An Introduction to the Mathematical Theory of Waves

Roger Knobel

Linear and nonlinear waves are a central part of the theory of PDEs. This book begins with a description of one-dimensional waves and their visualization through computer-aided techniques. Next, traveling waves are covered, such as solitary waves for the Klein-Gordon and KdV equations. Finally, the author gives a lucid discussion of waves arising from conservation laws, including shock and rarefaction waves. As an application, interesting models of traffic flow are used to illustrate conservation laws and wave phenomena.

This book is based on a course given by the author at the IAS/Park City Mathematics Institute. It is suitable for independent study by undergraduate students in mathematics, engineering, and science programs.

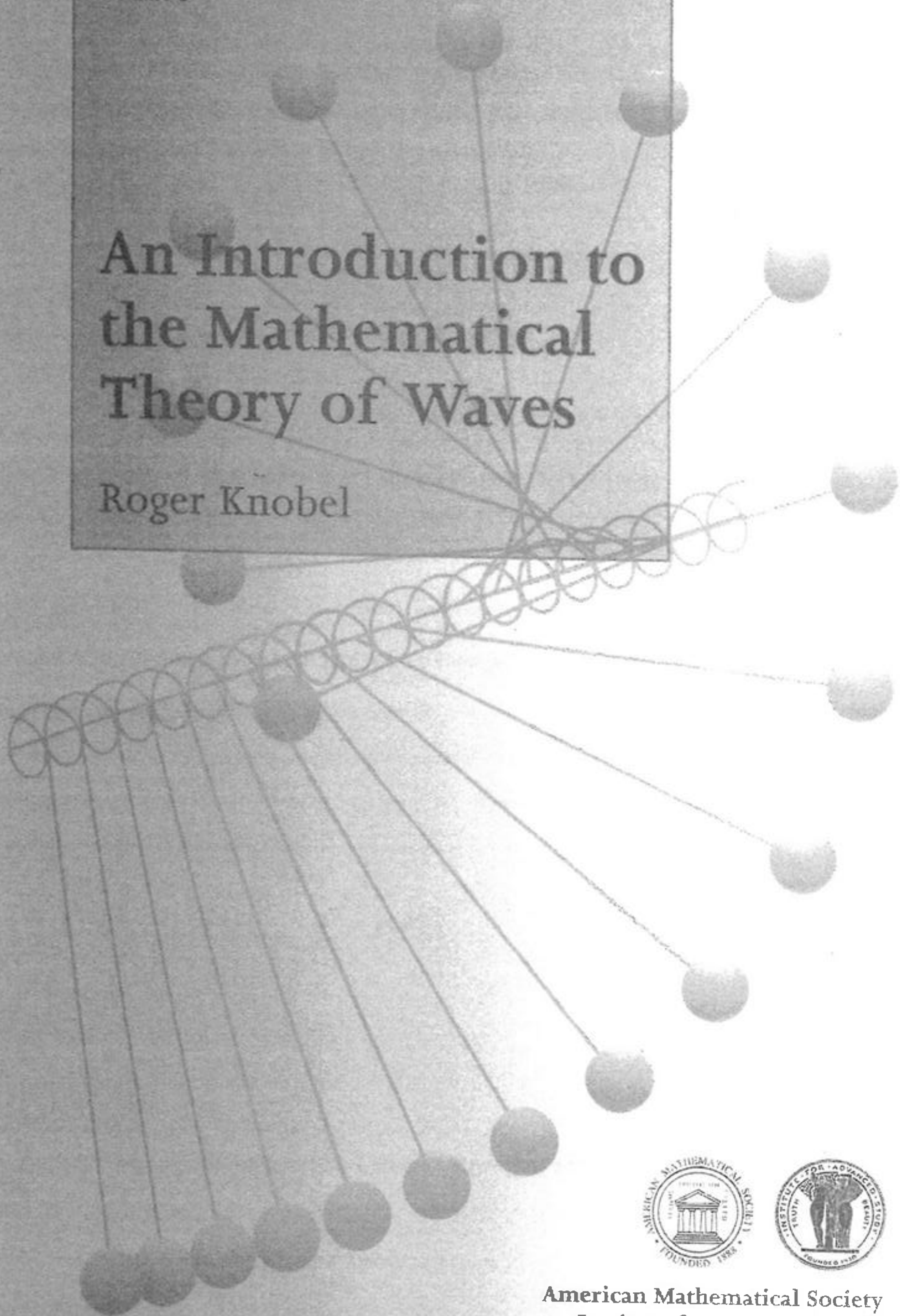
ISBN 0-8218-2039-7

9 780821 820391

STML/3

STML/IAS/PARK CITY 3 An Introduction to the Mathematical Theory of Waves Knobel AMS

STUDENT MATHEMATICAL LIBRARY
IAS/PARK CITY MATHEMATICAL SUBSERIES
Volume 3



An Introduction to the Mathematical Theory of Waves

Roger Knobel

American Mathematical Society
Institute for Advanced Study

Chapter 4

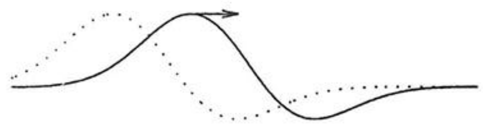
Traveling Waves

4.1. Traveling waves

One fundamental mathematical representation of a wave is

$$u(x,t) = f(x - ct)$$

where f is a function of one variable and c is a nonzero constant. The animation of such a function begins with the graph of the initial profile $u(x,0) = f(x)$. If c is positive, then the profile of $u(x,t) = f(x - ct)$ at a later time t is a translation of the initial profile by an amount ct in the positive x direction. Such a function represents a disturbance moving with constant speed c:

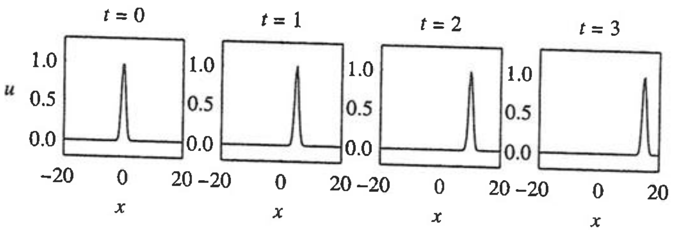


Similarly, $u(x,t) = f(x - ct)$ with $c < 0$ represents a disturbance moving in the negative x direction with speed $|c|$. In either case, the profile at each time t does not distort and remains a recognizable feature of a wave as it is translated along the x-axis.

Waves represented by functions of the form $u(x,t) = f(x - ct)$ are called **traveling waves**. The two basic features of any traveling wave are the underlying profile shape defined by f and the speed $|c|$

at which the profile is translated along the x-axis. It is assumed that the function f is not constant and c is not zero in order for $u(x,t)$ to represent the movement of a disturbance through a medium.

Example 4.1. The function $u(x,t) = e^{-(x-5t)^2}$ represents a traveling wave with initial profile $u(x,0) = e^{-x^2}$ moving in the positive x direction with speed 5. Four frames of the animation of this wave are



Example 4.2. The function $u(x,t) = \cos(2x+6t)$ can be seen to represent a traveling wave by writing it as $u(x,t) = \cos[2(x+3t)]$. The initial profile $u(x,0) = \cos(2x)$ is being displaced in the *negative* x direction with a speed of 3.

A **traveling wave solution** of a partial differential equation is a solution of the differential equation which has the form of a traveling wave $u(x,t) = f(x-ct)$. Finding traveling wave solutions generally begins by assuming $u(x,t) = f(x-ct)$ and then determining which functions f and constants c yield a solution to the differential equation.

Example 4.3. Here we will find traveling wave solutions of the wave equation

$$u_{tt} = au_{xx}, \quad a > 0 \text{ constant}.$$

Assuming that $u(x,t) = f(x-ct)$, the chain rule gives

$$\begin{aligned} u_t(x,t) &= \big[f'(x-ct)\big](x-ct)_t = -cf'(x-ct), \\ u_x(x,t) &= \big[f'(x-ct)\big](x-ct)_x = f'(x-ct). \end{aligned}$$

Applying the chain rule a second time

$$\begin{aligned} u_{tt}(x,t) &= \big[-cf''(x-ct)\big](x-ct)_t = c^2f''(x-ct), \\ u_{xx}(x,t) &= \big[f''(x-ct)\big](x-ct)_x = f''(x-ct) \end{aligned}$$

and substituting into the wave equation implies

$$c^2 f''(x-ct) = af''(x-ct).$$

Letting $z = x - ct$ and rearranging shows that we need to find c and $f(z)$ so that

$$(c^2 - a)f''(z) = 0$$

for all z.

One possibility is for $c^2 = a$. In this case f can be any twice differentiable function; taking any such nonconstant f and $c = \pm\sqrt{a}$, the two functions

$$u(x,t) = f(x - \sqrt{a}t), \quad u(x,t) = f(x + \sqrt{a}t)$$

are traveling wave solutions of the wave equation. Special examples include $u(x,t) = \sin(x - \sqrt{a}t)$, $u(x,t) = (x + \sqrt{a}t)^4$, and $u(x,t) = e^{-(x-\sqrt{a}t)^2}$. Another possibility is for $f'' = 0$, in which case f must be a linear function $f(z) = A + Bz$. The coefficient B should not be zero to ensure that the profile f is not constant. In this case

$$u(x,t) = A + B(x - ct)$$

is a traveling wave solution of the wave equation for any choice of A, B, c as long as $B \neq 0$ and $c \neq 0$.

Exercise 4.4. Find traveling wave solutions of the following equations.

(a) The advection equation $u_t + au_x = 0$ where a is a nonzero constant.

(b) The Klein-Gordon equation $u_{tt} = au_{xx} - bu$ where a and b are positive constants.

Exercise 4.5. Consider the Sine-Gordon equation $u_{tt} = u_{xx} - \sin u$.

(a) Show that the profile shape f of a traveling wave solution $u(x,t) = f(x-ct)$ of the Sine-Gordon equation must satisfy the differential equation

$$(1-c^2)f''(z) = \sin(f(z))$$

where $z = x - ct$.

(b) The differential equation in part (a) is a second order nonlinear equation. Since this equation does not explicitly involve $f'(z)$, it can be reduced to a first order equation with the following technique. Multiply both sides of the differential equation in part (a) by $f'(z)$ and integrate both sides with respect to z to show that

$$(1-c^2)\,(f'(z))^2 = A - 2\cos(f(z))$$

where A is an arbitrary constant of integration.

(c) In the special case $A = 2$ and $0 < c < 1$, show that the first order equation in part (b) can be rewritten in the form

$$(f'(z))^2 = \frac{4}{1-c^2}\sin^2(f(z)/2)\,.$$

Then verify that

$$f(z) = 4\arctan\left[\exp\left(\frac{z}{\sqrt{1-c^2}}\right)\right]$$

is a solution of this equation. Thus for any speed $0 < c < 1$,

$$u(x,t) = f(x-ct) = 4\arctan\left[\exp\left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right]$$

is a traveling wave solution of the Sine-Gordon equation.

Exercise 4.6. The previous exercise shows that

$$u(x,t) = 4\arctan\left[\exp\left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right]$$

is a traveling wave solution of the Sine-Gordon equation for any speed $0 < c < 1$. Animate this traveling wave three times using three different choices of c. How does the profile of the traveling wave change with c?

4.2. Wave fronts and pulses

A sudden change in weather occurs when a cold front passes through a region. The temperature at points ahead of the front appear to be at a relatively constant k_1 degrees, while behind the disturbance the temperature has dropped (sometimes by more than 30°F) to a new temperature k_2. On a weather map, the sudden drop in temperature

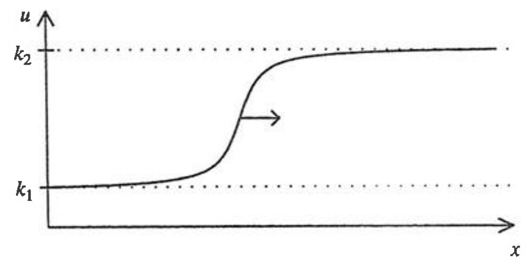


Figure 4.1. The profile of a wave front at time t.

is a recognizable feature which identifies the location and movement of this disturbance, so a cold front is an example of a wave.

A traveling wave such as the one profiled in Figure 4.1 is an example of a *wave front.* A traveling wave represented by $u(x,t)$ is said to be a **wave front** if for any fixed t,

$$u(x,t) \to k_1 \text{ as } x \to -\infty, \quad u(x,t) \to k_2 \text{ as } x \to \infty$$

for some constants k_1 and k_2. In general, the values k_1 and k_2 are not necessarily the same. In the particular case that the measure of u is approximately the same on both sides of the disturbance ($k_1 = k_2$), then the wave front is called a **pulse**. A pulse disturbance temporarily changes the value of u at position x before it settles back to its original value.

Example 4.7. The traveling wave $u(x,t) = e^{-(x-5t)^2}$ in Example 4.1 is a pulse since $\lim_{x\to\infty} e^{-(x-5t)^2} = 0$ and $\lim_{x\to-\infty} e^{-(x-5t)^2} = 0$. The traveling wave $u(x,t) = \cos(2x+6t)$ in Example 4.2 is not a wave front or a pulse since $\lim_{x\to\infty} u(x,t)$ does not exist.

Exercise 4.8. Is the traveling wave in Exercise 4.6 a wave front, pulse, or neither?

4.3. Wave trains and dispersion

The traveling wave $u(x,t) = \cos(2x+6t)$ from Example 4.2 is not a wave front or pulse, but rather an example of another type of wave.

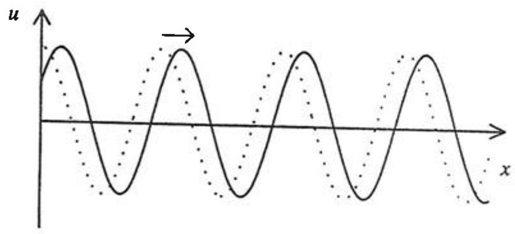


Figure 4.2. A wave train.

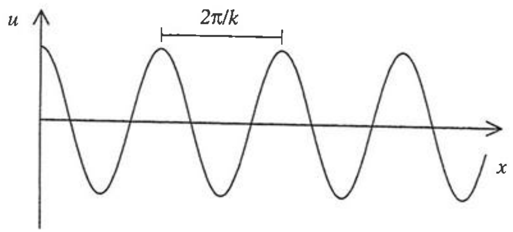


Figure 4.3. One cycle of a wave train.

A traveling wave which can be written in the form

$$u(x,t) = A\cos(kx - \omega t) \quad \text{or} \quad u(x,t) = A\cos(kx + \omega t)$$

where $A \neq 0$, $k > 0$ and $\omega > 0$ are constants is called a **wave train**. By rewriting $u(x,t)$ as

$$u(x,t) = A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]$$

one can see these are in fact traveling waves with profile shape $f(z) = A\cos(kz)$ moving with speed $c = \omega/k$ (see Figure 4.2). More generally, wave trains are represented as $u(x,t) = f(kx - \omega t)$ where $f(z)$ is a periodic function.

In a wave train $u(x,t) = A\cos(kx - \omega t)$, the number k is called the **wave number** and represents the number of cycles of this periodic wave that appear in a window of length 2π on the x–axis (Figure 4.3). The number ω is called the **circular frequency** and represents the number of cycles of the wave that pass by any fixed point x on the x–axis during a time interval of 2π.

A partial differential equation may have solutions which are wave trains, but not necessarily for every possible wave number k or frequency ω. To find which wave numbers and frequencies are permitted, one can substitute the form of a wave train such as $u(x,t) = A\cos(kx - \omega t)$ into the differential equation and reduce it to a relationship between k and ω. This relationship is called a **dispersion relation** and indicates which values of k and ω may be selected in order for $u(x,t)$ to be a wave train solution.

Example 4.9. Here we will look for wave train solutions of the form $u(x,t) = A\cos(kx - \omega t)$ for the advection equation

$$u_t + au_x = 0.$$

Computing the partial derivatives u_t and u_x of this wave train form shows $u(x,t)$ will be a solution of the advection equation if

$$\omega A\sin(kx - \omega t) + a\left[-kA\sin(kx - \omega t)\right] = 0,$$

or

$$A(\omega - ak)\sin(kx - \omega t) = 0.$$

The dispersion relation here is $\omega = ak$. Thus for any wave number k, $u(x,t) = \cos[k(x - at)]$ is a wave train solution traveling to the right with speed $c = a$.

Example 4.10. The Klein-Gordon Equation $u_{tt} = au_{xx} - bu$ (a, b positive constants) models the transverse vibration of a string with a linear restoring force. The wave train $u(x,t) = A\cos(kx - \omega t)$ is a solution of this equation if

$$-\omega^2 A\cos(kx - \omega t) = a\left[-k^2 A\cos(kx - \omega t)\right] - bA\cos(kx - \omega t)$$

or

$$A(\omega^2 - ak^2 - b)\cos(kx - \omega t) = 0.$$

Thus $u(x,t) = A\cos(kx - \omega t)$ is a solution of the Klein-Gordon equation if k and ω satisfy the dispersion relation $\omega^2 = ak^2 + b$. When $\omega = \sqrt{ak^2 + b}$, the wave train solution takes the traveling wave form

$$u(x,t) = A\cos\left(kx - \sqrt{ak^2 + b}\, t\right) = A\cos\left[k\left(x - \sqrt{\frac{ak^2 + b}{k^2}}\, t\right)\right]$$

with speed

$$c = \sqrt{\frac{ak^2+b}{k^2}} = \sqrt{a + \frac{b}{k^2}} = \sqrt{a + \frac{ab}{\omega^2 - b}}. \tag{4.1}$$

There is a fundamental difference between the previous two examples. In the advection equation, all wave train solutions travel with the same speed $c = a$. In the Klein-Gordon example, equation (4.1) shows that wave trains with higher frequency ω travel with lower speed c. A partial differential equation which has wave train solutions is said to be **dispersive** if waves trains of different frequencies ω propagate through the medium with different speeds. The Klein-Gordon equation is dispersive while the advection equation is not.

Exercise 4.11. Suppose that waves in a medium are governed by the Klein-Gordon equation. Based on (4.1), what are the possible speeds that a wave train can move through the medium? In particular, how fast and how slow can a wave train move through the medium?

Exercise 4.12. In each of the following partial differential equations, find the dispersion relation for wave train solutions of the form $u(x,t) = A\cos(kx - \omega t)$, then determine if each equation is dispersive or not. Assume a is a positive constant.

(a) $u_{tt} = au_{xx}$ The wave equation
(b) $u_{tt} + au_{xxxx} = 0$ The beam equation
(c) $u_t + u_x + u_{xxx} = 0$ The linearized KdV equation

Exercise 4.13. It is sometimes easier to find a dispersion relation using the complex wave train

$$u(x,t) = \cos(kx - \omega t) + i\sin(kx - \omega t) = e^{i(kx-\omega t)}$$

where i is the imaginary unit. In this case $u_x(x,t) = ike^{i(kx-\omega t)}$ and $u_t(x,t) = -i\omega e^{i(kx-\omega t)}$. Use this form of a wave train to find a dispersion relation for the following partial differential equations. Assume a and d are positive constants.

(a) $u_t + au_x = du_{xx}$ The linearized Burgers equation
(b) $iu_t + u_{xx} = 0$ The Schrödinger equation
(c) $u_{tt} = au_{xx}$ The wave equation

Chapter 5

The Korteweg-deVries Equation

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded smooth and well-defined heap of water which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.[1]

J.S. Russell, 1844.

[1] John Scott Russell, *Report on waves*, Report of the 14th Meeting of the British Association for the Advancement of Science, 1844, pp. 311–390.

5.1. The KdV equation

In 1834, J.S. Russell observed the phenomena of a large bulge of water slowly traveling along a channel of water. The ability of this water wave to retain its shape for such a long period of time was quite remarkable and led Russell to study this disturbance by conducting numerous detailed experiments. He later came to call this phenomena a Wave of Translation, highly suggestive of a traveling wave. Russell's work on the Wave of Translation is now considered the beginning study of what are now called *solitary waves* or *solitons.*

Russell's experiments and observations drew the attention of notable scientists such as Boussinesq, Rayleigh, and Stokes. In 1895, Korteweg and deVries derived a partial differential equation to model the height of the surface of shallow water in the presence of long gravity waves [**KdV**]. In these waves, the length of the wave is large compared to the depth of the water, as was the case in Russell's Wave of Translation. The differential equation of Korteweg and deVries,

$$U_t + (a_1 + a_2U)U_x + a_3U_{xxx} = 0, \quad a_2, a_3 \neq 0,$$

is a third order nonlinear equation now known as the Korteweg-deVries equation or **KdV equation.** A substitution of $u = a_1 + a_2U$ and a scaling of the independent variables x and t results in the reduced form of the KdV equation,

$$\boxed{u_t + uu_x + u_{xxx} = 0.} \tag{5.1}$$

For a more complete reading on the history of solitons, the KdV equation, and other fundamental equations from which solitons arise, see the monograph *Solitons in Mathematics and Physics* by Alan C. Newell [**New**].

5.2. Solitary wave solutions

In this section we will look for traveling wave solutions of the reduced KdV equation (5.1). The solutions that will be found are called *solitons* and model the wave phenomena observed by Russell.

In the spirit of a "heap" of water forming a Wave of Translation, we will look for a traveling wave solution $u(x,t) = f(x - ct)$ in the form of a pulse, where $c > 0$, and $f(z)$, $f'(z)$, and $f''(z)$ tend to 0

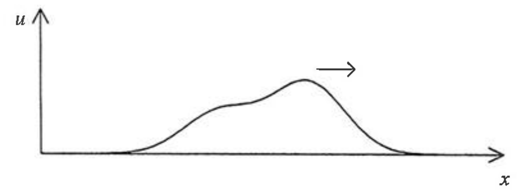


Figure 5.1. A pulse profile in which $u(x,t)$, $u_x(x,t)$, and $u_{xx}(x,t)$ approach 0 as $x \to \pm\infty$.

as $z \to \pm\infty$ (see Figure 5.1). Substituting $u(x,t) = f(x - ct)$ into the KdV equation $u_t + uu_x + u_{xxx} = 0$ forms a third order nonlinear ordinary differential equation for $f(z)$,

$$-cf' + ff' + f''' = 0.$$

This particular equation can be integrated once to get

$$-cf + \tfrac{1}{2}f^2 + f'' = a$$

where a is a constant of integration. From the assumptions that $f(z)$ and $f''(z) \to 0$ as $z \to \infty$, the value of a is zero. Multiplying by f'

$$-cff' + \tfrac{1}{2}f^2f' + f'f'' = 0$$

and integrating again results in the first order equation

$$-\tfrac{1}{2}cf^2 + \tfrac{1}{6}f^3 + \tfrac{1}{2}(f')^2 = b.$$

Since $f(z), f'(z) \to 0$ as $z \to \infty$ from the form of the pulse, the constant of integration b is zero. Solving for $(f')^2$ gives

$$3\,(f')^2 = (3c - f)f^2.$$

From here we will require $0 < f(z) < 3c$ in order to have a positive right-hand side; taking the positive square root yields

$$\frac{\sqrt{3}}{\sqrt{3c - f}\, f}\, f' = 1.$$

To integrate the left hand side, make the rationalizing substitution $g^2 = 3c - f$; substituting $f = 3c - g^2$ and $f' = -2gg'$ results in

$$\frac{2\sqrt{3}}{3c - g^2}\, g' = -1.$$

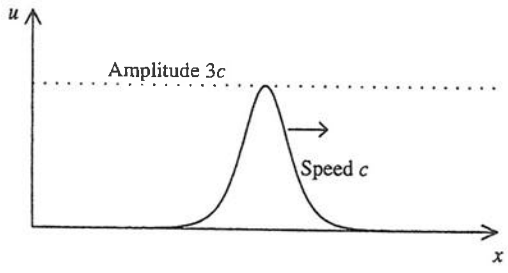


Figure 5.2. A profile of a solitary wave solution of the KdV equation.

By the method of partial fractions, integration of both sides with respect to z gives

$$\ln\left(\frac{\sqrt{3c}+g}{\sqrt{3c}-g}\right) = -\sqrt{c}z + d$$

for some constant of integration d. Solving for g yields

$$g(z) = \sqrt{3c}\,\frac{\exp(-\sqrt{c}z+d)-1}{\exp(-\sqrt{c}z+d)+1} = -\sqrt{3c}\tanh\left[\tfrac{1}{2}(\sqrt{c}z-d)\right],$$

and then computing $f = 3c - g^2$ results in

$$f(z) = 3c\,\mathrm{sech}^2\left[\tfrac{1}{2}(\sqrt{c}z-d)\right].$$

Recall that $\mathrm{sech}(z) = 1/\cosh(z)$, where $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$.

Since the arbitrary constant d is simply a shift of the shape

$$f(z) = 3c\,\mathrm{sech}^2\left[\tfrac{1}{2}\sqrt{c}z\right],$$

we can get a good idea of what this traveling wave looks like with $d = 0$. The resulting traveling wave solution to the *KdV* equation is

$$\boxed{u(x,t) = 3c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right].} \tag{5.2}$$

A profile of this wave is shown in Figure 5.2. This pulse is called a **solitary wave** or **soliton**.

Russell observed in his experiments that Waves of Translation with greater height moved with a greater velocity. This is borne out in the solution (5.2) of the KdV equation by observing that the amplitude of the wave is $3c$, three times the wave speed c.

Exercise 5.1. Find a pulse traveling wave solution of the *modified* KdV equation $u_t + u^2u_x + u_{xxx} = 0$. This equation appears in electric circuit theory and multicomponent plasmas [**IR**].

Exercise 5.2. Since the KdV equation is nonlinear, the sum of two of its solutions is not necessarily another solution. To illustrate this, let v and w represent two solutions of $u_t + uu_x + u_{xxx} = 0$. Show that $u = v + w$ is a solution only when the product vw does not depend on x.

Exercise 5.3. (Interacting Solitary Waves) Suppose k_1 and k_2 are positive numbers and set

$$\begin{aligned} u_1(x,t) &= \exp(k_1^3t - k_1x), \\ u_2(x,t) &= \exp(k_2^3t - k_2x), \\ A &= (k_1-k_2)^2/(k_1+k_2)^2. \end{aligned}$$

Let

$$u = 12\,\frac{k_1^2u_1 + k_2^2u_2 + 2(k_1-k_2)^2u_1u_2 + Au_1u_2(k_1^2u_2 + k_2^2u_1)}{(1+u_1+u_2+Au_1u_2)^2}.$$

This is a solution of $u_t + uu_x + u_{xxx} = 0$ derived using a method described in [**Whi**, pp. 580–583]. Taking $k_1 = 1$ and $k_2 = 2$, animate $u(x,t)$ for $-10 \le x \le 10$ and time $-10 \le t \le 10$ to observe the behavior of this double soliton solution. If animating using the script `wvmovie` provided with the companion MATLAB software (see page xiii), setting the $u(x,t)$ field to `kdv2(x,t)` will view this solution.

Chapter 6

The Sine-Gordon Equation

In this chapter we will derive the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

as a description of a *mechanical transmission line*, and look for traveling wave solutions of this equation.

6.1. A mechanical transmission line

In the late 1960's, A.C. Scott constructed a mechanical analogue of an electrical transmission line. This device consists of a series of pendula connected by a steel spring and supported horizontally by a thin wire (Figure 6.1). Each pendulum is free to swing in a plane perpendicular to the wire, however in doing so, the spring coils and provides a torque on the two neighboring pendula. This interaction between adjacent pendula permits a disturbance in one part of the device to propagate, mechanically transmitting a signal down the line of pendula. If a pendulum at one end of the device is disturbed slightly, then the transmitted disturbance results in a small "wavy" motion (Figure 6.2). A more dramatic effect occurs if a single pendulum at one end is quickly turned one full revolution around the wire (Figure 6.3).

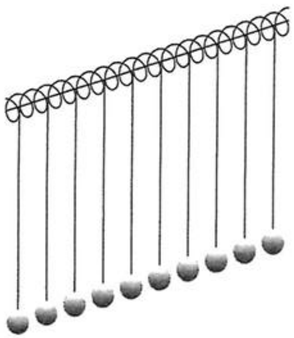


Figure 6.1. Pendula attached to a horizontal spring.

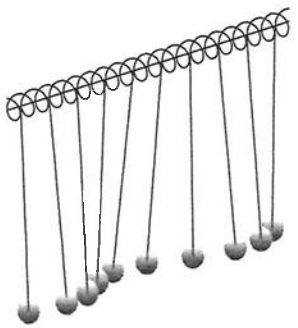


Figure 6.2. A small disturbance moving down the pendulum line.

6.2. The Sine-Gordon equation

In [Sc, pp. 48–49], the Sine-Gordon equation

$$\boxed{u_{tt} - u_{xx} + \sin u = 0} \tag{6.1}$$

is derived as a continuous model for describing motions of the pendula, where $u(x,t)$ represents the angle of rotation of the pendulum at position x and time t. The Sine-Gordon equation also arises in the study of superconductor transmission lines, crystals, laser pulses, and the geometry of surfaces. See [Sc, p. 250] for references.

We will now follow Scott's derivation of the Sine-Gordon equation (6.1) from a system of ordinary differential equations which models

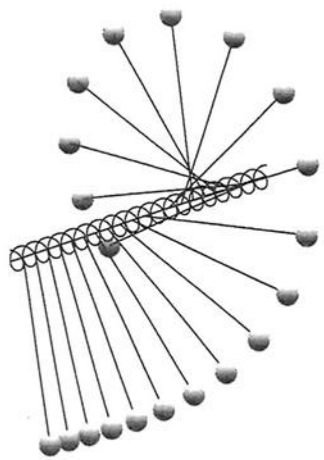


Figure 6.3. A large disturbance moving down the pendulum line.

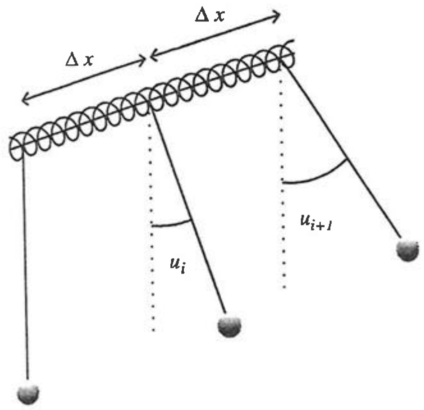


Figure 6.4

the angles of rotation of the pendula. Here we will assume that each pendulum has mass m and length l, and the pendula are equally spaced along the spring with a separation distance of Δx. Let $u_i(t)$ measure the angle of rotation of the i^{th} pendulum at time t, with $u_i = 0$ being the down position (Figure 6.4).

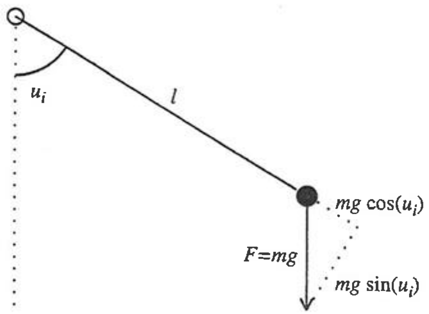


Figure 6.5

The mathematical model for the motion of the pendula is based on Newton's second law of motion in rotational form,

$$I\frac{d^2u_i}{dt^2} = \text{net torque acting on the } i^{th} \text{ pendulum} \tag{6.2}$$

where I is the moment of inertia of the pendulum, $I = ml^2$, and torque is the measure of the turning effect of a force. In this case there are three torques which will be taken into account—the torque due to gravity, the torque due to the twisting of the spring coiled between pendula i and $(i-1)$, and the torque due to the spring between pendula i and $(i+1)$.

Looking first at torque due to gravity, the gravitational force acting on the i^{th} pendulum tries to rotate the pendulum downward. As shown in Figure 6.5, the resulting torque is $\pm(mg\sin u_i)(l)$, where $mg\sin u_i$ is the amount of gravitational force perpendicular to the pendulum, l is the distance from the pivot point to the mass, and g is the acceleration due to gravity. Figure 6.5 also indicates the sign (direction) of the torque. If the pendulum has swung to the right $(0 < u_i < \pi/2)$, then $\sin u_i > 0$. The gravitational torque, however, will try to rotate the pendulum back to the left in the negative u_i direction. If the pendulum has swung to the left $(-\pi/2 < u_i < 0)$, then $\sin u_i < 0$. The gravitational torque, however, will try to rotate the pendulum back to the right in the direction of positive u_i. The portion of the net torque due to gravity is then $-mgl\sin u_i$ to account for the correct sign.

The next torque to be accounted for is the turning effect due to the portion of the spring between the i and $(i+1)$ pendula. Intuition suggests the strength of this turning effect depends on three major factors—the amount of twisting of the spring, the length of that part of the spring, and the stiffness of the spring's material. One model for this torque is

$$\text{Spring torque} = K\frac{u_{i+1} - u_i}{\Delta x}$$

where $u_{i+1} - u_i$ is the amount of twist in the part of the spring between the i and $(i+1)$ pendula, Δx is the length of that part of the spring, and $K > 0$ is a spring constant depending upon the spring's material. If $u_{i+1} - u_i = 0$, then both ends of that segment of spring have been rotated the same amount and so no twisting between pendula i and $(i+1)$ has taken place. Large values of $u_{i+1} - u_i$ correspond to one end of this segment of spring being rotated much more than the other, coiling the spring and resulting in a large torque on the i^{th} pendulum. Long sections of spring (large Δx) result in smaller torques since there are more coils of the spring to absorb twisting of the spring.

Similarly, the torque applied to the i^{th} pendulum due to the twisting of the spring between the i and $(i-1)$ pendula will be assumed to be

$$K\frac{(u_{i-1} - u_i)}{\Delta x}.$$

Putting the gravitation and spring torques in Newton's second law (6.2) results in

$$ml^2\frac{d^2u_i}{dt^2} = K\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x} - mgl\sin u_i. \tag{6.3}$$

Now suppose the number of pendula is increased while decreasing their mass in such a way that $m/\Delta x \to M$ as $\Delta x \to 0$. This forms a continuous "sheet" of material with mass density M. Let $u(x,t)$ denote the angle of rotation of this continuous sheet at position x and time t. Dividing (6.3) by another factor of Δx gives

$$\frac{ml^2}{\Delta x}\frac{d^2u_i}{dt^2} = K\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \frac{mgl}{\Delta x}\sin u_i,$$

and so taking the limit $\Delta x \to 0$ results in

$$Ml^2 u_{tt} = K u_{xx} - Mgl \sin u.$$

Setting $A = Ml^2$ and $T = Mgl$ puts this in the form of the Sine-Gordon equation

$$\boxed{A u_{tt} - K u_{xx} + T \sin u = 0.} \tag{6.4}$$

Exercise 6.1. The more general Sine-Gordon equation (6.4) can be reduced to the form (6.1) through a change of independent variables. Suppose $u(x,t)$ is a solution of

$$A u_{tt} - K u_{xx} + T \sin u = 0.$$

Let ξ and τ be a new set of independent variables formed by the scaling $\xi = ax$ and $\tau = bt$. Letting $U(\xi, \tau)$ be defined by $U(\xi, \tau) = u(x,t)$, find scaling constants a and b so that $U(\xi,\tau)$ is a solution of

$$U_{\tau\tau} - U_{\xi\xi} + \sin U = 0.$$

Exercise 6.2. The Sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$ is a special case of the more general form $u_{tt} - u_{xx} + V'(u) = 0$ where $V(u)$ represents potential energy. What is the potential energy function $V(u)$ for the Sine-Gordon equation?

6.3. Traveling wave solutions

In this section we will look for traveling wave solutions of the Sine-Gordon equation (6.1),

$$u_{tt} - u_{xx} + \sin u = 0.$$

Letting $u(x,t) = f(x - ct)$ and substituting into the Sine-Gordon equation gives

$$c^2 f'' - f'' + \sin f = 0.$$

The equation formed after multiplying by f',

$$(c^2 - 1) f'' f' + (\sin f) f' = 0,$$

can be integrated to produce the first order equation

$$\tfrac{1}{2}(c^2 - 1)(f')^2 - \cos f = a.$$

Additional conditions are needed to find the constant of integration a. With an eye towards the pendulum problem, we will look for a solution f which satisfies $f(z) \to 0$ and $f'(z) \to 0$ as $z \to \infty$ to approximate the notion of undisturbed pendula ahead of a moving disturbance. In this case $a = -1$, so

$$(f')^2 = \frac{2}{1 - c^2}(1 - \cos f) = \frac{4}{1 - c^2} \sin^2(f/2).$$

Here the speed c of the traveling wave will need to satisfy $c^2 < 1$ to ensure that the right hand side is positive. One solution of this equation is (see Exercise 4.5)

$$f(z) = 4 \arctan\left[\exp\left(-\frac{z}{\sqrt{1-c^2}}\right)\right],$$

resulting in the traveling wave solution

$$\boxed{u(x,t) = 4 \arctan\left[\exp\left(-\frac{x - ct}{\sqrt{1-c^2}}\right)\right].}$$

Four frames of animation of this traveling wave are shown in Figure 6.6. Since $u(x,t) \to 0$ as $x \to \infty$ and $u(x,t) \to 2\pi$ as $x \to -\infty$, this traveling wave is a wave front. Ahead of the wave front the pendula are in their undisturbed state (angle u near 0) while behind the wave front the pendula are near an angle of 2π, indicating that these pendula have rotated completely around the horizontal spring exactly once.

Exercise 6.3. Locate a traveling wave solution $u(x,t) = f(x - ct)$ of $u_{tt} - u_{xx} + \sin u = 0$ where $f(z) \to \pi$ and $f'(z) \to 0$ as $z \to \infty$. In terms of the pendula problem, what is a physical interpretation of this solution?

Exercise 6.4. Verify by direct substitution that the following is a solution of $u_{tt} - u_{xx} + \sin u = 0$:

$$u(x,t) = 4 \arctan\left[\frac{\sinh(ct/\sqrt{1-c^2})}{c \cosh(x/\sqrt{1-c^2})}\right].$$

Animate this solution with $c = 1/2$, $-20 \le x \le 20$, and $-50 \le t \le 50$. This solution is called a *particle-antiparticle* collision [**PS**].

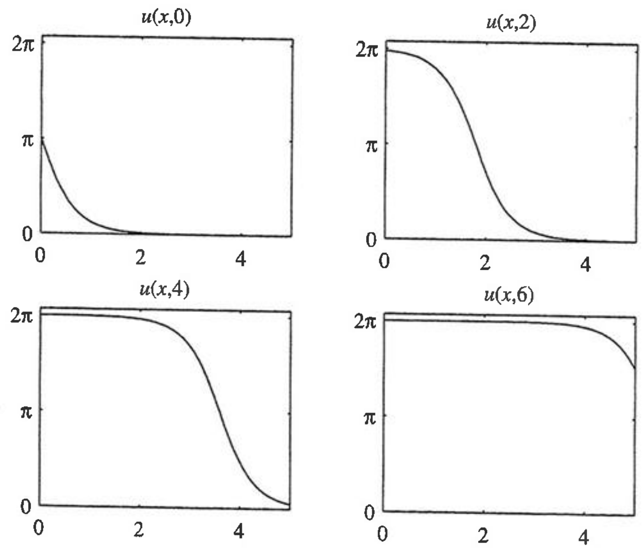


Figure 6.6. A Sine-Gordon traveling wave.

Exercise 6.5. If the motion of the pendula sheet is small (angle u remains close to zero), then one may make the approximation $\sin u \approx u$ in the Sine-Gordon equation (6.4). This results in a linear equation,

$$Au_{tt} - Ku_{xx} + Tu = 0.$$

This equation is called the Klein-Gordon equation.

(a) Find all traveling wave solutions for this linear equation.

(b) If the motion of the pendula is indeed small, then u must remain bounded. Which speeds c admit a traveling wave solution which is bounded?

(c) The bounded traveling wave solutions from part (b) are wave trains. Is the Klein-Gordon equation dispersive? In particular, do wave train solutions with high frequency travel with faster, slower, or same speed as solutions with low frequency?

(d) Show that there is a cutoff frequency ω_0 such that solutions with frequency $\omega \leq \omega_0$ are not permitted.