A GENERALIZATION OF THE SUBSPACE THEOREM WITH POLYNOMIALS OF HIGHER DEGREE

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To Professor Wolfgang Schmidt on his 70th birthday

Abstract. Recently, Corvaja and Zannier [2, Theorem 3] proved an extension of the Subspace Theorem with polynomials of arbitrary degree instead of linear forms. Their result states that the set of solutions in $\mathbb{P}^n(K)$ (K number field) of the inequality being considered is not Zariski dense.

In this paper we prove, by a different method, a generalization of their result, in which the solutions are taken from an arbitrary projective variety X instead of $\mathbb{P}^n$. Further we give a quantitative version, which states in a precise form that the solutions with large height lie in a finite number of proper subvarieties of X, with explicit upper bounds for the number and for the degrees of these subvarieties (Theorem 1.3 below).

We deduce our generalization from a general result on twisted heights on projective varieties (Theorem 2.1 in Section 2). Our main tools are the quantitative version of the Absolute Parametric Subspace Theorem by Evertse and Schlickewei [5, Theorem 1.2], as well as a lower bound by Evertse and Ferretti [4, Theorem 4.1] for the normalized Chow weight of a projective variety in terms of its m-th normalized Hilbert weight.

1. Introduction

1.1. The Subspace Theorem can be stated as follows. Let K be a number field (assumed to be contained in some given algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$), n a positive integer, $0 < \delta \leqslant 1$ and S a finite set of places of K. For $v \in S$, let $L_0^{(v)}, \dots, L_n^{(v)}$ be linearly independent linear forms in $\overline{\mathbb{Q}}[x_0, \dots, x_n]$. Then

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the set of solutions $\mathbf{x} \in \mathbb{P}^n(K)$ of

$$\log\left(\prod_{v\in S}\prod_{i=0}^{n}\frac{|L_i^{(v)}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}\right) \leqslant -(n+1+\delta)h(\mathbf{x}) \tag{1.1}$$

is contained in the union of finitely many proper linear subspaces of $\mathbb{P}^n$.

Here, $h(\cdot)$ denotes the absolute logarithmic height on $\mathbb{P}^n(\overline{\mathbb{Q}})$, $|\cdot|_v$, $\|\cdot\|_v$ ($v \in S$) denote normalized absolute values on K and normalized norms on K^{n+1}, and each $|\cdot|_v$ has been extended to $\overline{\mathbb{Q}}$ (see **§1.4** below). The Subspace Theorem was first proved by Schmidt [14],[15] for the case that S consists of the archimedean places of K, and then later extended by Schlickewei [13] to the general case.

1.2. We state a generalization of the Subspace Theorem in which the linear forms $L_i^{(v)}$ are replaced by homogeneous polynomials of arbitrary degree, and in which the solutions are taken from an n-dimensional projective subvariety of $\mathbb{P}^N$ where $N \geqslant n \geqslant 1$.

By a projective subvariety of $\mathbb{P}^N$ we mean a geometrically irreducible Zariski-closed subset of $\mathbb{P}^N$. For a Zariski-closed subset X of $\mathbb{P}^N$ and for a field Ω, we denote by $X(\Omega)$ the set of Ω-rational points of X. For homogeneous polynomials $f_1,\ldots,f_r$ in the variables $x_0,\ldots,x_N$ we denote by $\{f_1=0,\ldots,f_r=0\}$ the Zariski-closed subset of $\mathbb{P}^N$ given by $f_1=0,\ldots,$ $f_r=0$.

Then our result reads as follows:

Theorem 1.1. *Let K be a number field, S a finite set of places of K and X a projective subvariety of $\mathbb{P}^N$ defined over K of dimension $n \geqslant 1$ and degree d. Let $0 < \delta \leqslant 1$. Further, for $v \in S$ let $f_0^{(v)},\ldots,f_n^{(v)}$ be a system of homogeneous polynomials in $\overline{\mathbb{Q}}[x_0,\ldots,x_N]$ such that*

$$X(\overline{\mathbb{Q}}) \cap \left\{f_0^{(v)}=0,\ldots,f_n^{(v)}=0\right\} = \emptyset \text{ for } v \in S. \tag{1.2}$$

Then the set of solutions $\mathbf{x} \in X(K)$ of the inequality

$$\log\left(\prod_{v\in S}\prod_{i=0}^{n}\frac{|f_i^{(v)}(\mathbf{x})|_v^{1/\deg f_i^{(v)}}}{\|\mathbf{x}\|_v}\right) \leqslant -(n+1+\delta)h(\mathbf{x}) \tag{1.3}$$

is contained in a finite union $\bigcup_{i=1}^{u}\Big(X\cap\{G_i=0\}\Big)$, *where* $G_1,\dots,G_u$ *are homogeneous polynomials in* $K[x_0,\dots,x_N]$ *not vanishing identically on* X *of degree at most*

$$(8n+6)(n+2)^2 d\Delta^{n+1}\delta^{-1}\quad \textit{with } \Delta := \mathrm{lcm}\big(\deg f_i^{(v)} : v\in S,\ 0\leqslant i\leqslant n\big).$$

It should be noted that if $N=n$, $X=\mathbb{P}^n$ and $f_0^{(v)},\dots,f_n^{(v)}$ are linear forms, then condition (1.2) means precisely that $f_0^{(v)},\dots,f_n^{(v)}$ are linearly independent.

We give an immediate consequence:

Corollary 1.2. *Let* $f_0,\dots,f_n$ *be homogeneous polynomials in* $\overline{\mathbb{Q}}[x_0,\dots,x_n]$ *such that*

$$\big\{\mathbf{x}\in\overline{\mathbb{Q}}^{n+1} : f_0(\mathbf{x})=\cdots=f_n(\mathbf{x})=0\big\} = \{\mathbf{0}\}.$$

Let $0<\delta\leqslant 1$. *Then the set of solutions* $\mathbf{x}=(x_0,\dots,x_n)\in\mathbb{Z}^{n+1}$ *of*

$$\prod_{i=0}^{n}|f_i(\mathbf{x})|^{1/deg f_i}\leqslant\Big(\max_{0\leqslant i\leqslant n}|x_i|\Big)^{-\delta}$$

is contained in some finite union of hypersurfaces $\{G_1=0\}\cup\cdots\cup\{G_u=0\}$, *where each* G_i *is a homogeneous polynomial in* $\mathbb{Q}[x_0,\dots,x_n]$ *of degree at most* $(8n+6)(n+2)^2\Delta^{n+1}\delta^{-1}$ *with* $\Delta := \mathrm{lcm}\big(\deg f_i : 0\leqslant i\leqslant n\big)$.

1.3. In their paper [6], Faltings and Wüstholz introduced a new method to prove the Subspace Theorem, and gave some examples showing that their method enables to prove extensions of the Subspace Theorem with higher degree polynomials instead of linear forms, and with solutions from an arbitrary projective variety. Ferretti [7],[8] observed the role of Mumford's degree of contact [10] (or the Chow weight, see **§2.3** below) in the work of Faltings and Wüstholz and worked out several other cases. Evertse and Ferretti [4] showed that the extensions of the Subspace Theorem as proposed by Faltings and Wüstholz in [6] can be deduced directly from the Subspace Theorem itself.

Recently, Corvaja and Zannier [2, Theorem 3] obtained a result similar to our Theorem 1.1 with $X=\mathbb{P}^n$. (More precisely, Corvaja and Zannier gave an essentially equivalent affine formulation, in which the polynomials

$f_i^{(v)}$ need not be homogeneous and in which the solutions $\mathbf{x}$ have S-integer coordinates). In fact, Corvaja and Zannier showed that the set of solutions of (1.3) is contained in a finite union of hypersurfaces in $\mathbb{P}^n$ and gave some further information about the structure of these hypersurfaces, on the other hand they did not provide an explicit bound for their degrees. Corvaja and Zannier stated their result only for the case $X = \mathbb{P}^n$ but with their methods this may be extended to the case that X is a complete intersection. In contrast, our result is valid for arbitrary projective subvarieties X of $\mathbb{P}^N$.

In their paper [2], Corvaja and Zannier proved also finiteness results for several classes of Diophantine equations. It is likely, that similar results can be deduced by means of our approach, but we have not gone into this.

1.4. Below we state a quantitative version of Theorem 1.1. We first introduce the necessary notation. All number fields considered in this paper are contained in a given algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Let K be a number field and denote by G_K the Galois group of $\overline{\mathbb{Q}}$ over K. For $\mathbf{x} = (x_0, \dots, x_N) \in \overline{\mathbb{Q}}^{N+1}$, $\sigma \in G_K$ we write $\sigma(\mathbf{x}) = (\sigma(x_0), \dots, \sigma(x_N))$. Denote by M_K the set of places of K. For $v \in M_K$, choose an absolute value $|.|_v$ normalized such that the restriction of $|.|_v$ to $\mathbb{Q}$ is $|.|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ if v is archimedean and $|.|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ if v lies above the prime number p. Here $|.|$ is the ordinary absolute value, and $|.|_p$ is the p-adic absolute value with $|p|_p = p^{-1}$. These absolute values satisfy the product formula $\prod_{v \in M_K} |x|_v = 1$ for $x \in K^*$.

Given $\mathbf{x} = (x_0, \dots, x_N) \in K^{N+1}$ we put $\|\mathbf{x}\|_v := \max(|x_0|_v, \dots, |x_N|_v)$ for $v \in M_K$. Then the absolute logarithmic height of $\mathbf{x}$ is defined by $h(\mathbf{x}) = \log\left(\prod_{v \in M_K} \|\mathbf{x}\|_v\right)$. By the product formula, $h(\lambda\mathbf{x}) = h(\mathbf{x})$ for $\lambda \in K^*$. Moreover, $h(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the particular number field K containing $x_0, \dots, x_N$. Thus, this function h gives rise to a height on $\mathbb{P}^N(\overline{\mathbb{Q}})$.

Given a system $f_0, \dots, f_m$ of polynomials with coefficients in $\overline{\mathbb{Q}}$ we define $h(f_0, \dots, f_m) := h(\mathbf{a})$, where $\mathbf{a}$ is a vector consisting of the non-zero coefficients of $f_0, \dots, f_m$. Further by $K(f_0, \dots, f_m)$ we denote the extension of K generated by the coefficients of $f_0, \dots, f_m$. The height of a projective subvariety X of $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$ is defined by $h(X) := h(F_X)$, where F_X is the Chow form of X (see **§2.3** below).

For every $v \in M_K$ we choose an extension of $|\cdot|_v$ to $\overline{\mathbb{Q}}$ (this amounts to extending $|\cdot|_v$ to the algebraic closure $\overline{K}_v$ of K_v and choosing an embedding of $\overline{\mathbb{Q}}$ into $\overline{K}_v$). Further for $v \in M_K$, $\mathbf{x} = (x_0, \dots, x_N) \in \overline{\mathbb{Q}}^{N+1}$ we put $\|\mathbf{x}\|_v := \max(|x_0|_v, \dots, |x_N|_v)$.

1.5. Schmidt [16] was the first to obtain a quantitative version of the Subspace Theorem, giving an explicit upper bound for the number of subspaces containing all solutions with 'large' height. Since then his basic result has been improved and generalized in various directions. Evertse and Schlickewei [5, Theorem 3.1] deduced a quantitative version of the Absolute Subspace Theorem, dealing with solutions in $\mathbb{P}^n(\overline{\mathbb{Q}})$ of some absolute extension of (1.1). Their result can be stated as follows.

Let again K be a number field, and S a finite set of places of K of cardinality s. Let $n \geqslant 1$, $0 < \delta \leqslant 1$. For $v \in S$, let $L_0^{(v)}, \dots, L_n^{(v)}$ be linearly independent linear forms in $\overline{\mathbb{Q}}[x_0, \dots, x_n]$. Put $\mathcal{D} := \prod_{v \in S} |\det(L_0^{(v)}, \dots, L_n^{(v)})|_v$ and assume that $[K(L_i^{(v)}) : K] \leqslant C$ for $v \in S$, $i = 0, \dots, n$. Then the set of $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ with

$$\log\left(\mathcal{D}^{-1} \prod_{v \in S} \prod_{i=0}^{n} \max_{\sigma \in G_K} \frac{|L_i^{(v)}(\sigma(\mathbf{x}))|_v}{\|\sigma(\mathbf{x})\|_v}\right) \leqslant -(n+1+\delta)h(\mathbf{x}),$$

$$h(\mathbf{x}) \geqslant 9(n+1)\delta^{-1}\log(n+1) + \max\bigl(h(L_i^{(v)}) : v \in S, 0 \leqslant i \leqslant n\bigr)$$

is contained in the union of not more than

$$(3n+3)^{(2n+2)s} 8^{(n+10)^2} \delta^{-(n+1)s-n-5} \log(4C) \log\log(4C)$$

proper linear subspaces of $\mathbb{P}^n(\overline{\mathbb{Q}})$ which are all defined over K.

Typically, the lower bound for $h(\mathbf{x})$ depends on the linear forms $L_i^{(v)}$, while the upper bound for the number of subspaces does not depend on the $L_i^{(v)}$.

1.6. We now state an analogue for inequalities with higher degree polynomials instead of linear forms. We first list some notation:

δ is a real with $0 < \delta \leqslant 1$, K is a number field, S is a finite set of places of K of cardinality s, X is a projective subvariety of $\mathbb{P}^N$ defined over K of dimension $n \geqslant 1$ and degree d, $f_0^{(v)}, \dots, f_n^{(v)}$ $(v \in S)$ are systems of homogeneous polynomials in $\overline{\mathbb{Q}}[x_0,\dots,x_N]$,

$$(1.4)\qquad \begin{cases} C := \max\big([K(f_i^{(v)}):K]:\, v\in S,\, i=0,\dots,n\big),\\ \Delta := \mathrm{lcm}\big(\deg f_i^{(v)}:\, v\in S,\, i=0,\dots,n\big),\end{cases}$$

$$(1.5)\qquad \begin{cases} A_1 := (20n\delta^{-1})^{(n+1)s}\cdot \exp\Big(2^{12n+16}n^{4n}\delta^{-2n}d^{2n+2}\Delta^{n(2n+2)}\Big)\cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdot\log(4C)\log\log(4C),\\ A_2 := (8n+6)(n+2)^2 d\Delta^{n+1}\delta^{-1},\\ A_3 := \exp\Big(2^{6n+20}n^{2n+3}\delta^{-n-1}d^{n+2}\Delta^{n(n+2)}\log(2Cs)\Big),\\ H := \log(2N)+h(X)+\max\big(h(1,f_i^{(v)}):\, v\in S,\, 0\leqslant i\leqslant n\big).\end{cases}$$

Theorem 1.3. *Assume that*

$$(1.2)\qquad X(\overline{\mathbb{Q}})\cap\big\{f_0^{(v)}=0,\dots,f_n^{(v)}=0\big\} = \emptyset \quad \textit{for } v\in S.$$

Then there are homogeneous polynomials $G_1,\dots,G_u\in K[x_0,\dots,x_N]$ *with*

$$u\leqslant A_1\,,\quad \deg G_i\leqslant A_2 \ \textit{ for } i=1,\dots,u$$

which do not vanish identically on X*, such that the set of* $\mathbf{x}\in X(\overline{\mathbb{Q}})$ *with*

$$(1.6)\qquad \log\left(\prod_{v\in S}\prod_{i=0}^{n}\max_{\sigma\in G_K}\frac{|f_i^{(v)}(\sigma(\mathbf{x}))|_v^{1/\deg f_i^{(v)}}}{\|\sigma(\mathbf{x})\|_v}\right)\leqslant -(n+1+\delta)h(\mathbf{x})\,,$$

$$(1.7)\qquad h(\mathbf{x})\geqslant A_3\cdot H$$

is contained in $\bigcup_{i=1}^{u}\Big(X\cap\{G_i=0\}\Big)$.

Clearly, the bounds in Theorem 1.3 are much worse than those in the result of Evertse and Schlickewei. It would be very interesting if one could replace A_1, A_3 by quantities which are at most exponential in (some power of) n and which are polynomial in δ^{-1}, d, Δ. Further, we do not know whether the dependence of A_2 on δ is needed.

1.7. Our starting point is a result for twisted heights on $\mathbb{P}^n$ (a quantitative version of the Absolute Parametric Subspace Theorem), due to Evertse and Schlickewei [5, Theorem 2.1] (see also Proposition 3.1 in Section 3 below).

From this, we deduce an analogous result for twisted heights on arbitrary projective varieties; the statement of this result is in Section 2 (Theorem 2.1) and its proof in Section 3. The proof involves some arguments from Evertse and Ferretti [4], in particular an explicit lower bound of the normalized Chow weight of a projective variety in terms of the m-th normalized Hilbert weight of that variety. In Section 4 we give some height estimates; here we use heavily Rémond's exposé [12]. Then in Section 5 we deduce Theorem 1.3. Using that $\mathbb{P}^N(K)$ has only finitely many points with height below any given bound, Theorem 1.1 follows at once from Theorem 1.3.

2. Twisted heights

2.1. The quantitative version of the Absolute Parametric Subspace Theorem of Evertse and Schlickewei mentioned in the previous section deals with a class of twisted heights defined on $\mathbb{P}^n(\overline{\mathbb{Q}})$ parametrized by a real $Q \geqslant 1$. Roughly speaking, this result states that there are a finite number of proper linear subspaces of $\mathbb{P}^n$ such that for every sufficiently large Q, the set of points in $\mathbb{P}^n(\overline{\mathbb{Q}})$ with small Q-height is contained in one of these subspaces. Theorem 2.1 stated below is an analogue in which the points are taken from an arbitrary projective variety instead of $\mathbb{P}^n$. Loosely speaking, Theorem 1.3 stated in the previous section is proved by defining a suitable finite morphism φ from X to a projective variety $Y \subset \mathbb{P}^R$ and a finite number of classes of twisted heights on Y as above, and applying Theorem 2.1 to each of these classes.

2.2. Let K be a number field. For finite extensions of K we define normalized absolute values similarly as for K. Thus, if L is a finite extension of K, w is a place of L, and v is the place of K lying below w, then

$$|x|_w = |x|_v^{d(w|v)} \text{ for } x \in K, \text{ with } d(w|v) := \frac{[L_w : K_v]}{[L : K]}, \tag{2.1}$$

where K_v, L_w denote the completions at v, w, respectively.

We denote points on $\mathbb{P}^R$ by $\mathbf{y} = (y_0, \ldots, y_R)$. For $v \in M_K$, let $\mathbf{c}_v = (c_{0v}, \ldots, c_{Rv})$ be a tuple of reals such that $c_{0v} = \cdots = c_{Rv} = 0$ for all but finitely many places $v \in M_K$ and put $\mathbf{c} = (\mathbf{c}_v : v \in M_K)$. Further, let Q be a real $\geqslant 1$. We define a twisted height on $\mathbb{P}^R(\overline{\mathbb{Q}})$ as follows. First put

$$H_{Q,\mathbf{c}}(\mathbf{y}) := \prod_{v\in M_K} \max_{0\leqslant i\leqslant R} \left(|y_i|_v Q^{c_{iv}}\right) \quad \text{for } \mathbf{y}=(y_0,\dots,y_R)\in\mathbb{P}^R(K);$$

by the product formula, this is well-defined on $\mathbb{P}^R(K)$. For any finite extension L of K we put

$$c_{iw} := c_{iv}\cdot d(w|v) \quad \text{for } w\in M_L, \tag{2.2}$$

where M_L is the set of places of L and v the place of K lying below w. Then for $\mathbf{y}\in\mathbb{P}^R(\overline{\mathbb{Q}})$, we define

$$H_{Q,\mathbf{c}}(\mathbf{y}) := \prod_{w\in M_L} \max_{0\leqslant i\leqslant R} \left(|y_i|_w Q^{c_{iw}}\right) \tag{2.3}$$

where L is any finite extension of K such that $\mathbf{y}\in\mathbb{P}^R(L)$. In view of (2.1) this definition does not depend on L.

2.3. Let Y be a (by definition irreducible) projective subvariety of $\mathbb{P}^R$ of dimension n and degree D, defined over K. We recall that up to a constant factor there is a unique polynomial $F_Y(\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)})$ with coefficients in K in blocks of variables $\mathbf{u}^{(0)}=(u_0^{(0)},\dots,u_R^{(0)}),\dots,\ \mathbf{u}^{(n)}=(u_0^{(n)},\dots,u_R^{(n)})$, called the *Chow form* of Y, with the following properties:

F_Y is irreducible over $\overline{\mathbb{Q}}$; F_Y is homogeneous in each block $\mathbf{u}^{(h)}$ ($h=0,\dots,n$); and $F_Y(\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)})=0$ if and only if Y and the hyperplanes $\sum_{i=0}^R u_i^{(h)}y_i=0$ ($h=0,\dots,n$) have a $\overline{\mathbb{Q}}$-rational point in common.

It is well-known that the degree of F_Y in each block $\mathbf{u}^{(h)}$ is D.

Let $\mathbf{c}=(c_0,\dots,c_R)$ be a tuple of reals. Introduce an auxiliary variable t and substitute $t^{c_i}u_i^{(h)}$ for $u_i^{(h)}$ in F_Y for $h=0,\dots,n$, $i=0,\dots,R$. Thus we obtain an expression

$$\begin{aligned}&F_Y(t^{c_0}u_0^{(0)},\dots,t^{c_R}u_R^{(0)};\dots;t^{c_0}u_0^{(n)},\dots,t^{c_R}u_R^{(n)})\\ &\qquad = t^{e_0}G_0(\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)})+\cdots+t^{e_r}G_r(\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)}),\end{aligned} \tag{2.4}$$

with $G_0, \dots, G_r \in K[\mathbf{u}^{(0)}, \dots, \mathbf{u}^{(n)}]$ and $e_0 > e_1 > \cdots > e_r$. Now we define the *Chow weight* of Y with respect to $\mathbf{c}$ [1] by

$$e_Y(\mathbf{c}) := e_0. \tag{2.5}$$

2.4. We formulate our main result for twisted heights. Below, Y is a projective subvariety of $\mathbb{P}^R$ of dimension $n \geqslant 1$ and degree D, defined over K, and $\mathbf{c}_v = (c_{0v}, \dots, c_{Rv})$ $(v \in M_K)$ are tuples of reals such that

$$c_{iv} \geqslant 0 \ \text{ for } v \in M_K,\ i = 0, \dots, R; \tag{2.6}$$

$$c_{0v} = \cdots = c_{Rv} = 0 \ \text{ for all but finitely many } v \in M_K; \tag{2.7}$$

$$\sum_{v \in M_K} \max(c_{0v}, \dots, c_{Rv}) \leqslant 1. \tag{2.8}$$

Put

$$E_Y(\mathbf{c}) := \frac{1}{(n+1)D} \left(\sum_{v \in M_K} e_Y(\mathbf{c}_v) \right). \tag{2.9}$$

Further, let $0 < \delta \leqslant 1$, and put

$$\begin{cases} B_1 := \exp\left(2^{10n+4}\delta^{-2n}D^{2n+2}\right) \cdot \log(4R)\log\log(4R), \\ B_2 := (4n+3)D\delta^{-1}, \\ B_3 := \exp\left(2^{5n+4}\delta^{-n-1}D^{n+2}\log(4R)\right). \end{cases} \tag{2.10}$$

Theorem 2.1. *There are homogeneous polynomials $F_1, \dots, F_t \in K[y_0, \dots, y_R]$ with*

$$t \leqslant B_1, \quad \deg F_i \leqslant B_2 \ \text{ for } i = 1, \dots, t,$$

which do not vanish identically on Y, such that for every real number Q with

$$\log Q \geqslant B_3 \cdot (h(Y) + 1)$$

[1]The Chow weight was introduced in [4], and named such because of its relation to the Chow form. It is an adaptation of the *degree of contact* earlier introduced by Mumford [10], so perhaps the naming 'Mumford weight' would have been a happier choice. Roughly speaking, the degree of contact of Y with respect to $\mathbf{c}$ is defined for integer tuples $\mathbf{c}$ and it is equal to e_r instead of e_0.

there is $F_i \in \{F_1, \dots, F_t\}$ *with*

$$\big\{\mathbf{y} \in Y(\overline{\mathbb{Q}}) : H_{Q,\mathbf{c}}(\mathbf{y}) \leqslant Q^{E_Y(\mathbf{c})-\delta}\big\} \subset Y \cap \big\{F_i = 0\big\}. \tag{2.11}$$

3. Proof of Theorem 2.1

3.1. We first recall the quantitative version of the Absolute Parametric Subspace Theorem of Evertse and Schlickewei. As before, K is an algebraic number field and R, n are integers with $R \geqslant n \geqslant 1$. We denote the coordinates on $\mathbb{P}^n$ by $(x_0, \dots, x_n)$. Given an index set $I = \{i_0, \dots, i_n\}$ with $i_0 < \cdots < i_n$ and linear forms $L_j = \sum_{i=0}^n a_{ij} x_i$ $(j \in I)$ we write $\det(L_j : j \in I) := \det(a_{i,i_j})_{i,j=0,\dots,n}$.

Let $L_0, \dots, L_R$ be linear forms in $K[x_0, \dots, x_n]$ with $\mathrm{rank}\{L_0, \dots, L_R\} = n+1$. Further, let I_v $(v \in M_K)$ be subsets of $\{0, \dots, R\}$ of cardinality $n+1$ such that

$$\mathrm{rank}\{L_i : i \in I_v\} = n+1 \quad \text{for } v \in M_K. \tag{3.1}$$

Define

$$\mathcal{H} := \prod_{v \in M_K} \max_I |\det(L_i : i \in I)|_v\,, \quad \mathcal{D} := \prod_{v \in M_K} |\det(L_i : i \in I_v)|_v\,; \tag{3.2}$$

here the maximum is taken over all subsets I of $\{0, \dots, R\}$ of cardinality $n+1$. According to [4, Lemma 7.2] we have

$$\mathcal{D} \geqslant \mathcal{H}^{1-\binom{R+1}{n+1}}. \tag{3.3}$$

Let $\mathbf{d}_v = (d_{iv} : i \in I_v)$ $(v \in M_K)$ be tuples of reals such that

$$d_{iv} = 0 \ \text{ for } i \in I_v \text{ and for all but finitely many } v \in M_K, \tag{3.4}$$

$$\sum_{v \in M_K} \sum_{i \in I_v} d_{iv} = 0, \tag{3.5}$$

$$\sum_{v \in M_K} \max(d_{iv} : i \in I_v) \leqslant 1 \tag{3.6}$$

and write $\mathbf{d} = (\mathbf{d}_v : v \in M_K)$.

We define a twisted height on $\mathbb{P}^n(\overline{\mathbb{Q}})$ as follows. For any real number $Q \geqslant 1$ we first put

$$H^*_{Q,\mathbf{d}}(\mathbf{x}) = \prod_{v \in M_K} \left(\max_{i \in I_v} |L_i(\mathbf{x})|_v Q^{-d_{iv}} \right) \quad \text{for } \mathbf{x} \in \mathbb{P}^n(K).$$

More generally, if L is any finite extension of K, put

$$d_{iw} := d(w|v) d_{iv}, \;\; I_w := I_v \tag{3.7}$$

where v is the place of K lying below w. Then for $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ we define

$$H^*_{Q,\mathbf{d}}(\mathbf{x}) = \prod_{w \in M_L} \left(\max_{i \in I_w} |L_i(\mathbf{x})|_v Q^{-d_{iw}} \right) \tag{3.8}$$

where L is any finite extension of K such that $\mathbf{x} \in \mathbb{P}^n(L)$. This is independent of the choice of L.

Now the result of Evertse and Schlickewei [5, Theorem 2.1] is as follows:

Proposition 3.1. *Let I_v $(v \in M_K)$, $\mathbf{d} = (\mathbf{d}_v : v \in M_K)$, satisfy* (3.1), (3.4)*, respectively, and let $0 < \varepsilon \leqslant 1$.*
There are proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{P}^n$, defined over K, with

$$t \leqslant 4^{(n+9)^2} \varepsilon^{-n-5} \log(3R) \log\log(3R), \tag{3.9}$$

such that for every real number Q with

$$Q \geqslant \max\left(\mathcal{H}^{1/\binom{R+1}{n+1}}, (n+1)^{2/\varepsilon} \right) \tag{3.10}$$

there is $T_i \in \{T_1, \ldots, T_t\}$ with

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : \, H^*_{Q,\mathbf{d}}(\mathbf{x}) \leqslant \mathcal{D}^{1/(n+1)} Q^{-\varepsilon}\} \subset T_i \, . \tag{3.11}$$

3.2. We recall some results from [4]. As in Section 2, we denote the coordinates on $\mathbb{P}^R$ by $(y_0, \ldots, y_R)$. Let Y be a projective variety of $\mathbb{P}^R$ defined over K of dimension n and degree D. Let I_Y be the prime ideal of Y, i.e. the ideal of polynomials from $\overline{\mathbb{Q}}[y_0, \ldots, y_R]$ vanishing identically on Y. For $m \in \mathbb{N}$, denote by $\overline{\mathbb{Q}}[y_0, \ldots, y_R]_m$ the vector space of homogeneous polynomials in $\overline{\mathbb{Q}}[y_0, \ldots, y_R]$ of degree m, and put $(I_Y)_m := \overline{\mathbb{Q}}[y_0, \ldots, y_R]_m \cap I_Y$. Then the Hilbert function of Y is defined by

$$H_Y(m) := \dim_{\overline{\mathbb{Q}}} \left(\overline{\mathbb{Q}}[y_0, \ldots, y_R]_m / (I_Y)_m \right).$$

The scalar product of $\mathbf{a} = (a_0, \dots, a_R)$, $\mathbf{b} = (b_0, \dots, b_R) \in \mathbb{R}^{R+1}$ is given by $\mathbf{a} \cdot \mathbf{b} := a_0 b_0 + \cdots + a_R b_R$. For $\mathbf{a} = (a_0, \dots, a_R) \in (\mathbb{Z}_{\geqslant 0})^{R+1}$, denote by $\mathbf{y}^{\mathbf{a}}$ the monomial $y_0^{a_0} \cdots y_R^{a_R}$. Then the m-th Hilbert weight of Y with respect to a tuple $\mathbf{c} = (c_0, \dots, c_R) \in \mathbb{R}^{R+1}$ is defined by

$$s_Y(m, \mathbf{c}) := \max \left(\sum_{i=1}^{H_Y(m)} \mathbf{a}_i \cdot \mathbf{c} \right), \tag{3.12}$$

where the maximum is taken over all sets of monomials $\{\mathbf{y}^{\mathbf{a}_1}, \dots, \mathbf{y}^{\mathbf{a}_{H_Y(m)}}\}$, whose residue classes modulo $(I_Y)_m$ form a basis of $\overline{\mathbb{Q}}[y_0, \dots, y_R]_m/(I_Y)_m$.

We recall Evertse and Ferretti [4, Theorem 4.1]:

Proposition 3.2. *Let $\mathbf{c} = (c_0, \dots, c_R)$ be a tuple of non-negative reals. Let $m > D$ be an integer. Then*

$$\tfrac{1}{mH_Y(m)} \cdot s_Y(m, \mathbf{c}) \geqslant \tfrac{1}{(n+1)D} \cdot e_Y(\mathbf{c}) - \tfrac{(2n+1)D}{m} \cdot \max(c_0, \dots, c_R). \tag{3.13}$$

Let m be a positive integer. Put

$$n_m := H_Y(m) - 1, \quad R_m := \tbinom{R+m}{m} - 1,$$

and let $\mathbf{y}^{\mathbf{a}_0}, \dots, \mathbf{y}^{\mathbf{a}_{R_m}}$ be the monomials of degree m in $y_0, \dots, y_R$, in some order. Denote by φ_m the Veronese map of degree m, $\mathbf{y} \mapsto (\mathbf{y}^{\mathbf{a}_0}, \dots, \mathbf{y}^{\mathbf{a}_{R_m}})$. Lastly, denote by Y_m the smallest linear subspace of $\mathbb{P}^{R_m}$ containing $\varphi_m(Y)$.

Lemma 3.3. (i) *Y_m is defined over K;*
(ii) $\dim Y_m = n_m \leqslant D\binom{m+n}{n}$*;*
(iii) $h(Y_m) \leqslant Dm\binom{m+n}{n}\Big(D^{-1}h(Y) + (3n+4)\log(R+1)\Big)$.

Proof. (i),(iii) [4, Lemma 8.3]; (ii) Chardin [1, Théorème 1]. □

3.3. Let $\mathbf{c}_v \in \mathbb{R}^R$ $(v \in M_K)$ be tuples with (2.6) and (2.8). For a suitable value of m, we link the twisted height $H_{Q,\mathbf{c}}$ from Theorem 2.1 to a twisted height on $\mathbb{P}^{n_m}$ to which Proposition 3.1 is applicable. Put

$$m := [(4n+3)D\delta^{-1}]. \tag{3.14}$$

Then by Proposition 3.2 and (2.6) we have

$$\frac{1}{mH_Y(m)}\cdot\left(\sum_{v\in M_K} s_Y(m,\mathbf{c}_v)\right)\geqslant\frac{1}{(n+1)D}\cdot\left(\sum_{v\in M_K} e_Y(\mathbf{c}_v)\right)-\frac{\delta}{2}\,. \tag{3.15}$$

Denote as before the coordinates on $\mathbb{P}^R$ by $\mathbf{y}=(y_0,\dots,y_R)$, those on $\mathbb{P}^{n_m}=\mathbb{P}^{H_Y(m)-1}$ by $\mathbf{x}=(x_0,\dots,x_{n_m})$, and those on $\mathbb{P}^{R_m}=\mathbb{P}^{\binom{R+m}{m}-1}$ by $\mathbf{z}=(z_0,\dots,z_{R_m})$. Since Y_m is an n_m-dimensional linear subspace of $\mathbb{P}^{R_m}$ defined over K, there are linear forms $L_0,\dots,L_{R_m}\in K[x_0,\dots,x_{n_m}]$ such that the map

$$\psi_m:\,\mathbf{x}\mapsto(L_0(\mathbf{x}),\dots,L_{R_m}(\mathbf{x}))$$

is a linear isomorphism from $\mathbb{P}^{n_m}$ to Y_m. Thus, $\psi_m^{-1}\varphi_m$ is an injective map from Y into $\mathbb{P}^{n_m}$.

For $v\in M_K$ there is a subset I_v of $\{0,\dots,R_m\}$ of cardinality $n_m+1=H_Y(m)$ such that $\{\mathbf{y}^{\mathbf{a}_i}:\,i\in I_v\}$ is a basis of $\overline{\mathbb{Q}}[y_0,\dots,y_R]_m/(I_Y)_m$ and

$$s_Y(m,\mathbf{c}_v)=\sum_{i\in I_v}\mathbf{a}_i\cdot\mathbf{c}_v\,. \tag{3.16}$$

Now define the tuples $\mathbf{d}_v=(d_{iv}\,,i\in I_v)$ $(v\in M_K)$ by

$$\begin{aligned} d_{iv} &= -\frac{1}{m}\cdot\mathbf{a}_i\cdot\mathbf{c}_v+\frac{1}{m(n_m+1)}\left(\sum_{j\in I_v}\mathbf{a}_j\cdot\mathbf{c}_v\right) \\ &= -\frac{1}{m}\cdot\mathbf{a}_i\cdot\mathbf{c}_v+\frac{1}{mH_Y(m)}\cdot s_Y(m,\mathbf{c}_v)\,, \end{aligned} \tag{3.17}$$

and put $\mathbf{d}=(\mathbf{d}_v:\,v\in M_K)$. Similarly to (3.2) we define

$$\mathcal{H}:=\prod_{v\in M_K}\max_I|\det(L_i:\,i\in I)|_v,\quad \mathcal{D}:=\prod_{v\in M_K}|\det(L_i:\,i\in I_v)|_v\,,$$

where the maximum is taken over all subsets I of $\{0,\dots,R_m\}$ of cardinality n_m+1. Then by, e.g., [4, page 1300] we have

$$\log\mathcal{H}=h(Y_m)\,. \tag{3.18}$$

We define in a usual manner a twisted height on $\mathbb{P}^{n_m}(\overline{\mathbb{Q}})$ by putting

$$H^*_{Q,\mathbf{d}}(\mathbf{x})=\prod_{w\in M_L}\max_{i\in I_w}\left(|L_i(\mathbf{x})|_w Q^{-d_{iw}}\right)$$

for $\mathbf{x} \in \mathbb{P}^{n_m}(\overline{\mathbb{Q}})$, where L is any finite extension of K such that $\mathbf{x} \in \mathbb{P}^{n_m}(L)$, $Q \geqslant 1$ is a real number, and $d_{iw} = d(w|v)d_{iv}$, $I_w = I_v$ with v the place of K below w. It follows at once from (2.7) that $d_{iv} = 0$ for all but finitely many v and for $i \in I_v$. Therefore this height is well-defined.

Lemma 3.4. *Assume that*

$$Q \geqslant \mathcal{D}^{6/\delta m(n_m+1)}. \tag{3.19}$$

Let $\mathbf{y} \in Y(\overline{\mathbb{Q}})$ *be such that*

$$H_{Q,\mathbf{c}}(\mathbf{y}) \leqslant Q^{E_Y(\mathbf{c})-\delta}, \tag{3.20}$$

where $E_Y(\mathbf{c}) = \frac{1}{(n+1)D}\left(\sum_{v\in M_K} e_Y(\mathbf{c}_v)\right)$. *Let* $\mathbf{x} = \psi_m^{-1}\varphi_m(\mathbf{y})$. *Then*

$$H^*_{Q^m,\mathbf{d}}(\mathbf{x}) \leqslant \mathcal{D}^{1/(n_m+1)}(Q^m)^{-\delta/3}. \tag{3.21}$$

Proof. Put $s_v := \frac{1}{mH_Y(m)} s_Y(m, \mathbf{c}_v)$, $s := \sum_{v\in M_K} s_v$. We first show that

$$H^*_{Q^m,\mathbf{d}}(\mathbf{x}) \leqslant Q^{-ms}\big(H_{Q,\mathbf{c}}(\mathbf{y})\big)^m. \tag{3.22}$$

Take a finite extension L of K such that $\mathbf{y} \in Y(L)$. We have $\mathbf{x} \in \mathbb{P}^{n_m}(L)$ and $L_i(\mathbf{x}) = \mathbf{y}^{\mathbf{a}_i}$ for $i = 0, \dots, R_m$. So for $w \in M_L$ we have (putting $s_w := d(w|v)s_v$, with v the place of K below w),

$$\begin{aligned}\max_{i\in I_w}\left(|L_i(\mathbf{x})|_w (Q^m)^{-d_{iw}}\right) &= \max_{i\in I_w}\left(|\mathbf{y}^{\mathbf{a}_i}|_w Q^{\mathbf{a}_i\cdot\mathbf{c}_w - ms_w}\right)\\ &\leqslant \max_{i=0,\dots,R_m}\left(|\mathbf{y}^{\mathbf{a}_i}|_w Q^{\mathbf{a}_i\cdot\mathbf{c}_w - ms_w}\right) \leqslant \left(Q^{-s_w}\max_{i=0,\dots,R}\left(|y_i|_w Q^{c_{iw}}\right)\right)^m.\end{aligned}$$

By taking the product over all $w \in M_L$, (3.22) follows.

Now a successive application of (3.19), (3.22), (3.20), (3.15) gives

$$H^*_{Q^m,\mathbf{d}}(\mathbf{x}) \leqslant \mathcal{D}^{1/(n_m+1)} Q^{m\delta/6}\cdot Q^{-ms}Q^{mE_Y(\mathbf{c})-m\delta} \leqslant \mathcal{D}^{1/(n_m+1)}(Q^m)^{-\delta/3}.$$

□

3.4. To complete the proof of Theorem 2.1 we apply Proposition 3.1 to (3.21); that is, we apply Proposition 3.1 with $n = n_m$, $R = R_m$, $\varepsilon = \delta/3$, and with Q^m in place of Q. For the moment we assume

$$\log Q \geqslant \frac{6}{(n_m+1)m\delta}(R_m+1)^{n_m+1}(h(Y_m)+1). \tag{3.23}$$

In view of (3.18), this is precisely (3.10) with $R = R_m, n = n_m, \varepsilon = \delta/3$ and with Q^m in place of Q.

We have to verify that (3.1), (3.4), (3.5), (3.6) are satisfied with n_m, R_m in place of n, R. First, (3.1) follows at once from the definition of I_v and the fact that ψ_m is a linear isomorphism. Secondly, (3.4) follows from (2.7) and (3.17). Thirdly, (3.5) follows from (3.17), (3.16). Finally, (3.6) is consequence of (2.6), (2.8) and the fact that $\frac{1}{mH_Y(m)} \cdot s_Y(m, \mathbf{c}_v)$ can be expressed as a maximum of linear forms in $c_{0v}, \dots, c_{Rv}$, whose coefficients are non-negative and have sum equal to 1.

Thus, there are proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{P}^{n_m}$, defined over K, with

$$t \leqslant 4^{(n_m+9)^2}(3/\delta)^{n_m+5}\log(3R_m)\log\log(3R_m) \tag{3.24}$$

such that for every Q with (3.23) there is $T_i \in \{T_1, \dots, T_t\}$ with

$$\{\mathbf{x} \in \mathbb{P}^{n_m}(\overline{\mathbb{Q}}) : H^*_{Q^m,\mathbf{d}}(\mathbf{x}) \leqslant \mathcal{D}^{1/(n_m+1)}(Q^m)^{-\delta/3}\} \subset T_i\,.$$

For each space T_i there is a linear form $L_i \in K[z_0, \dots, z_{R_m}]$ vanishing identically on $\psi_m(T_i)$ but not on Y_m. Since by definition, Y_m is the smallest linear subvariety of $\mathbb{P}^{R_m}$ containing $\varphi_m(Y)$, the linear form L_i does not vanish identically on $\varphi_m(Y)$. Replacing in L_i the coordinate z_j by $\mathbf{y}^{\mathbf{a}_j}$ for $j = 0, \dots, R_m$, we obtain a homogeneous polynomial $F_i \in K[y_0, \dots, y_R]$ of degree m not vanishing identically on Y such that if $\mathbf{x} = \psi_m^{-1}\varphi_m(\mathbf{y}) \in T_i$, then $F_i(\mathbf{y}) = 0$.

It is easily seen that assumption (3.23), together with (3.18) and (3.3), implies (3.19); hence Lemma 3.4 is applicable. Thus, we infer that there are homogeneous polynomials $F_1, \dots, F_t \in K[y_0, \dots, y_R]$ of degree m, with t satisfying (3.24), such that for every Q with (3.23) there is $F_i \in \{F_1, \dots, F_t\}$ with

$$\{\mathbf{y} \in Y(\overline{\mathbb{Q}}) : H_{Q,\mathbf{c}}(\mathbf{y}) \leqslant Q^{E_Y(\mathbf{c})-\delta}\} \subset Y \cap \{F_i = 0\}\,.$$

By (3.14) we have $m \leqslant (4n+3)D\delta^{-1}$, which is the quantity B_2 from (2.10). So to complete the proof of Theorem 2.1, it suffices to show that the right-hand side of (3.24) is at most B_1 and that the right-hand side of (3.23) is at most $B_3 \cdot (h(Y) + 1)$, where B_1, B_3 are given by (2.10).

Using $m \geqslant 7$ and the inequality

$$(3.25)\qquad \binom{x+y}{y} \leqslant \frac{(x+y)^{x+y}}{x^x y^y} = \Big(1+\frac{y}{x}\Big)^x \cdot \Big(1+\frac{x}{y}\Big)^y \leqslant \Big(e\big(1+\frac{x}{y}\big)\Big)^y$$

for positive integers x, y, we infer

$$(3.26)\qquad R_m = \binom{R+m}{m} - 1 \leqslant \Big(e\big(1+\frac{R}{m}\big)\Big)^m \leqslant (4R)^m \,.$$

So by (3.14),

$$\begin{aligned} \log(3R_m)\log\log(3R_m) &\leqslant 2m^2\log(4R)\log\log(4R) \\ &\leqslant 2(8n+6)^2 D^2\delta^{-2}\log(4R)\log\log(4R)\,. \end{aligned}$$

Further, by Lemma 3.3, (ii),

$$(3.27)\qquad \begin{aligned} n_m &\leqslant D\binom{m+n}{n} \leqslant D\big(e(1+\frac{m}{n})\big)^n \\ &\leqslant D\big(e(1+7D\delta^{-1})\big)^n \leqslant 2^{5n}\delta^{-n}D^{n+1}\,. \end{aligned}$$

Hence the right-hand side of (3.24) is at most

$$\begin{aligned} & 4^{(2^{5n}\delta^{-n}D^{n+1}+9)^2}(3\delta^{-1})^{2^{5n}\delta^{-n}D^{n+1}+5} \times \\ & \qquad \times 2(8n+6)^2D^2\delta^{-2}\log(4R)\log\log(4R) \\ \leqslant\ & \exp\Big(2^{10n+4}\delta^{-2n}D^{2n+2}\Big)\cdot\log(4R)\log\log(4R) = B_1\,, \end{aligned}$$

while by Lemma 3.3, (3.14), (3.26), (3.27), the right-hand side of (3.23) is at most

$$\begin{aligned} & \frac{6}{(n_m+1)m\delta}\Big((4R)^m+1\Big)^{n_m+1} \times \\ & \qquad \times\Big(1+Dm\binom{m+n}{n}\Big)\Big(D^{-1}h(Y)+(3n+4)\log(R+1)\Big) \\ \leqslant\ & \delta^{-1}\Big((4R)^{(4n+3)D\delta^{-1}}+1\Big)^{2^{5n}\delta^{-n}D^{n+1}+1} \times \\ & \qquad \times 2^{5n}\delta^{-n}D^{n+1}(3n+1)\log(R+1)\cdot(h(Y)+1) \\ <\ & \exp\Big(2^{5n+4}\delta^{-n-1}D^{n+2}\log(4R)\Big)\cdot(h(Y)+1) = B_3\cdot(h(Y)+1)\,. \end{aligned}$$

This completes the proof of Theorem 2.1. □

4. Height estimates

4.1. In this section we deduce some height estimates, using results from Rémond's paper [12].

Let K be a number field. Denote as before the set of places of K by M_K, and denote the sets of archimedean and non-archimedean places of K by M_K^∞ and M_K^0, respectively. We use the normalized absolute values $|\cdot|_v$ introduced in **§1.4**. Recall that for each of these absolute values we have chosen an extension to $\overline{\mathbb{Q}}$. In particular, for each $v \in M_K^\infty$ there is an isomorphic embedding $\sigma_v : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ such that $|x|_v = |\sigma_v(x)|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ for $x \in \overline{\mathbb{Q}}$.

We represent polynomials as $f = \sum_{\mathbf{m}\in M_f} c_f(\mathbf{m})\mathbf{m}$, where the symbol $\mathbf{m}$ denotes a monomial, M_f is a finite set of monomials, and $c_f(\mathbf{m})$ ($\mathbf{m} \in M_f$) are the coefficients. For any map σ on the field of definition of f we put $\sigma(f) := \sum_{\mathbf{m}\in M_f} \sigma(c_f(\mathbf{m}))\mathbf{m}$.

We define norms for polynomials $f_i = \sum_{\mathbf{m}\in M_{f_i}} c_{f_i}(\mathbf{m})\mathbf{m}$ ($i = 1, \ldots, r$) with complex coefficients:

$$\begin{aligned} \|f_1, \ldots, f_r\| &:= \max\left(|c_{f_i}(\mathbf{m})| : 1 \leqslant i \leqslant r,\, \mathbf{m} \in M_{f_i}\right), \\ \|f_1, \ldots, f_r\|_1 &:= \sum_{i=1}^{r} \sum_{\mathbf{m}\in M_{f_i}} |c_{f_i}(\mathbf{m})| \end{aligned}$$

and for polynomials $f_1, \ldots, f_r$ with coefficients in $\overline{\mathbb{Q}}$:

$$\begin{aligned} &\|f_1, \ldots, f_r\|_v := \max\left(|c_{f_i}(\mathbf{m})|_v : 1 \leqslant i \leqslant r,\, \mathbf{m} \in M_{f_i}\right) \ (v \in M_K), \\ &\|f_1, \ldots, f_r\|_{v,1} := \|\sigma_v(f_1), \ldots, \sigma_v(f_r)\|_1^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} \ (v \in M_K^\infty), \\ &\|f_1, \ldots, f_r\|_{v,1} := \|f_1, \ldots, f_r\|_v \ (v \in M_K^0). \end{aligned} \tag{4.1}$$

Lastly, for polynomials $f_1, \ldots, f_r$ with coefficients in K we define heights

$$\begin{aligned} h(f_1, \ldots, f_r) &:= \log\left(\prod_{v\in M_K} \|f_1, \ldots, f_r\|_v\right), \\ h_1(f_1, \ldots, f_r) &:= \log\left(\prod_{v\in M_K} \|f_1, \ldots, f_r\|_{v,1}\right). \end{aligned}$$

More generally, for polynomials $f_1, \dots, f_r$ with coefficients in $\overline{\mathbb{Q}}$ we define $h(f_1, \dots, f_r)$, $h_1(f_1, \dots, f_r)$ by choosing a number field K containing the coefficients of $f_1, \dots, f_r$ and using the above definitions; this is independent of the choice of K.

We state without proof some easy inequalities. First, for $\mathbf{x} \in \overline{\mathbb{Q}}^{n+1}$ and $f \in \overline{\mathbb{Q}}[x_0, \dots, x_n]$ homogeneous of degree D we have

$$\|f(\mathbf{x})\|_v \leqslant \|f\|_{v,1} \|\mathbf{x}\|_v^D \quad \text{for } v \in M_K. \tag{4.2}$$

Secondly, for $\mathbf{x} \in \mathbb{P}^N(\overline{\mathbb{Q}})$ and $f_0, \dots, f_r \in \overline{\mathbb{Q}}[x_0, \dots, x_N]$ homogeneous of degree D we have

$$h(\mathbf{y}) \leqslant Dh(\mathbf{x}) + h_1(f_0, \dots, f_r)\,, \tag{4.3}$$

where $\mathbf{y} = (f_0(\mathbf{x}), \dots, f_r(\mathbf{x}))$.

Thirdly, if $f \in \overline{\mathbb{Q}}[x_0, \dots, x_n]$ is homogeneous of degree D, and if $g_0, \dots, g_n \in \overline{\mathbb{Q}}[x_0, \dots, x_m]$ are homogeneous of equal degree, then for the polynomial $f(g_0, \dots, g_n)$, obtained by substituting the polynomial $g_i(x_0, \dots, x_m)$ for x_i in f for $i = 0, \dots, n$, we have

$$h_1\big(f(g_0, \dots, g_n)\big) \leqslant h_1(f) + Dh_1(g_0, \dots, g_n)\,. \tag{4.4}$$

Finally, for $f_1, \dots, f_r \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$ we have

$$h(f_1, \dots, f_r) \leqslant h_1(f_1, \dots, f_r) \leqslant h(f_1, \dots, f_r) + \log M\,, \tag{4.5}$$

where M is the number of non-zero coefficients in $f_1, \dots, f_r$.

4.2. We define another height for multihomogeneous polynomials. Given a field Ω and tuples of non-negative integers $\mathbf{l} = (l_0, \dots, l_m)$, we write $\Omega[\mathbf{l}]$ for the set of polynomials with coefficients in Ω in blocks of variables $\mathbf{z}^{(0)} = (z_0^{(0)}, \dots, z_{l_0}^{(0)}), \dots, \mathbf{z}^{(m)} = (z_0^{(m)}, \dots, z_{l_m}^{(m)})$ which are homogeneous in block $\mathbf{z}^{(h)}$ for $h = 0, \dots, m$. For $f \in \Omega[\mathbf{l}]$ we denote by $\deg_h f$ the degree of f in block $\mathbf{z}^{(h)}$.

Let

$$S(l+1) := \{(z_0, \dots, z_l) \in \mathbb{C}^{l+1} : |z_0|^2 + \cdots + |z_l|^2 = 1\}\,,$$
$$S(\mathbf{l}) := S(l_0 + 1) \times \cdots \times S(l_m + 1)\,.$$

Denote by μ_{l+1} the unique $U(l+1,\mathbb{C})$-invariant measure on $S(l+1)$ normalized such that $\mu_{l+1}(S(l+1))=1$, and let $\mu_{\mathbf{l}}=\mu_{l_0+1}\times\cdots\times\mu_{l_m+1}$ be the product measure on $S(\mathbf{l})$. Then for $f\in\mathbb{C}[\mathbf{l}]$ we set

$$(4.6)\quad m(f):=\int_{S(\mathbf{l})}\log|f(\mathbf{z}^{(0)},\dots,\mathbf{z}^{(m)})|\cdot\mu_{\mathbf{l}}+\frac{1}{2}\sum_{h=0}^{m}\deg_h f\left(\sum_{j=1}^{l_h}\frac{1}{2j}\right).$$

Given a number field K, we define for $f\in K[\mathbf{l}]$,

$$(4.7)\qquad h^*(f):=\sum_{v\in M_K^\infty}\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}m(\sigma_v(f))+\sum_{v\in M_K^0}\log\|f\|_v\,.$$

Again, this does not depend on the choice of the number field K containing the coefficients of f, so it defines a height on $\overline{\mathbb{Q}}[\mathbf{l}]$. It is not difficult to verify that

$$(4.8)\qquad h^*(f_1\cdots f_r)=\sum_{i=1}^{r}h^*(f_i)\quad\text{for } f_1,\dots,f_r\in\overline{\mathbb{Q}}[\mathbf{l}].$$

Lemma 4.1. *Let $\mathbf{l}=(l_0,\dots,l_m)$ be a tuple of non-negative integers, and $f\in\overline{\mathbb{Q}}[\mathbf{l}]$, $f\neq 0$. Then*

$$|h^*(f)-h_1(f)|\leqslant\sum_{h=0}^{m}(\deg_h f)\log(l_h+1)\,.$$

Proof. Put $A:=\prod_{h=0}^{m}(l_h+1)^{\deg_h f}$. According to the definitions of h^* and h_1, it suffices to prove that for $f\in\mathbb{C}[\mathbf{l}]$,

$$(4.9)\qquad |m(f)-\log\|f\|_1|\leqslant\log A.$$

Using $|f(\mathbf{z}^{(0)},\dots,\mathbf{z}^{(m)})|\leqslant\|f\|_1$ for $(\mathbf{z}^{(0)},\dots,\mathbf{z}^{(m)})\in S(\mathbf{l})$ we obtain at once

$$m(f)\leqslant\log\|f\|_1+\frac{1}{2}\sum_{h=0}^{m}\deg_h f\left(\sum_{j=1}^{l_h}\frac{1}{2j}\right)\leqslant\log\|f\|_1+\log A\,.$$

To prove the inequality in the other direction, write $f=\sum_{\mathbf{m}\in M_f}c(\mathbf{m})\mathbf{m}$, where the sum is over a finite number of monomials $\mathbf{m}=\prod_{h=0}^{m}\prod_{j=0}^{l_h}(z_j^{(h)})^{a_{hj}}$ with $\sum_{j=0}^{l_h}a_{hj}=\deg_h f$ for $h=0,\dots,m$. For each such monomial we put

$$\alpha(\mathbf{m}):=\prod_{h=0}^{m}\frac{(\deg_h f)!}{a_{h0}!\cdots a_{h,l_h}!}\,.$$

Then by an argument on [12, pp. 111,112],

$$\left(\sum_{\mathbf{m}\in M_f}\alpha(\mathbf{m})^{-1}|c(\mathbf{m})|^2\right)^{1/2}\leqslant A^{1/2}\exp(m(f))\,.$$

On combining this with the Cauchy-Schwarz inequality and $\sum_{\mathbf{m}}\alpha(\mathbf{m})\leqslant A$, we obtain

$$\begin{aligned}\|f\|_1 &= \sum_{\mathbf{m}\in M_f}|c(\mathbf{m})|\leqslant\left(\sum_{\mathbf{m}\in M_f}\alpha(\mathbf{m})\right)^{1/2}\cdot\left(\sum_{\mathbf{m}\in M_f}\alpha(\mathbf{m})^{-1}|c(\mathbf{m})|^2\right)^{1/2}\\ &\leqslant A\exp(m(f))\,.\end{aligned}$$

This proves $\log\|f\|_1\leqslant m(f)+\log A$, hence (4.9). □

Lemma 4.2. *Let* $f_1,\dots,f_r\in\overline{\mathbb{Q}}[\mathbf{l}]$ *and* $f=\prod_{i=1}^r f_i$. *Then*

$$h_1(f)\leqslant\sum_{i=1}^r h_1(f_i)\leqslant h_1(f)+2\sum_{h=0}^m(\deg_h f)\log(l_h+1)\,.$$

Proof. The first inequality is straightforward while the second follows from Lemma 4.1 and (4.8). □

4.3. In this subsection, X is a projective subvariety of $\mathbb{P}^N$ of dimension $n\geqslant 1$ and degree d defined over $\overline{\mathbb{Q}}$.

Let Δ be a positive integer. Denote by M_Δ the collection of all monomials of degree Δ in the variables $x_0,\dots,x_N$. Let $\mathbf{u}^{(h)}=(u_{\mathbf{m}}^{(h)}:\mathbf{m}\in M_\Delta)$ $(h=0,\dots,n)$ be blocks of variables. Up to a constant factor there is a unique, irreducible polynomial $F_{X,\Delta}\in\overline{\mathbb{Q}}[\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)}]$, called the Δ-Chow form of X, having the following property (see [11]):

$F_{X,\Delta}(\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n)})=0$ if and only if there is a $\overline{\mathbb{Q}}$-rational point in the intersection of X and the hypersurfaces $\sum_{\mathbf{m}\in M_\Delta}u_{\mathbf{m}}^{(h)}\mathbf{m}=0$ $(h=0,\dots,n)$.

Notice that $F_{X,1}$ is none other than the Chow form F_X of X. The form $F_{X,\Delta}$ corresponds to the Chow form $F_{\varphi_\Delta(X)}$ of the image of X under the Veronese embedding φ_Δ of degree Δ. It is known that $F_{X,\Delta}$ is homogeneous of degree $\Delta^n d$ in $\mathbf{u}^{(h)}$ for $h=0,\dots,n$.

For a monomial $\mathbf{m} = x_0^{a_0} \cdots x_N^{a_N}$ of degree Δ, put $\beta(\mathbf{m}) = \Delta!/a_0! \cdots a_N!$. Then the modified Chow form $G_{X,\Delta}(\mathbf{u}^{(0)}, \dots, \mathbf{u}^{(n)})$ is obtained by substituting $\beta(\mathbf{m})^{1/2} u_{\mathbf{m}}^{(h)}$ for the variable $u_{\mathbf{m}}^{(h)}$ in the polynomial $F_{X,\Delta}(\mathbf{u}^{(0)}, \dots, \mathbf{u}^{(n)})$. Notice that $G_{X,1} = F_{X,1} = F_X$. Further, using the estimates $|\beta(\mathbf{m})| \leqslant \Delta!$, $|\beta(\mathbf{m})|_p \geqslant |\Delta!|_p$ for each prime number p, one easily obtains

$$
\begin{aligned}
|h_1(F_{X,\Delta}) - h_1(G_{X,\Delta})| &\leqslant \frac{1}{2}(n+1)d\Delta^n \log(\Delta!) \\
&\leqslant \frac{1}{2}(n+1)d\Delta^{n+1} \log \Delta\,.
\end{aligned}
\tag{4.10}
$$

The following is a special case of a fundamental result of Rémond [12, Thm. 2, pp. 99,100]:

Lemma 4.3. $h^*(G_{X,\Delta}) = \Delta^{n+1} h^*(G_{X,1}) = \Delta^{n+1} h^*(F_X)$.

From this we deduce:

Lemma 4.4. $h_1(F_{X,\Delta}) \leqslant \Delta^{n+1} h(F_X) + 5(n+1)d\Delta^{n+1} \log(N+\Delta)$.

Proof. Recall that $F_{X,\Delta}$ and $G_{X,\Delta}$ are homogeneous of degree $\Delta^n d$ in each block of variables $\mathbf{u}^{(h)}$ $(h = 0, \dots, n)$ and that each of these blocks has $\binom{N+\Delta}{\Delta} \leqslant (N+\Delta)^\Delta$ variables (that is, the number of coefficients of a homogeneous polynomial of degree Δ in $N+1$ variables). So by (4.10) and Lemma 4.1,

$$
\begin{aligned}
h_1(F_{X,\Delta}) &\leqslant h_1(G_{X,\Delta}) + \frac{1}{2}(n+1)d\Delta^{n+1} \log \Delta \\
&\leqslant h^*(G_{X,\Delta}) + \frac{1}{2}(n+1)d\Delta^{n+1} \log \Delta + (n+1)d\Delta^n \log \tbinom{N+\Delta}{\Delta} \\
&\leqslant h^*(G_{X,\Delta}) + \frac{3}{2}(n+1)d\Delta^{n+1} \log(N+\Delta)\,.
\end{aligned}
$$

Then using Lemma 4.3, again Lemma 4.1 and inequality (4.5) we obtain

$$
\begin{aligned}
h_1(F_{X,\Delta}) &\leqslant \Delta^{n+1} h^*(F_X) + \frac{3}{2}(n+1)d\Delta^{n+1} \log(N+\Delta) \\
&\leqslant \Delta^{n+1} h_1(F_X) + \frac{5}{2}(n+1)d\Delta^{n+1} \log(N+\Delta) \\
&\leqslant \Delta^{n+1} h(F_X) + \frac{5}{2}(n+1)d\Delta^{n+1} \log(N+\Delta) + \Delta^{n+1} \log M\,,
\end{aligned}
$$

where M is the number of non-zero coefficients of F_X. Since F_X is a polynomial in $n+1$ blocks of $N+1$ variables, and homogeneous of degree d in each block, we have, using (3.25)

$$\begin{aligned} M &\leqslant \binom{N+d}{d}^{n+1} \leqslant \big(e(N+1)\big)^{(n+1)d} \\ &\leqslant \exp\left(\frac{5}{2}(n+1)d\log(N+\Delta)\right). \end{aligned}$$

By inserting this into the last inequality, our lemma follows. □

We arrive at the following:

Proposition 4.5. *Let $g_0,\dots,g_R$ be homogeneous polynomials of degree Δ in $\overline{\mathbb{Q}}[x_0,\dots,x_N]$ such that*

$$X(\overline{\mathbb{Q}})\cap\big\{g_0=0,\dots,g_R=0\big\} = \emptyset\,.$$

Let $Y=\varphi(X)$, where φ is the morphism on X given by $\mathbf{x}\mapsto(g_0(\mathbf{x}),\dots,g_R(\mathbf{x}))$. Then

$$\begin{aligned} h(Y) \;\leqslant\; & \Delta^{n+1}h(X)+(n+1)d\Delta^n h_1(g_0,\dots,g_R)+ \\ & +5(n+1)d\Delta^{n+1}\log(N+\Delta)+3(n+1)d\Delta^n\log(R+1)\,. \end{aligned}$$

Proof. For $j=0,\dots,R$ write y_j for $g_j(\mathbf{x})$ and denote by $\mathbf{g}_j$ the vector of coefficients of g_j, i.e., $g_j=\sum_{\mathbf{m}\in M_\Delta}c_{g_j}(\mathbf{m})\mathbf{m}$ and $\mathbf{g}_j=(c_{g_j}(\mathbf{m}):\,\mathbf{m}\in M_\Delta)$. Introduce blocks of variables $\mathbf{v}^{(h)}=(v_0^{(h)},\dots,v_R^{(h)})$ $(h=0,\dots,n)$ and define the polynomial

$$G(\mathbf{v}^{(0)},\dots,\mathbf{v}^{(n)}):=F_{X,\Delta}\Big(\sum_{j=0}^{R}v_j^{(0)}\mathbf{g}_j,\dots,\sum_{j=0}^{R}v_j^{(n)}\mathbf{g}_j\Big)\,.$$

Then $G(\mathbf{v}^{(0)},\dots,\mathbf{v}^{(n)})=0$ if and only if X and the hypersurfaces $\sum_{j=0}^{R}v_j^{(h)}g_j=0$ $(h=0,\dots,n)$ have a $\overline{\mathbb{Q}}$-rational point in common, if and only if Y and the hyperplanes $\sum_{j=0}^{R}v_j^{(h)}y_j=0$ $(h=0,\dots,n)$ have a $\overline{\mathbb{Q}}$-rational point in common, if and only if $F_Y(\mathbf{v}^{(0)},\dots,\mathbf{v}^{(n)})=0$, where F_Y is the Chow form of Y. Therefore, G is up to a constant factor equal to a power of F_Y.

Put $A:=(n+1)d\Delta^{n+1}\log(N+\Delta)$, $B:=(n+1)d\Delta^n\log(R+1)$. Notice that G has degree $d\Delta^n$ in each block $\mathbf{v}^{(h)}$. Further, by (4.4) we have

$h_1(G) \leqslant h_1(F_{X,\Delta})+(n+1)d\Delta^n h_1(g_0,\ldots,g_R)+B$. Together with Lemma 4.2, Lemma 4.1, this implies

$$\begin{aligned} h(Y) &= h(F_Y) \leqslant h_1(F_Y) \leqslant h_1(G)+2B \\ &\leqslant h_1(F_{X,\Delta})+(n+1)d\Delta^n h_1(g_0,\ldots,g_R)+3B \\ &\leqslant \Delta^{n+1}h(X)+(n+1)d\Delta^n h_1(g_0,\ldots,g_R)+5A+3B\,, \end{aligned}$$

proving our Proposition. □

5. Proof of Theorem 1.3.

5.1. We start with some auxiliary results. We denote the coordinates of $\mathbb{P}^R$ by $\mathbf{y}=(y_0,\ldots,y_R)$.

Lemma 5.1. *Let Y be a projective subvariety of $\mathbb{P}^R$ of dimension $n \geqslant 1$ and degree D, defined over $\overline{\mathbb{Q}}$. Let $\mathbf{c}=(c_0,\ldots,c_R)$ be a tuple of reals. Let $\{i_0,\ldots,i_n\}$ be a subset of $\{0,\ldots,R\}$ such that*

$$Y(\overline{\mathbb{Q}}) \cap \{y_{i_0}=0,\ldots,y_{i_n}=0\} = \emptyset\,. \tag{5.1}$$

Then

$$e_Y(\mathbf{c}) \geqslant D(c_{i_0}+\cdots+c_{i_n}). \tag{5.2}$$

Proof. For a subset $I=\{k_0,\ldots,k_n\}$ of $\{0,\ldots,R\}$ with $k_0<k_1<\cdots<k_n$, define the *bracket*

$$[I]=[I](\mathbf{u}^{(0)},\ldots,\mathbf{u}^{(n)}) := \det\left(u^{(i)}_{k_j}\right)_{i,j=0,\ldots,n},$$

where again $\mathbf{u}^{(h)}$ denotes the block of variables $(u_0^{(h)},\ldots,u_R^{(h)})$. Let $I_1,\ldots,I_S$ with $S=\binom{R+1}{n+1}$ be all subsets of $\{0,\ldots,R\}$ of cardinality $n+1$. Then the Chow form F_Y of Y can be written as a homogeneous polynomial of degree D in $[I_1],\ldots,[I_S]$:

$$F_Y=\sum_{\mathbf{a}\in A} C(\mathbf{a})[I_1]^{a_1}\cdots[I_S]^{a_S}, \tag{5.3}$$

where A is the set of tuples of non-negative integers $\mathbf{a}=(a_1,\ldots,a_S)$ with $a_1+\cdots+a_S=D$ and where $C(\mathbf{a})\in\overline{\mathbb{Q}}$ for $\mathbf{a}\in A$ [9, p. 41, Theorem IV].

For each bracket $[I]$ we have

$$[I](t^{c_0}u_0^{(0)},\dots,t^{c_R}u_R^{(0)};\dots;t^{c_0}u_0^{(n)},\dots,t^{c_R}u_R^{(n)})=t^{\sum_{i\in I}c_i}[I],$$

therefore,

$$\begin{aligned}(5.4)\qquad & F_Y(t^{c_0}u_0^{(0)},\dots,t^{c_R}u_R^{(0)};\dots;t^{c_0}u_0^{(n)},\dots,t^{c_R}u_R^{(n)})\\ &\qquad =\sum_{\mathbf{a}\in A}C(\mathbf{a})t^{\sum_{j=1}^{S}a_j(\sum_{i\in I_j}c_i)}[I_1]^{a_1}\cdots[I_S]^{a_S}\,.\end{aligned}$$

Put $\mathbf{e}_0:=(1,0,\dots,0)$, $\mathbf{e}_1:=(0,1,\dots,0)$, ..., $\mathbf{e}_R:=(0,0,\dots,1)$. Write $\{i_0,\dots,i_n\}=:I_1$. By (5.1) we have $F_Y(\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n})\neq 0$. Further,

$$[I_1](\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n})=1,\quad [I](\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n})=0 \text{ for } I\neq I_1.$$

Hence in expression (5.3) there is a term $C\cdot[I_1]^D$ with $C\in\overline{\mathbb{Q}}^*$, and if we substitute $\mathbf{u}^{(j)}=\mathbf{e}_{i_j}$ $(j=0,\dots,n)$ in (5.4) we obtain $C\cdot t^{D(c_{i_0}+\cdots+c_{i_n})}$. That is, one of the numbers e_i in (2.4) is equal to $D\,(c_{i_0}+\cdots+c_{i_n})$. This implies (5.2) at once. □

In addition, we need the following combinatorial lemma, which is a consequence of [3, Lemma 4].

Lemma 5.2. *Let θ be a real with $0<\theta\leqslant\frac{1}{2}$ and let q be a positive integer. Then there exists a set $\mathcal{W}$ of cardinality at most $(e/\theta)^{q-1}$, consisting of tuples $(c_1,\dots,c_q)$ of non-negative reals with $c_1+\cdots+c_q=1$, with the following property:*
for every set of reals $A_1,\dots,A_q$ and Λ with $A_j\leqslant 0$ for $j=1,\dots,q$ and $\sum_{j=1}^q A_j\leqslant-\Lambda$, there exists a tuple $(c_1,\dots,c_q)\in\mathcal{W}$ such that

$$A_j\leqslant -c_j(1-\theta)\Lambda \text{ for } j=1,\dots,q.$$

5.2. In what follows, K is a number field, S a finite set of places of K, and X, N, n, d, s, C, $f_i^{(v)}$ $(v\in S,\ i=0,\dots,n)$, C, Δ, A_1, A_2, A_3, H are as in Theorem 1.3. We denote the coordinates on $\mathbb{P}^N$ by $\mathbf{x}=(x_0,\dots,x_N)$.

Let $f_0,\dots,f_R$ be the distinct polynomials among $\sigma(f_j^{(v)})$ $(v\in S,\ j=0,\dots,n,\ \sigma\in G_K)$. Then by (1.4),

$$(5.5)\qquad R\leqslant C(n+1)s-1\,.$$

Let K' be the extension of K generated by the coefficients of $f_0,\dots,f_R$. Put $g_i := f_i^{\Delta/\deg f_i}$ for $i=0,\dots,R$. Thus, $g_0,\dots,g_R$ are homogenenous polynomials in $K'[x_0,\dots,x_N]$ of degree Δ. Define

$$\varphi:\ \mathbf{x}\mapsto (g_0(\mathbf{x}),\dots,g_R(\mathbf{x})),\quad Y:=\varphi(X)\,.$$

By assumption (1.2), φ is a finite morphism on X, and Y is a projective subvariety of $\mathbb{P}^R$ defined over K'. We have

$$\dim Y = n,\quad \deg Y =: D \leqslant d\Delta^n\,. \tag{5.6}$$

We denote places on K' by v' and define normalized absolute values $|\cdot|_{v'}$ on K' similarly to **§1.4**. Further, for every $v'\in M_{K'}$ we choose an extension of $|\cdot|_{v'}$ to $\overline{\mathbb{Q}}$. Since K'/K is a normal extension, for every $v'\in M_{K'}$ there is $\tau_{v'}\in G_K$ such that

$$|x|_{v'} = |\tau_{v'}(x)|_v^{1/g(v)}\quad \text{for } x\in\overline{\mathbb{Q}} \tag{5.7}$$

where $v\in M_K$ is the place below v' and $g(v)$ is the number of places of K' lying above v. For each $v'\in M_{K'}^{\infty}$ there is an isomorphic embedding $\sigma_{v'}:\ K'\hookrightarrow\mathbb{C}$ such that $|x|_{v'} = |\sigma_{v'}(x)|^{[K_{v'}:\mathbb{R}]/[K':\mathbb{Q}]}$ for $x\in\overline{\mathbb{Q}}$. We define norms $\|\cdot\|_{v'}$, $\|\cdot\|_{v',1}$ for polynomials similarly as in (4.1), with $K',v',\sigma_{v'}$ in place of K,v,σ_v.

5.3. For later purposes we estimate from above $h_1(1,g_0,\dots,g_R)$ and $h(Y)$. By a straightforward computation we have for $v'\in M_{K'}^{\infty}$,

$$\begin{aligned}
&\|1,\sigma_{v'}(g_0),\dots,\sigma_{v'}(g_R)\|_1\\
&\quad = 1+\sum_{i=0}^{R}\|\sigma_{v'}(g_i)\|_1 \leqslant 1+\sum_{i=0}^{R}\|\sigma_{v'}(f_i)\|_1^{\Delta/\deg f_i}\\
&\quad \leqslant 1+\sum_{i=0}^{R}\left(\binom{\deg f_i+N}{\deg f_i}\|\sigma_{v'}(f_i)\|\right)^{\Delta/\deg f_i}\\
&\quad \leqslant (R+2)(N+\Delta)^{\Delta}\|1,\sigma_{v'}(f_0),\dots,\sigma_{v'}(f_R)\|^{\Delta}\\
&\quad \leqslant (R+2)(N+\Delta)^{\Delta}\prod_{i=0}^{R}\|1,\sigma_{v'}(f_i)\|^{\Delta}\,.
\end{aligned}$$

So for $v' \in M_{K'}^{\infty}$ we have

$$\|1, g_0, \ldots, g_R\|_{v',1} \leqslant \big((R+2)(N+\Delta)^{\Delta}\big)^{\frac{[K'_{v'}:\mathbb{R}]}{[K':\mathbb{Q}]}} \cdot \prod_{i=0}^{R} \|1, f_i\|_{v'}^{\Delta}.$$

In an easier manner one obtains for $v' \in M_{K'}^{0}$,

$$\|1, g_0, \ldots, g_R\|_{v',1} \leqslant \prod_{i=0}^{R} \|1, f_i\|_{v'}^{\Delta}.$$

So by taking the product over $v' \in M_{K'}$, substituting (5.5), and using that polynomials with conjugate sets of coefficients have the same height,

$$\begin{aligned} h_1(1, g_0, \ldots, g_R) &\leqslant \Delta\big(\sum_{i=0}^{R} h(1, f_i)\big) + \Delta \log\big((R+2)(N+\Delta)^{\Delta}\big) \\ &\leqslant \Delta C\big(\sum_{v\in S}\sum_{j=0}^{n} h(1, f_j^{(v)})\big) + \Delta\log(N+\Delta) + \log(3Cns)\,, \end{aligned}$$

and by inserting this estimate into Proposition 4.5 we infer

$$\begin{aligned} h(Y) \leqslant \Delta^{n+1}h(X) + (n+1)d\Delta^{n+1}C\sum_{v\in S}\sum_{j=0}^{n} h(1, f_j^{(v)}) + \\ + 6(n+1)d\Delta^{n+1}\log(N+\Delta) + 4(n+1)d\Delta^{n}\log(3Cns)\,. \end{aligned}$$

A straightforward computation gives the more tractable estimates

$$h_1(g_0, \ldots, g_R) \leqslant 6\Delta^2 Cns \cdot H\,, \tag{5.8}$$

$$h(Y) \leqslant 25n^2 d\Delta^{n+2} Cs \cdot H\,, \tag{5.9}$$

where H is defined by (1.5).

5.4. We reduce (1.6) to a finite number of systems of inequalities, and then show that each such system leads to an inequality involving a twisted height.

Let $\mathbf{x} \in X(\overline{\mathbb{Q}})$ be a solution of (1.6). For $v \in S$, let I_v be the subset of $\{0, \ldots, R\}$ such that $\{f_j^{(v)} : j = 0, \ldots, n\} = \{f_i : i \in I_v\}$. Put $G_v := \|1, g_0, \ldots, g_R\|_{v,1}$ for $v \in S$. Then

$$\sum_{v\in S}\sum_{i\in I_v} \log\left(\max_{\sigma\in G_K} \frac{|g_i(\mathbf{x})|_v}{G_v\|\sigma(\mathbf{x})\|_v^{\Delta}}\right) \leqslant -(n+1+\delta)\Delta h(\mathbf{x})\,.$$

By (4.2), the terms in the sum are $\leqslant 0$. We apply Lemma 5.2 with $q = (n+1)s$ and $\theta = \frac{\delta}{(2n+2+2\delta)} = 1 - \frac{n+1+\delta/2}{n+1+\delta}$. We infer that there is a set $\mathcal{W}$ with

$$\#\mathcal{W} \leqslant \left(\frac{e(2n+2+2\delta)}{\delta}\right)^{(n+1)s-1} \leqslant (17n\delta^{-1})^{(n+1)s-1} \tag{5.10}$$

consisting of tuples of non-negative reals $(c_{iv} : v \in S,\, i \in I_v)$ with

$$\sum_{v\in S}\sum_{i\in I_v} c_{iv} = 1\,, \tag{5.11}$$

such that for every solution $\mathbf{x} \in X(\overline{\mathbb{Q}})$ of (1.6) there is a tuple $(c_{iv} : v \in S,\, i \in I_v) \in \mathcal{W}$ with

$$\log\left(\max_{\sigma\in G_K} \frac{|g_i(\sigma(\mathbf{x}))|_v}{G_v \cdot \|\sigma(\mathbf{x})\|_v^{\Delta}}\right) \;\leqslant\; -c_{iv}\left(n+1+\frac{\delta}{2}\right)\Delta h(\mathbf{x}) \qquad (v \in S,\, i \in I_v). \tag{5.12}$$

Denote by S' the set of places of K' lying above the places in S. Notice that each element of G_K acts as a permutation on $g_0, \ldots, g_R$. Let $v' \in S'$. Write v for the place of K lying below v' and let $\tau_{v'} \in G_K$ be given by (5.7). Then we define $I_{v'} \subset \{0, \ldots, R\}$, $c_{i,v'}$ $(i \in I_{v'})$ by

$$\begin{aligned} &\{g_i : i \in I_{v'}\} = \{\tau_{v'}^{-1}(g_j) : j \in I_v\} \quad \text{for } v' \in S', \\ &c_{i,v'} := c_{jv}/g(v) \quad \text{for } v' \in S',\, i \in I_{v'}, \end{aligned}$$

where $j \in I_v$ is the index such that $g_i = \tau_{v'}^{-1}(g_j)$. Further, we put

$$G_{v'} := \|1, g_0, \ldots, g_R\|_{v',1} \quad \text{for } v' \in M_{K'}.$$

Then in view of (5.7), we can rewrite system (5.12) as

$$\log\left(\max_{\sigma\in G_K} \frac{|g_i(\sigma(\mathbf{x}))|_{v'}}{G_{v'} \cdot \|\sigma(\mathbf{x})\|_{v'}^{\Delta}}\right) \;\leqslant\; -c_{i,v'}\left(n+1+\frac{\delta}{2}\right)\Delta h(\mathbf{x}) \qquad (v' \in S',\, i \in I_{v'}). \tag{5.13}$$

Invoking (5.10), (5.11) we obtain the following:

Lemma 5.3. *There is a set $\mathcal{W}'$ of cardinality at most $(17n\delta^{-1})^{(n+1)s-1}$, consisting of tuples of non-negative reals $(c_{i,v'} : v' \in S', i \in I_{v'})$ with*

$$\sum_{v\in S'}\sum_{i\in I_{v'}} c_{i,v'} = 1\,, \tag{5.14}$$

with the property that for every $\mathbf{x} \in X(\overline{\mathbb{Q}})$ with (1.6) *there is a tuple in $\mathcal{W}'$ such that $\mathbf{x}$ satisfies* (5.13).

We consider the solutions of a fixed system (5.13). Put

$$\begin{aligned} c_{i,v'} = 0 \quad & \text{for } v' \in S',\, i \in \{0,\dots,R\}\backslash I_{v'} \\ & \text{and } v' \in M_{K'}\backslash S',\, i = 0,\dots,R \end{aligned} \tag{5.15}$$

and put $\mathbf{c}_{v'} := (c_{0,v'},\dots,c_{R,v'})$ for $v' \in M_{K'}$, $\mathbf{c} := (\mathbf{c}_{v'} : v' \in M_{K'})$. Denote by $\mathbf{y} = (y_0,\dots,y_R)$ the coordinates of $\mathbb{P}^R$. We define $H_{Q,\mathbf{c}}(\mathbf{y})$, $E_Y(\mathbf{c})$ similarly as (2.3), (2.9), respectively, but with K' in place of K.

Lemma 5.4. *Let $\mathbf{x} \in X(\overline{\mathbb{Q}})$ be a solution of* (5.13) *satisfying* (1.7) *and let $\sigma \in G_K$. Put*

$$\mathbf{y} := \varphi(\sigma(\mathbf{x})), \quad Q := \exp\Big((n+1+\delta/2)\Delta h(\mathbf{x})\Big).$$

Then

$$H_{Q,\mathbf{c}}(\mathbf{y}) \leqslant Q^{E_Y(\mathbf{c}) - \frac{\delta}{2(n+2)^2}}\,. \tag{5.16}$$

Proof. We first estimate from below $E_Y(\mathbf{c})$. Let $v' \in S'$ and write $I_{v'} = \{i_0,\dots,i_n\}$. From assumption (1.2), and from the fact that X is defined over K and that $g_{i_0},\dots,g_{i_n}$ are conjugate over K to powers of $f_0^{(v)},\dots,f_n^{(v)}$ where $v \in S$ is the place below v', it follows that $X(\overline{\mathbb{Q}})\cap\big\{g_{i_0}=0,\dots,g_{i_n}=0\big\} = \emptyset$. Since $Y = \varphi(X)$, for $\mathbf{y} \in Y(\overline{\mathbb{Q}})$ there is $\mathbf{x} \in X(\overline{\mathbb{Q}})$ with $y_i = g_i(\mathbf{x})$ for $i = 0,\dots,R$. Hence

$$Y(\overline{\mathbb{Q}}) \cap \big\{y_{i_0}=0,\dots,y_{i_n}=0\big\} = \emptyset\,.$$

Now Lemma 5.1 implies

$$\frac{1}{(n+1)D}\cdot e_Y(\mathbf{c}_{v'}) \geqslant \frac{1}{n+1}(c_{i_0,v'}+\cdots+c_{i_n,v'}) = \frac{1}{n+1}\cdot\sum_{i\in I_{v'}} c_{i,v'}\,.$$

This holds for $v' \in S'$. For $v' \notin S'$ we have $e_Y(\mathbf{c}_{v'}) = 0$ by (5.15). By summing over $v' \in S'$ and using (5.14), we arrive at

$$E_Y(\mathbf{c}) \geqslant \frac{1}{n+1}\,. \tag{5.17}$$

Now let $\mathbf{x} \in X(\overline{\mathbb{Q}})$ be a solution of (5.13) with (1.7) and let $\sigma \in G_K$. Then $\sigma(\mathbf{x})$ is also a solution of (5.13). In fact, by (5.15), $\sigma(\mathbf{x})$ satisfies (5.13) for $v \in M_K$, $i = 0, \ldots, R$. Write $\mathbf{y} = \varphi(\sigma(\mathbf{x}))$ so that $y_i = g_i(\sigma(\mathbf{x}))$ for $i = 0, \ldots, R$. Let L be a finite normal extension of K' such that $\sigma(\mathbf{x}) \in X(L)$. Pick $w \in M_L$ and let v' be the place of K' below w. Then there is $\tau_w \in \mathrm{Gal}(\overline{\mathbb{Q}}/K')$ such that $|x|_w = |\tau_w(x)|_{v'}^{d(w|v')}$ for $x \in L$, where $d(w|v') = [L_w : K'_{v'}]/[L : K']$. Hence for $i = 0, \ldots, R$, with the usual notation $c_{iw} = d(w|v')c_{i,v'}$,

$$\begin{aligned} |y_i|_w Q^{c_{iw}} &= |g_i(\sigma(\mathbf{x}))|_w Q^{c_{iw}} = \left(|g_i(\tau_w\sigma(\mathbf{x}))|_{v'} Q^{c_{i,v'}}\right)^{d(w|v')} \\ &\leqslant \left(G_{v'} \|\tau_w\sigma(\mathbf{x})\|_{v'}^{\Delta}\right)^{d(w|v')} = G_{v'}^{d(w|v)} \|\sigma(\mathbf{x})\|_w^{\Delta}\,. \end{aligned}$$

By taking the product over $w \in M_L$ and using $h(\sigma(\mathbf{x})) = h(\mathbf{x})$ we obtain

$$H_{Q,\mathbf{c}}(\mathbf{y}) \leqslant \exp(h_1(1, g_0, \ldots, g_R)) \cdot Q^{\frac{1}{n+1+\delta/2}}\,.$$

Now (5.16) follows by observing that by (5.17), assumption (1.7), and (5.8),

$$\begin{aligned} &\left(E_Y(\mathbf{c}) - \frac{\delta}{2(n+2)^2} - \frac{1}{n+1+\delta/2}\right) \log Q \\ &\geqslant \left(\frac{1}{n+1} - \frac{\delta}{2(n+2)^2} - \frac{1}{n+1+\delta/2}\right) \log Q \\ &= \frac{\delta(4n+6-\delta(n+1))}{4(n+1)(n+2)^2} \cdot \Delta h(\mathbf{x}) \geqslant \frac{\delta\Delta}{2(n+2)^2} A_3 H \\ &\geqslant 6\Delta^2 CnsH \geqslant h_1(1, g_0, \ldots, g_R)\,. \end{aligned}$$

□

5.5. We finish the proof of Theorem 1.3. We apply Theorem 2.1 with K', $\frac{\delta}{2(n+2)^2}$ in place of K, δ and, in view of (5.5) and (5.6), with $D \leqslant d\Delta^n$ and $R = C(n+1)s - 1$. Notice that by (5.14),(5.15), the conditions (2.6), (2.7), (2.8) (with K' in place of K) are satisfied. Denote by B_1', B_2', B_3' the quantities obtained by substituting $\frac{\delta}{2(n+2)^2}$ for δ, $C(n+1)s - 1$ for R, and $d\Delta^n$ for D in the quantities B_1, B_2, B_3, respectively, defined by (2.10). Recall that if $\mathbf{x}$ satisfies (1.7) then Lemma 5.4 is applicable. Moreover,

$$\begin{aligned}
\log Q &= \left(n+1+\frac{\delta}{2}\right)\Delta h(\mathbf{x}) \geqslant A_3 H \\
&= \exp\left(2^{6n+20}n^{2n+3}\delta^{-n-1}d^{n+2}\Delta^{n(n+2)}\log(2Cs)\right)\cdot H \\
&\geqslant \exp\left(2^{5n+4}(2(n+2)^2\delta^{-1})^{n+1}(d\Delta^n)^{n+2}\log(4C(n+1)s)\right)\cdot \\
&\qquad \cdot\left(26n^2 d\Delta^{n+2}Cs\right)\cdot H \\
&= B_3'\cdot\left(26n^2 d\Delta^{n+2}Cs\right)\cdot H \geqslant B_3'(h(Y)+1)\,,
\end{aligned}$$

where the last inequality follows from (5.9). Hence Theorem 2.1 is applicable.

Now Theorem 2.1 and Lemma 5.4 imply that there are homogeneous polynomials $F_1,\dots,F_t \in K'[y_0,\dots,y_R]$ not vanishing identically on Y, with $t \leqslant B_1'$ and $\deg F_i \leqslant B_2'$ for $i=1,\dots,t$, with the property that for every solution $\mathbf{x}\in X(\overline{\mathbb{Q}})$ of (5.13) with (1.7), there is $F_i \in \{F_1,\dots,F_t\}$ such that $F_i(\varphi(\sigma(\mathbf{x})))=0$ for every $\sigma\in G_K$. (In fact, taking $Q=\exp\big((n+1+\delta/2)\Delta h(\mathbf{x})\big)$ it follows from Theorem 2.1 that there is F_i with $F_i(\mathbf{y})=0$ for every $\mathbf{y}\in Y(\overline{\mathbb{Q}})$ with $H_{Q,\mathbf{c}}(\mathbf{y})\leqslant Q^{E_Y(\mathbf{c})-\delta/2(n+2)^2}$, and then by Lemma 5.4 this holds in particular for all points $\mathbf{y}=\varphi(\sigma(\mathbf{x}))$, $\sigma\in G_K$.)

This means that $\tilde{F}_i(\sigma(\mathbf{x}))=0$ for $\sigma\in G_K$, where $\tilde{F}_i$ is the polynomial obtained by substituting g_j for y_j in F_i for $j=0,\dots,R$. Notice that $\tilde{F}_i\in K'[x_0,\dots,x_N]$, $\deg\tilde{F}_i\leqslant B_2'\Delta$, and that $\tilde{F}_i$ does not vanish identically on X. Write $\tilde{F}_i=\sum_{k=1}^M \omega_k\tilde{F}_{ik}$ where $\omega_1,\dots,\omega_M$ is a K-basis of K', and the $\tilde{F}_{ik}$ are polynomials with coefficients in K. We can choose $G_i\in\{\tilde{F}_{ik}: k=1,\dots,M\}$ which does not vanish identically on X. Now $\sigma(\tilde{F}_i)(\mathbf{x})=0$ for $\sigma\in G_K$. Since the polynomials $\tilde{F}_{ik}$ are linear combinations of the polynomials $\sigma(\tilde{F}_i)$ ($\sigma\in G_K$) it follows that $\tilde{F}_{ik}(\mathbf{x})=0$ for $k=1,\dots,M$, so in particular $G_i(\mathbf{x})=0$.

It follows that there are homogeneous polynomials $G_1,\dots,G_t\in K[x_0,\dots,x_N]$ with $t\leqslant B_1'$ and $\deg G_i\leqslant B_2'\Delta$ for $i=1,\dots,t$, not vanishing identically on X, such that the set of $\mathbf{x}\in X(\overline{\mathbb{Q}})$ with (5.13) and with (1.7) is contained in $\bigcup_{i=1}^t\Big(X\cap\{G_i=0\}\Big)$.

According to Lemma 5.3, there are at most $T := (17n\delta^{-1})^{(n+1)s-1}$ different systems (5.13), such that every solution $\mathbf{x} \in X(\overline{\mathbb{Q}})$ of (1.6) satisfies one of these systems. Consequently, there are homogeneous polynomials $G_1, \dots, G_u \in K[x_0, \dots, x_N]$ not vanishing identically on X, with $u \leqslant B_1'T$ and with $\deg G_i \leqslant B_2'\Delta$ for $i = 1, \dots, u$, such that the set of $\mathbf{x} \in X(\overline{\mathbb{Q}})$ with (1.6), (1.7) is contained in $\bigcup_{i=1}^u \Big(X \cap \{G_i = 0\}\Big)$.

Now the proof of Theorem 1.3 is completed by observing that in view of (2.10),

$$B_2'\Delta = (4n+3)(d\Delta^n)(2(n+2)^2\delta^{-1})\Delta = (8n+6)(n+2)^2 d\Delta^{n+1}\delta^{-1} = A_2$$

and

$$\begin{aligned} B_1'T &\leqslant \exp\left(2^{10n+4}(2(n+2)^2)^{2n}\delta^{-2n}(d\Delta^n)^{2n+2}\right) \cdot \\ &\qquad \cdot \log(4(n+1)Cs)\log\log(4(n+1)Cs) \cdot \Big(17n\delta^{-1}\Big)^{(n+1)s-1} \\ &\leqslant \exp\left(2^{12n+16}n^{4n}\delta^{-2n}d^{2n+2}\Delta^{n(2n+2)}\right) \cdot \\ &\qquad \cdot (20n\delta^{-1})^{(n+1)s} \cdot \log(4C)\log\log(4C) \\ &= A_1 . \end{aligned}$$

□

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