APPROXIMATION OF COMPLEX ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

This note is a result of a discussion with Yann Bugeaud.

Denote by $H(\xi)$ the naive height, that is the maximum of the absolute values of the coefficients of the minimal polynomial of an algebraic number ξ. Schmidt proved that for every real algebraic number $\alpha \in \mathbb{R}$ and every $\varepsilon > 0$ there are only finitely many algebraic numbers ξ of degree d such that $|\alpha - \xi| < H(\xi)^{-d-1-\varepsilon}$. For algebraic numbers $\alpha \in \mathbb{C}\backslash\mathbb{R}$ one expects a similar result but with exponent $-\frac{1}{2}(d+1) - \varepsilon$. In this note we prove such a type of result, but unfortunately we have to impose some technical condition on α.

We start with an auxiliary result. Given a linear form $L(\mathbf{X}) = \alpha_1 X_1 + \cdots + \alpha_n X_n$ with algebraic coefficients in $\mathbb{C}$, define the complex conjugate linear form $\overline{L}(\mathbf{X}) = \overline{\alpha_1} X_1 + \cdots + \overline{\alpha_n} X_n$. Further, we define the norm of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ by $\|\mathbf{x}\| := \max(|x_1|, \ldots, |x_n|)$.

Theorem 1. *Let $n \geqslant 2$. Let $L(\mathbf{X}) = \alpha_1 X_1 + \cdots + \alpha_n X_n$ be a linear form with algebraic coefficients in $\mathbb{C}$ satisfying the following technical hypothesis:*

(0.1) *For any $\mathbb{Q}$-linear subspace T of $\mathbb{Q}^n$ of dimension $> n/2$, the restrictions of $L, \overline{L}$ to T are linearly independent.*

Then for any $\varepsilon > 0$, the inequality

$$0 < |L(\mathbf{x})| < \|\mathbf{x}\|^{1-(n/2)-\varepsilon} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n \tag{0.2}$$

has only finitely many solutions.

Proof. Write $L(\mathbf{x}) = L_1(\mathbf{x}) + iL_2(\mathbf{x})$, where L_1 consists of the real parts of the coefficients of L, and L_2 of the imaginary parts. We apply Theorem 2A on p. 157 of [W.M. Schmidt, Diophantine approximation, Springer Verlag LNM 785, 1980] to L_1, L_2. Thus in Schmidt's notation, $u = 2$, $v = n - 2$. Our assumption on L implies that for every d-dimensional $\mathbb{Q}$-linear subspace T of $\mathbb{Q}^n$, the restrictions of L_1, L_2 to T have rank $\geqslant d \cdot 2/n$. This is precisely the condition to be satisfied in Schmidt's theorem. Thus, it follows that for every

$\varepsilon > 0$, the system of inequalities

$$|L_1(\mathbf{x})| < |\mathbf{x}|^{-\frac{n-2}{2}-\varepsilon}, \quad |L_2(\mathbf{x})| < |\mathbf{x}|^{-\frac{n-2}{2}-\varepsilon}$$

has only finitely many solutions in $\mathbf{x} \in \mathbb{Z}^n$. It follows that (0.2) has only finitely many solutions. □

Denote by V_d the vector space of polynomials in $\mathbb{Q}[X]$ of degree $\leqslant d$.

Theorem 2. *Let $\varepsilon > 0$. Let α be an algebraic number in $\mathbb{C}\backslash\mathbb{R}$ satisfying the following technical hypothesis:*

(0.3) *if T is any $\mathbb{Q}$-linear subspace of V_d with the property that $h_1(\alpha)h_2(\overline{\alpha}) \in \mathbb{R}$ for each pair of polynomials $h_1, h_2 \in T$, then $\dim T \leqslant (d+1)/2$.*

Then the inequality

$$|\alpha - \xi| < H(\xi)^{-\frac{1}{2}(d+1)-\varepsilon} \tag{0.4}$$

has only finitely many solutions in algebraic numbers ξ of degree d.

Proof. Denote by f the minimal polynomial of ξ (with coefficients in $\mathbb{Z}$ having gcd 1 and with positive leading coefficient). Let ξ be a solution of (0.4). Then $|f(\alpha)| \ll H(\xi)|\alpha - \xi|$ and so

$$|f(\alpha)| \ll H(f)^{1-((d+1)/2)-\varepsilon}. \tag{0.5}$$

We may view $f(\alpha)$ as a linear form on V_d in $d+1$ variables with algebraic coefficients in $\mathbb{C}$. We claim that if T is a $\mathbb{Q}$-linear subspace of V_d of dimension $> (d+1)/2$, then the restrictions of $f(\alpha)$, $f(\overline{\alpha})$ to T are linearly independent. Then by Theorem 1, inequality (0.5) has only finitely many solutions f, and this gives only finitely many possibilities for ξ.

So it remains to prove our claim. Choose a basis $\{g_1, \ldots, g_t\}$ of T. We have to show that the vectors $(g_1(\alpha), \ldots, g_t(\alpha))$, $(g_1(\overline{\alpha}), \ldots, g_t(\overline{\alpha}))$ are linearly independent. But if this is not the case, then each of the determinants $g_i(\alpha)g_j(\overline{\alpha}) - g_j(\alpha)g_i(\overline{\alpha}) = 0$, i.e., $g_i(\alpha)g_j(\overline{\alpha}) \in \mathbb{R}$ for each pair i, j. But then by $\mathbb{Q}$-linearity, $h_1(\alpha)h_2(\overline{\alpha}) \in \mathbb{R}$ for each $h_1, h_2 \in T$. By assumption (0.3) this is possible only if $t \leqslant (d+1)/2$. This proves our claim, hence Theorem 2. □

Corollary. *Let α be an algebraic number in $\mathbb{C}\backslash\mathbb{R}$ such that either*

$$[\mathbb{Q}(\alpha):\mathbb{Q}(\alpha)\cap\mathbb{R}]\geqslant\lceil\frac{1}{2}(d+3)\rceil \tag{0.6}$$

or

$$[\mathbb{Q}(\alpha)\cap\mathbb{R}:\mathbb{Q}]\leqslant[\frac{1}{2}(d+1)]\,. \tag{0.7}$$

Then for any $\varepsilon>0$, (0.4) has only finitely many solutions in algebraic numbers ξ of degree d.

Proof. We first show that there is no loss of generality to assume $[\mathbb{Q}(\alpha):\mathbb{Q}]>d+1$. Suppose that α has degree $r\leqslant d+1$, and let ξ be a non-real algebraic number of degree d. Let h,f denote the minimal polynomials of h,f, respectively. Let $\alpha_1=\alpha,\alpha_2=\overline{\alpha},\alpha_3,\ldots,\alpha_r$ denote the conjugates of α and $\xi_1=\xi,\xi_2,\xi_3,\ldots,\xi_d$ those of ξ. Suppose that ξ is not equal to a conjugate of α. Then, using some basic facts about the resultant $R(h,f)$ of h,f,

$$\begin{aligned}1 &\leqslant |R(h,f)|=M(h)^dM(f)^r\cdot\prod_{i=1}^{r}\prod_{j=1}^{d}\frac{|\alpha_i-\xi_j|}{\max(1,|\alpha_i|)\max(1,|\xi_j|)}\\ &\ll H(\alpha)^dH(\xi)^r|\alpha-\xi|\cdot|\overline{\alpha}-\overline{\xi}|=H(\alpha)^dH(\xi)^r|\alpha-\xi|^2,\end{aligned}$$

where $M(h)$, $M(f)$ denote the Mahler measures of h,f, respectively. Therefore,

$$|\alpha-\xi|\gg H(\xi)^{-r/2}$$

where the constant implied by $\gg$ depends only on α. Since $r\leqslant d+1$, this trivially implies that (0.4) has only finitely many solutions in algebraic numbers ξ of degree d.

Now assume that $[\mathbb{Q}(\alpha):\mathbb{Q}]>d+1$ and that either (0.6), or (0.7) is satisfied. We have to verify (0.3). Let T be a $\mathbb{Q}$-linear subspace of V_d such that $h_1(\alpha)h_2(\overline{\alpha})\in\mathbb{R}$ for each $h_1,h_2\in T$. Suppose T has dimension t and choose a basis $\{g_1,\ldots,g_t\}$ of T. Then $g_i(\alpha)/g_1(\alpha)=g_i(\alpha)g_1(\overline{\alpha})/|g_1(\overline{\alpha})|^2\in\mathbb{R}$ for $i=1,\ldots,t$; we know that $g_1(\alpha)\neq 0$ since α has degree $>d+1$. Further, since α has degree $>d+1$, the numbers $1,g_2(\alpha)/g_1(\alpha),\ldots,g_t(\alpha)/g_1(\alpha)$ are $\mathbb{Q}$-linearly independent elements of $\mathbb{Q}(\alpha)\cap\mathbb{R}$. Therefore, $t\leqslant[\mathbb{Q}(\alpha)\cap\mathbb{R}:\mathbb{Q}]$. So if (0.7) holds, then (0.3) is satisfied.

After applying Gauss elimination or the like to a given basis of T, we obtain a basis $\{g_1, \dots, g_t\}$ with $\deg g_1 < \deg g_2 < \cdots < \deg g_t$. Thus, $\deg g_i \leqslant d - t + i$ for $i = 1, \dots, t$. Then similarly as above, $g_2(\alpha)/g_1(\alpha) \in \mathbb{R}$, i.e., there is a $\lambda \in \mathbb{Q}(\alpha) \cap \mathbb{R}$ such that $g_2(\alpha) - \lambda g_1(\alpha) = 0$, i.e., $h(\alpha) = 0$ where h is a non-zero polynomial of degree $\leqslant d - t + 2$ with coefficients in $\mathbb{Q}(\alpha) \cap \mathbb{R}$. Now if (0.6) holds, then $d-t+2 \geqslant \lceil \frac{1}{2}(d+3) \rceil$, i.e., $t \leqslant d+2-\lceil \frac{1}{2}(d+3) \rceil = [\frac{1}{2}(d+1)]$, which again implies (0.3). Our Corollary follows. $\square$