21. Consider the differential equation

$$y'' + \frac{\alpha}{x^s} y' + \frac{\beta}{x^t} y = 0, \tag{i}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and $s$ and $t$ are positive integers that for the moment are arbitrary.

(a) Show that if $s > 1$ or $t > 2$, then the point $x = 0$ is an irregular singular point.

(b) Try to find a solution of Eq. (i) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \qquad x > 0. \tag{ii}$$

Show that if $s = 2$ and $t = 2$, then there is only one possible value of $r$ for which there is a formal solution of Eq. (i) of the form (ii).

(c) Show that if $s = 1$ and $t = 3$, then there are no solutions of Eq. (i) of the form (ii).

(d) Show that the maximum values of $s$ and $t$ for which the indicial equation is quadratic in $r$ [and hence we can hope to find two solutions of the form (ii)] are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a “weak singularity,” or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution we should point out that while it is sometimes possible to obtain a formal series solution of the form (ii) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 20 for an example.

## 5.8 Bessel’s Equation

In this section we consider three special cases of Bessel’s[12] equation,

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0, \tag{1}$$

where $\nu$ is a constant, which illustrate the theory discussed in Section 5.7. It is easy to show that $x = 0$ is a regular singular point. For simplicity we consider only the case $x > 0$.

***Bessel Equation of Order Zero.*** This example illustrates the situation in which the roots of the indicial equation are equal. Setting $\nu = 0$ in Eq. (1) gives

$$L[y] = x^2 y'' + xy' + x^2 y = 0. \tag{2}$$

Substituting

$$y = \phi(r, x) = a_0 x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \tag{3}$$

[12] Friedrich Wilhelm Bessel (1784–1846) embarked on a career in business as a youth, but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of Eq. (1). He is also famous for making the first accurate determination (1838) of the distance from the earth to a star.

we obtain

$$\begin{aligned} L[\phi](r,x) &= \sum_{n=0}^{\infty} a_n[(r+n)(r+n-1)+(r+n)]x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0[r(r-1)+r]x^r + a_1[(r+1)r+(r+1)]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}\{a_n[(r+n)(r+n-1)+(r+n)]+a_{n-2}\}x^{r+n} = 0. \qquad (4) \end{aligned}$$

The roots of the indicial equation $F(r) = r(r-1) + r = 0$ are $r_1 = 0$ and $r_2 = 0$; hence we have the case of equal roots. The recurrence relation is

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \qquad n \geq 2. \qquad (5)$$

To determine $y_1(x)$ we set $r$ equal to 0. Then from Eq. (4) it follows that for the coefficient of $x^{r+1}$ to be zero we must choose $a_1 = 0$. Hence from Eq. (5), $a_3 = a_5 = a_7 = \cdots = 0$. Further,

$$a_n(0) = -a_{n-2}(0)/n^2, \qquad n = 2, 4, 6, 8, \ldots,$$

or letting $n = 2m$, we obtain

$$a_{2m}(0) = -a_{2m-2}(0)/(2m)^2, \qquad m = 1, 2, 3, \ldots .$$

Thus

$$a_2(0) = -\frac{a_0}{2^2}, \quad a_4(0) = \frac{a_0}{2^4 2^2}, \quad a_6(0) = -\frac{a_0}{2^6(3\cdot 2)^2},$$

and, in general,

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots . \qquad (6)$$

Hence

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \qquad x > 0. \qquad (7)$$

The function in brackets is known as the Bessel function of the first kind of order zero and is denoted by $J_0(x)$. It follows from Theorem 5.7.1 that the series converges for all $x$, and that $J_0$ is analytic at $x = 0$. Some of the important properties of $J_0$ are discussed in the problems. Figure 5.8.1 shows the graphs of $y = J_0(x)$ and some of the partial sums of the series (7).

To determine $y_2(x)$ we will calculate $a_n'(0)$. The alternative procedure in which we simply substitute the form (23) of Section 5.7 in Eq. (2) and then determine the $b_n$ is discussed in Problem 10. First we note from the coefficient of $x^{r+1}$ in Eq. (4) that $(r+1)^2 a_1(r) = 0$. It follows that not only does $a_1(0) = 0$ but also $a_1'(0) = 0$. It is easy to deduce from the recurrence relation (5) that $a_3'(0) = a_5'(0) = \cdots = a_{2n+1}'(0) = \cdots = 0$; hence we need only compute $a_{2m}'(0)$, $m = 1, 2, 3, \ldots$. From Eq. (5) we have

$$a_{2m}(r) = -a_{2m-2}(r)/(r+2m)^2, \qquad m = 1, 2, 3, \ldots .$$

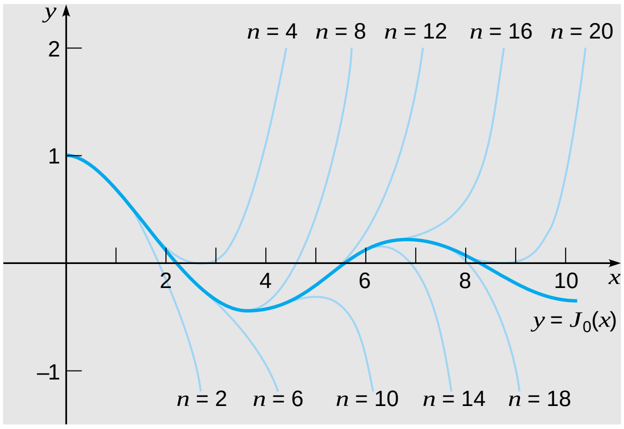


**FIGURE 5.8.1** Polynomial approximations to $J_0(x)$. The value of $n$ is the degree of the approximating polynomial.

By solving this recurrence relation we obtain

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2(r+4)^2\cdots(r+2m-2)^2(r+2m)^2}, \qquad m = 1, 2, 3, \ldots. \tag{8}$$

The computation of $a'_{2m}(r)$ can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n},$$

and if $x$ is not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Applying this result to $a_{2m}(r)$ from Eq. (8) we find that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right),$$

and, setting $r$ equal to 0, we obtain

$$a'_{2m}(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right]a_{2m}(0).$$

Substituting for $a_{2m}(0)$ from Eq. (6), and letting

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}, \tag{9}$$

we obtain, finally,

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots.$$

The second solution of the Bessel equation of order zero is found by setting $a_0 = 1$ and substituting for $y_1(x)$ and $a'_{2m}(0) = b_{2m}(0)$ in Eq. (23) of Section 5.7. We obtain

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \qquad x > 0. \tag{10}$$

Instead of $y_2$, the second solution is usually taken to be a certain linear combination of $J_0$ and $y_2$. It is known as the Bessel function of the second kind of order zero and is denoted by $Y_0$. Following Copson (Chapter 12), we define[13]

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2)J_0(x)]. \tag{11}$$

Here $\gamma$ is a constant, known as the Euler–Máscheroni (1750–1800) constant; it is defined by the equation

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \cong 0.5772. \tag{12}$$

Substituting for $y_2(x)$ in Eq. (11), we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right], \qquad x > 0. \tag{13}$$

The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

Note that $J_0(x) \to 1$ as $x \to 0$ and that $Y_0(x)$ has a logarithmic singularity at $x = 0$; that is, $Y_0(x)$ behaves as $(2/\pi)\ln x$ when $x \to 0$ through positive values. Thus if we are interested in solutions of Bessel's equation of order zero that are finite at the origin, which is often the case, we must discard $Y_0$. The graphs of the functions $J_0$ and $Y_0$ are shown in Figure 5.8.2.

It is interesting to note from Figure 5.8.2 that for $x$ large both $J_0(x)$ and $Y_0(x)$ are oscillatory. Such a behavior might be anticipated from the original equation; indeed it

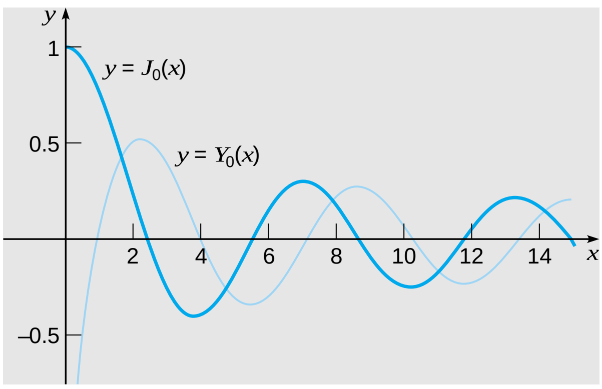


**FIGURE 5.8.2** The Bessel functions of order zero.

[13]Other authors use other definitions for $Y_0$. The present choice for $Y_0$ is also known as the Weber (1842–1913) function.

is true for the solutions of the Bessel equation of order $\nu$. If we divide Eq. (1) by $x^2$, we obtain

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

For $x$ very large it is reasonable to suspect that the terms $(1/x)y'$ and $(\nu^2/x^2)y$ are small and hence can be neglected. If this is true, then the Bessel equation of order $\nu$ can be approximated by

$$y'' + y = 0.$$

The solutions of this equation are $\sin x$ and $\cos x$; thus we might anticipate that the solutions of Bessel's equation for large $x$ are similar to linear combinations of $\sin x$ and $\cos x$. This is correct insofar as the Bessel functions are oscillatory; however, it is only partly correct. For $x$ large the functions $J_0$ and $Y_0$ also decay as $x$ increases; thus the equation $y'' + y = 0$ does not provide an adequate approximation to the Bessel equation for large $x$, and a more delicate analysis is required. In fact, it is possible to show that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty, \tag{14}$$

and that

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty. \tag{15}$$

These asymptotic approximations, as $x \to \infty$, are actually very good. For example, Figure 5.8.3 shows that the asymptotic approximation (14) to $J_0(x)$ is reasonably accurate for all $x \geq 1$. Thus to approximate $J_0(x)$ over the entire range from zero to infinity, one can use two or three terms of the series (7) for $x \leq 1$ and the asymptotic approximation (14) for $x \geq 1$.

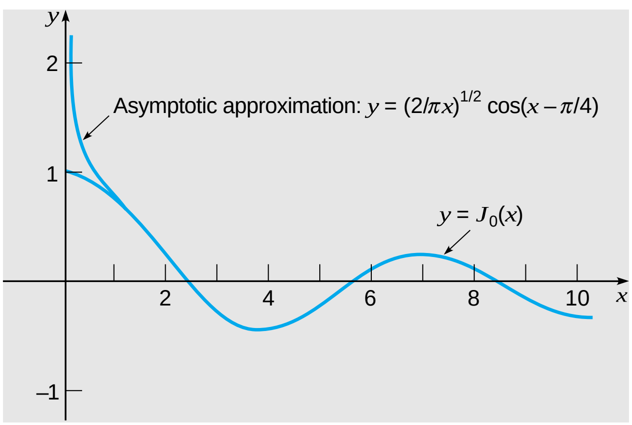


**FIGURE 5.8.3** Asymptotic approximation to $J_0(x)$.

***Bessel Equation of Order One-Half.*** This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting $\nu = \frac{1}{2}$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right) y = 0. \tag{16}$$

If we substitute the series (3) for $y = \phi(r, x)$, we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} \left[(r+n)(r+n-1) + (r+n) - \tfrac{1}{4}\right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= (r^2 - \tfrac{1}{4})a_0x^r + \left[(r+1)^2 - \tfrac{1}{4}\right] a_1 x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \left\{\left[(r+n)^2 - \tfrac{1}{4}\right] a_n + a_{n-2}\right\} x^{r+n} = 0. \end{aligned} \tag{17}$$

The roots of the indicial equation are $r_1 = \frac{1}{2}$, $r_2 = -\frac{1}{2}$; hence the roots differ by an integer. The recurrence relation is

$$\left[(r+n)^2 - \tfrac{1}{4}\right] a_n = -a_{n-2}, \qquad n \geq 2. \tag{18}$$

Corresponding to the larger root $r_1 = \frac{1}{2}$ we find from the coefficient of $x^{r+1}$ in Eq. (17) that $a_1 = 0$. Hence, from Eq. (18), $a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$. Further, for $r = \frac{1}{2}$,

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \qquad n = 2, 4, 6 \ldots,$$

or letting $n = 2m$, we obtain

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \qquad m = 1, 2, 3, \ldots.$$

By solving this recurrence relation we find that

$$a_2 = -\frac{a_0}{3!}, \qquad a_4 = \frac{a_0}{5!}, \ldots$$

and, in general,

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \qquad m = 1, 2, 3, \ldots.$$

Hence, taking $a_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!}\right] = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}, \qquad x > 0. \tag{19}$$

The power series in Eq. (19) is precisely the Taylor series for $\sin x$; hence one solution of the Bessel equation of order one-half is $x^{-1/2} \sin x$. The Bessel function of the first kind of order one-half, $J_{1/2}$, is defined as $(2/\pi)^{1/2} y_1$. Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \qquad x > 0. \tag{20}$$

Corresponding to the root $r_2 = -\frac{1}{2}$ it is possible that we may have difficulty in computing $a_1$ since $N = r_1 - r_2 = 1$. However, from Eq. (17) for $r = -\frac{1}{2}$ the coefficients of $x^r$ and $x^{r+1}$ are both zero regardless of the choice of $a_0$ and $a_1$. Hence $a_0$ and $a_1$ can be chosen arbitrarily. From the recurrence relation (18) we obtain a set of even-numbered coefficients corresponding to $a_0$ and a set of odd-numbered coefficients corresponding to $a_1$. Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for $r = -\frac{1}{2}$,

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \qquad n = 1, 2, \dots .$$

Hence

$$\begin{aligned} y_2(x) &= x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\ &= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \qquad x > 0. \end{aligned} \tag{21}$$

The constant $a_1$ simply introduces a multiple of $y_1(x)$. The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution for which $a_0 = (2/\pi)^{1/2}$ and $a_1 = 0$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad x > 0. \tag{22}$$

The general solution of Eq. (16) is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$.

By comparing Eqs. (20) and (22) with Eqs. (14) and (15) we see that, except for a phase shift of $\pi/4$, the functions $J_{-1/2}$ and $J_{1/2}$ resemble $J_0$ and $Y_0$, respectively, for large $x$. The graphs of $J_{1/2}$ and $J_{-1/2}$ are shown in Figure 5.8.4.

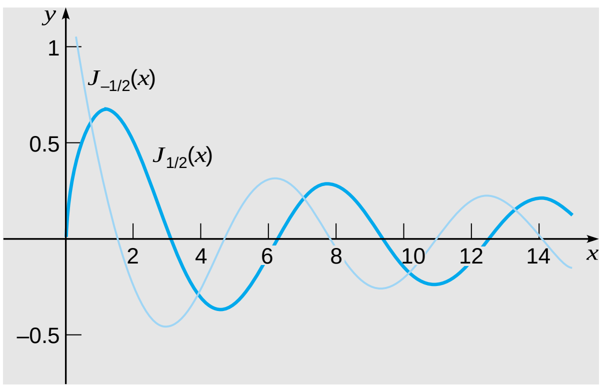


**FIGURE 5.8.4** The Bessel functions $J_{1/2}$ and $J_{-1/2}$.

***Bessel Equation of Order One.*** This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting $\nu = 1$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + (x^2 - 1)y = 0. \tag{23}$$

If we substitute the series (3) for $y = \phi(r, x)$ and collect terms as in the preceding cases, we obtain

$$\begin{aligned} L[\phi](r, x) = a_0(r^2 - 1)x^r + a_1[(r+1)^2 - 1]x^{r+1} \\ + \sum_{n=2}^{\infty}\{[(r+n)^2 - 1]a_n + a_{n-2}\}x^{r+n} = 0. \end{aligned} \tag{24}$$

The roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. The recurrence relation is

$$[(r+n)^2 - 1]a_n(r) = -a_{n-2}(r), \qquad n \geq 2. \tag{25}$$

Corresponding to the larger root $r = 1$ the recurrence relation becomes

$$a_n = -\frac{a_{n-2}}{(n+2)n}, \qquad n = 2, 3, 4, \ldots .$$

We also find from the coefficient of $x^{r+1}$ in Eq. (24) that $a_1 = 0$; hence from the recurrence relation $a_3 = a_5 = \cdots = 0$. For even values of $n$, let $n = 2m$; then

$$a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m)} = -\frac{a_{2m-2}}{2^2(m+1)m}, \qquad m = 1, 2, 3, \ldots .$$

By solving this recurrence relation we obtain

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!m!}, \qquad m = 1, 2, 3, \ldots . \tag{26}$$

The Bessel function of the first kind of order one, denoted by $J_1$, is obtained by choosing $a_0 = 1/2$. Hence

$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}. \tag{27}$$

The series converges absolutely for all $x$, so the function $J_1$ is analytic everywhere.

In determining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in Eq. (28) below is rather complicated, but the first few coefficients can be found fairly easily. According to Theorem 5.7.1 we assume that

$$y_2(x) = aJ_1(x)\ln x + x^{-1}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right], \qquad x > 0. \tag{28}$$

Computing $y_2'(x)$, $y_2''(x)$, substituting in Eq. (23), and making use of the fact that $J_1$ is a solution of Eq. (23) give

$$2axJ_1'(x) + \sum_{n=0}^{\infty}[(n-1)(n-2)c_n + (n-1)c_n - c_n]x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0, \tag{29}$$

where $c_0 = 1$. Substituting for $J_1(x)$ from Eq. (27), shifting the indices of summation in the two series, and carrying out several steps of algebra give

$$
\begin{aligned}
-c_1 + [0 \cdot c_2 + c_0]x + \sum_{n=2}^{\infty} [(n^2 - 1)c_{n+1} + c_{n-1}]x^n \\
= -a \left[ x + \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1) x^{2m+1}}{2^{2m}(m+1)!\, m!} \right]. \qquad (30)
\end{aligned}
$$

From Eq. (30) we observe first that $c_1 = 0$, and $a = -c_0 = -1$. Further, since there are only odd powers of $x$ on the right, the coefficient of each even power of $x$ on the left must be zero. Thus, since $c_1 = 0$, we have $c_3 = c_5 = \cdots = 0$. Corresponding to the odd powers of $x$ we obtain the recurrence relation [let $n = 2m + 1$ in the series on the left side of Eq. (30)]

$$
[(2m+1)^2 - 1]c_{2m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m}(m+1)!\, m!}, \qquad m = 1, 2, 3, \ldots . \qquad (31)
$$

When we set $m = 1$ in Eq. (31), we obtain

$$
(3^2 - 1)c_4 + c_2 = (-1)3/(2^2 \cdot 2!).
$$

Notice that $c_2$ can be selected *arbitrarily*, and then this equation determines $c_4$. Also notice that in the equation for the coefficient of $x$, $c_2$ appeared multiplied by 0, and that equation was used to determine $a$. That $c_2$ is arbitrary is not surprising, since $c_2$ is the coefficient of $x$ in the expression $x^{-1}[1 + \sum_{n=1}^{\infty} c_n x^n]$. Consequently, $c_2$ simply generates a multiple of $J_1$, and $y_2$ is only determined up to an additive multiple of $J_1$. In accord with the usual practice we choose $c_2 = 1/2^2$. Then we obtain

$$
\begin{aligned}
c_4 &= \frac{-1}{2^4 \cdot 2} \left[ \frac{3}{2} + 1 \right] = \frac{-1}{2^4 2!} \left[ \left(1 + \frac{1}{2}\right) + 1 \right] \\
&= \frac{(-1)}{2^4 \cdot 2!}(H_2 + H_1).
\end{aligned}
$$

It is possible to show that the solution of the recurrence relation (31) is

$$
c_{2m} = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m} m!(m-1)!}, \qquad m = 1, 2, \ldots
$$

with the understanding that $H_0 = 0$. Thus

$$
y_2(x) = -J_1(x) \ln x + \frac{1}{x} \left[ 1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m!(m-1)!} x^{2m} \right], \qquad x > 0. \qquad (32)
$$

The calculation of $y_2(x)$ using the alternative procedure [see Eqs. (19) and (20) of Section 5.7] in which we determine the $c_n(r_2)$ is slightly easier. In particular the latter procedure yields the general formula for $c_{2m}$ without the necessity of solving a recurrence relation of the form (31) (see Problem 11). In this regard the reader may also wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 10.

The second solution of Eq. (23), the Bessel function of the second kind of order one, $Y_1$, is usually taken to be a certain linear combination of $J_1$ and $y_2$. Following Copson (Chapter 12), $Y_1$ is defined as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)], \tag{33}$$

where $\gamma$ is defined in Eq. (12). The general solution of Eq. (23) for $x > 0$ is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that while $J_1$ is analytic at $x = 0$, the second solution $Y_1$ becomes unbounded in the same manner as $1/x$ as $x \to 0$. The graphs of $J_1$ and $Y_1$ are shown in Figure 5.8.5.

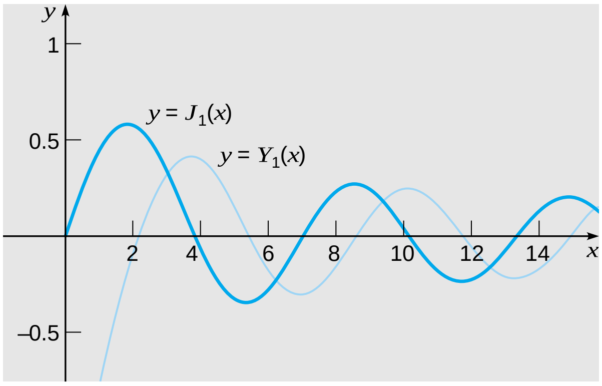


**FIGURE 5.8.5** The Bessel functions $J_1$ and $Y_1$.

## PROBLEMS

In each of Problems 1 through 4 show that the given differential equation has a regular singular point at $x = 0$, and determine two linearly independent solutions for $x > 0$.

1. $x^2y'' + 2xy' + xy = 0$
2. $x^2y'' + 3xy' + (1 + x)y = 0$
3. $x^2y'' + xy' + 2xy = 0$
4. $x^2y'' + 4xy' + (2 + x)y = 0$
5. Find two linearly independent solutions of the Bessel equation of order $\frac{3}{2}$,

$$x^2y'' + xy' + (x^2 - \tfrac{9}{4})y = 0, \qquad x > 0.$$

6. Show that the Bessel equation of order one-half,

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0, \qquad x > 0,$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

7. Show directly that the series for $J_0(x)$, Eq. (7), converges absolutely for all $x$.
8. Show directly that the series for $J_1(x)$, Eq. (27), converges absolutely for all $x$ and that $J_0'(x) = -J_1(x)$.