) is an orthonormal basis of \( \mathbb{C}_{\text{class}}(G) \) with respect to the inner product \( \langle \ , \ \rangle \).

1. Let \( G \) be a finite group, and let \( V \) be a finite-dimensional representation of \( G \). By Maschke’s theorem, \( V \) is isomorphic to a representation of the form \( \bigoplus_{S \in S} S^{n_S} \), where \( S \) is the set of irreducible representations of \( G \) up to isomorphism and the \( n_S \) are non-negative integers. Prove the identity
\[
\langle \chi_V, \chi_V \rangle = \sum_{S \in S} n_S^2.
\]

2. Let \( G \) be a finite group, and let \( f: G \to \mathbb{C} \) be a class function. Since \( X(G) \) is a basis of \( \mathbb{C}_{\text{class}}(G) \), we can write \( f = \sum_{\chi \in X(G)} a_\chi \chi \) with \( a_\chi \in \mathbb{C} \).
   (a) Show that for each \( \chi \in X(G) \), the coefficient \( a_\chi \) equals \( \langle \chi, f \rangle \).
   (b) Show that \( f \) is the character of a finite-dimensional representation of \( G \) if and only if all the \( a_\chi \) are non-negative integers.

3. Let \( G \) be a finite group, and consider the class function \( \chi: G \to \mathbb{C} \) defined by
\[
\chi(g) = \begin{cases} 
\#G & \text{if } g = 1, \\
0 & \text{if } g \neq 1.
\end{cases}
\]
Show that \( \chi \) is the character of a finite-dimensional representation of \( G \). Which representation is this?

The character table of \( G \) is a matrix with rows labelled by the irreducible representations of \( G \) up to isomorphism and columns labelled by the conjugacy classes of \( G \). The entry in the row labelled by an irreducible representation \( V \) and the column labelled by a conjugacy class \([g]\) is the complex number \( \chi_V(g) \).
4. Determine the character tables of the dihedral group $D_4$ and of the quaternion group $Q$, both of order 8. Do you notice anything remarkable?

5. Determine the character table of the dihedral group $D_5$ of order 10.

6. Determine the character table of the alternating group $A_4$ of order 12.

7. Determine the character table of the symmetric group $S_4$ of order 24.

8. Determine the character table of the alternating group $A_5$ of order 60.

(Hint for Exercises 4–8: use explicit descriptions of low-dimensional representations and constraints on the inner products between rows of the character table. For Exercises 4, 6 and 7, you may also use results from problem sheet 7.)

9. Let $G$ be the symmetric group $S_3$ of order 6. Let $V$ be the unique two-dimensional irreducible representation of $G$, and let $\chi_2: G \to \mathbb{C}$ be its character.

   (a) Express the class function $\chi_2^2 \in \mathbb{C}_{\text{class}}(G)$ as a linear combination of characters of irreducible representations of $G$.

   (b) From the result of (a), deduce how the 4-dimensional representation $V \otimes V$ of $G$ decomposes as a direct sum of irreducible representations.

10. As Exercise 9, but for $G = S_4$. (Note that $S_4$, like $S_3$, has a unique two-dimensional irreducible representation; see Exercise 8 of problem sheet 7).