THE SUBSPACE THEOREM AND TWISTED HEIGHTS

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ABSTRACT. In 1994, FALTINGS and WÜSTHOLZ [4] gave an entirely new proof of Schmidt's Subspace Theorem. They introduced several new concepts, among which so-called Harder-Narasimhan filtrations for vector spaces, and pointed out that these play a crucial role in a more refined analysis of the Subspace Theorem.

In this paper, we work out the refined analysis of the Subspace Theorem as suggested by Faltings and Wüstholz. However, we do not use their Diophantine approximation techniques but instead deduce everything from the Parametric Subspace Theorem by SCHLICKEWEI and the author [3], dealing with a parametrized class of twisted heights.

The main purpose of this paper is to integrate some results from the papers by Faltings and Wüstholz [4] and SCHMIDT [12]. We do not introduce essentially new ideas. We deduce some refinements of existing results concerning the Subspace Theorem. The hard core of our arguments is a limit result for the successive minima of twisted heights as mentioned above which may be of some independent interest. A special case of this limit result was proved earlier by FUJIMORI [5, Theorem 2.8].

1. INTRODUCTION

We fix some notation. Let K be a number field, M_K the set of places of K and $\{|\cdot|_v : v \in M_K\}$ the corresponding normalized absolute values, given by requiring that if v lies above ∞ then the restriction of $|\cdot|_v$ to $\mathbb{Q}$ is $|\cdot|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ where $|\cdot|$ is the standard absolute value, while if v lies above a prime p, then the restriction of $|\cdot|_v$ to $\mathbb{Q}$ is $|\cdot|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$, where $|\cdot|_p$ is the p-adic absolute value with $|p|_p = p^{-1}$. Here $\mathbb{Q}_p$, K_v

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denote the completions. These absolute values satisfy the product formula $\prod_{v\in M_K}|x|_v=1$ for $x\in K^*$. For $\mathbf{x}=(x_0,\dots,x_N)\in K^{N+1}$, $v\in M_K$, we put $\|\mathbf{x}\|_v:=\max(|x_0|_v,\dots,|x_N|_v)$. Lastly, for $\mathbf{x}\in K^{N+1}$ or $\mathbf{x}\in\mathbb{P}^N(K)$ we define the absolute height $H(\mathbf{x}):=\prod_{v\in M_K}\|\mathbf{x}\|_v$.

The Subspace Theorem states that if S is a finite set of places of K, $\{L_{0v},\dots,L_{Nv}\}$ ($v\in S$) are linearly independent sets of linear forms in $K[X_0,\dots,X_N]$ and $\delta>0$, then the set of solutions of

$$\prod_{v\in S}\prod_{i=0}^{N}\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}\leqslant H(\mathbf{x})^{-N-1-\delta}\quad\text{in }\mathbf{x}\in\mathbb{P}^N(K)\tag{1.1}$$

is contained in the union of finitely many proper linear subspaces of $\mathbb{P}^N(K)$. The Subspace Theorem was proved by SCHMIDT [10], [11] in the case that S consists of only archimedean places, and later extended by SCHLICKEWEI [8] to arbitrary sets of places S. Their proofs do not give a method to determine the subspaces effectively.

In 1989, VOJTA [13] proved the following refinement of the Subspace Theorem: There are a finite collection $\{T_1,\dots,T_t\}$ of proper linear subspaces of $\mathbb{P}^N(K)$ which is independent of δ, and a finite set F that may depend on δ, such that the set of solutions of (1.1) is contained in $T_1\cup\cdots\cup T_t\cup F$. In 1993, SCHMIDT [12] gave a rather different proof of Vojta's refinement. Both Vojta and Schmidt deduced their result from the basic Subspace Theorem mentioned above. Vojta's proof contains an effective procedure to determine the spaces $T_1,\dots,T_t$, whereas Schmidt's proof does not. On the other hand, Schmidt's proof gives information not provided by Vojta's. Neither with Vojta's proof nor with Schmidt's it is possible to determine the finite set F effectively or even to determine its cardinality.

By a standard combinatorial argument (see, e.g., [3, §21]) one can reduce inequality (1.1) to a finite number of systems of inequalities

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}\leqslant H(\mathbf{x})^{-c_{iv}}\quad(v\in S,\ i=0,\dots,N)\quad\text{in }\mathbf{x}\in\mathbb{P}^N(K),\tag{1.2}$$
$$\text{where }\sum_{v\in S}\sum_{i=0}^{N}c_{iv}>N+1\,.$$

Thus, an equivalent formulation of the Subspace Theorem is that the set of solutions of (1.2) is contained in the union of finitely many proper linear subspaces of $\mathbb{P}^N(K)$.

In 1994, Faltings and Wüstholz [4] gave another proof of the Subspace Theorem, based on Faltings' Product Theorem, in which they focused on (1.2) instead of (1.1). Faltings and Wüstholz proved the following refinement of Vojta's result: There are a proper linear subspace T of $\mathbb{P}^N(K)$ and a finite set F such that the set of solutions of (1.2) is contained in $T \cup F$. In their proof, Faltings and Wüstholz introduced the notion of *Harder-Narasimhan filtration* for vector spaces with a finite number of weighted filtrations, analogous to an already existing notion for vector bundles. As it turns out, the exceptional subspace T arises from the Harder-Narasimhan filtration related to system (1.2). This Harder-Narasimhan filtration, and hence T, can be chosen from a finite, effectively determinable collection which is independent of the exponents c_{iv} in (1.2). The set F may depend on the exponents c_{iv}. The proof of Faltings and Wüstholz does not give a method to determine F effectively or to estimate its cardinality.

In the present paper, we refine the result of Faltings and Wüstholz on (1.2). Under some mild conditions on the exponents c_{iv} (roughly speaking, the real vector $(c_{iv} : v \in S,\ i = 0, \ldots, N)$ should lie either in the interior or exterior of a particular polytope but not on its boundary) we show that it can be decided effectively whether or not (1.2) has only finitely solutions. Moreover, in case that (1.2) has infinitely many solutions, we show that the set of solutions of (1.2) is contained in some union $T \cup F$, where F is finite, and T is a proper linear subspace of $\mathbb{P}^N(K)$ such that the solutions of (1.2) lying in T are in fact Zariski dense in T. Our proof has the same drawback as the others mentioned above that it does not enable to determine F effectively or to estimate its cardinality.

In our proof, we do not use the method of Faltings and Wüstholz, but instead derive everything from the *Parametric Subspace Theorem* of Schlickewei and the author [3]. The latter can be stated as follows. Let d_{iv} ($v \in S$, $i = 0, \ldots, N$) be reals with $\sum_{v\in S} \sum_{i=0}^{N} d_{iv} = 0$. For $Q \in \mathbb{R}_{\geqslant 1}$ define the twisted height on $\mathbb{P}^N(K)$,

$$H_Q(\mathbf{x}) := \prod_{v\in S} \max_{0\leqslant i\leqslant N} \left(|L_{iv}(\mathbf{x})|_v Q^{d_{iv}}\right) \cdot \prod_{v\notin S} \|\mathbf{x}\|_v.$$

Then for every $\delta > 0$ there are $Q_1 > 1$ and finitely many proper linear subspaces $T_1, \dots, T_t$ of $\mathbb{P}^N(K)$ such that for every $Q > Q_1$ there is $T_i \in \{T_1, \dots, T_t\}$ for which

$$\{\mathbf{x} \in \mathbb{P}^N(K) :\ H_Q(\mathbf{x}) \leqslant Q^{-\delta}\} \subset T_i.$$

By applying this with $Q = H(\mathbf{x})$, $d_{iv} = c_{iv} - \frac{1}{N+1}\Big(\sum_{l=0}^N c_{lv}\Big)$ ($v \in S$, $i = 0, \dots, N$) and $\delta = \frac{1}{N+1}\Big(\sum_{v\in S}\sum_{l=0}^N c_{lv}\Big) - 1$, one obtains that the solutions of (1.2) with $H(\mathbf{x}) > Q_1$ lie in only finitely many proper linear subspaces of $\mathbb{P}^N(K)$, and by Northcott's Theorem, the remaining solutions lie in finitely many subspaces as well. In fact, Schlickewei and the author proved a quantitative version, with an explicit value for Q_1 and an explicit upper bound for t. We need only the qualitative version. We mention that the qualitative version of the above Parametric Subspace Theorem was not explicitly stated before [3], but it was implicit in various earlier papers. Schlickewei [9] was the first to formulate in a special case, a quantitative version of a Parametric Subspace Theorem, but for a parametrized class of parallelepipeds instead of twisted heights. Roy and Thunder [7] developed 'absolute' geometry of numbers for twisted heights, i.e., over the algebraic closure of $\mathbb{Q}$. To our knowledge, a twisted height like above was used for the first time by Dubois [2]. In fact, he introduced a function field analogue of our twisted height, and used this to prove a function field analogue of the Subspace Theorem.

From the Parametric Subspace Theorem we deduce a result which describes the limit behaviour of $\log \lambda_j(Q)/\log Q$ as $Q \to \infty$, where $\lambda_j(Q)$ ($j = 1, \dots, N+1$) denotes the j-th successive minimum of H_Q, i.e., the smallest λ such that the linear subspace spanned by the points $\mathbf{x} \in \mathbb{P}^N(K)$ with $H_Q(\mathbf{x}) \leqslant \lambda$ has (projective) dimension at least $j-1$. Our result generalizes a theorem of Fujimori [5, Theorem 2.8]. Our limit result on the successive minima $\lambda_j(Q)$ will be the main tool in our proof of the refinement of the result of Faltings and Wüstholz.

In our arguments we heavily use ideas from Schmidt [12].

2. Harder-Narasimhan filtrations for vector spaces.

We have collected from [4] some facts about Harder-Narasimhan filtrations for vector spaces which are needed in the statements and proofs of our results.

Let K be a field, V a finite dimensional K-vector space, and S a finite index set. Denote by $\mathrm{span}\{L_1,\dots,L_r\}$ the linear subspace of V generated by $L_1,\dots,L_r$. Let

$$\mathcal{L}=(L_{iv}:\ v\in S,\, i=0,\dots,N_v),\quad \mathbf{c}=(c_{iv}:\ v\in S,\, i=0,\dots,N_v)$$

be a tuple of elements of V and a tuple of reals, respectively, such that

$$\left.\begin{array}{l}\mathrm{rank}(L_{0v},\dots,L_{N_v,v})=\dim V\\ \text{or }\ \min(c_{0v},\dots,c_{N_v,v})\geqslant 0\end{array}\right\}\text{ for } v\in S \tag{2.1}$$

and

$$c_{0v}\geqslant\cdots\geqslant c_{N_v,v}\text{ for } v\in S. \tag{2.2}$$

Define the linear subspaces of V,

$$V_{-1,v}:=(\mathbf{0}),\ \ V_{iv}:=\mathrm{span}\{L_{0v},\dots,L_{iv}\}\text{ for } i=0,\dots,N_v.$$

We define the *weight* of a linear subspace U of V with respect to $(\mathcal{L},\mathbf{c})$ by

$$\begin{aligned} w_{\mathcal{L},\mathbf{c}}(U) &:= \sum_{v\in S}\sum_{i=0}^{N_v} c_{iv}\big(\dim(V_{iv}\cap U)-\dim(V_{i-1,v}\cap U)\big)\\ &= \sum_{v\in S}\sum_{l\geqslant 1} c_{i_{lv},v}\end{aligned} \tag{2.3}$$

where i_{lv} is the smallest index i such that $\dim V_{iv}\cap U=l$ for $v\in S$, $l=1,2,\dots$. Further, we define the *slope* of a non-zero linear subspace U of V with respect to $(\mathcal{L},\mathbf{c})$ by

$$\mu_{\mathcal{L},\mathbf{c}}(U):=\frac{w_{\mathcal{L},\mathbf{c}}(U)}{\dim U}. \tag{2.4}$$

We have

$$w_{\mathcal{L},\mathbf{c}}(U_1+U_2)+w_{\mathcal{L},\mathbf{c}}(U_1\cap U_2)\geqslant w_{\mathcal{L},\mathbf{c}}(U_1)+w_{\mathcal{L},\mathbf{c}}(U_2) \tag{2.5}$$

for any two linear subspaces U_1, U_2 of V. This can be proved by rewriting $w_{\mathcal{L},\mathbf{c}}(U)$ as

$$w_{\mathcal{L},\mathbf{c}}(U) = \sum_{v\in S}\Big(\sum_{i=0}^{N_v-1}(c_{iv}-c_{i+1,v})\dim(V_{iv}\cap U) + +c_{N_v,v}\dim(V_{N_v,v}\cap U)\Big), \tag{2.6}$$

and using (2.2), (2.1) (i.e., $c_{N_v,v}\geqslant 0$ if $V_{N_v,v}\subsetneq V$), and

$$\dim(W_1+W_2)+\dim(W_1\cap W_2) = \dim W_1 + \dim W_2,$$
$$W\cap(W_1+W_2)\supseteq (W\cap W_1)+(W\cap W_2)$$

for any linear subspaces W_1, W_2, W of V.

Let U_1 be a linear subspace of V. For $L\in V$ we denote by $\overline{L}$ the image of L under the canonical map $V\to V/U_1$, and further, $\overline{\mathcal{L}} := (\overline{L}_{iv} : v\in S,\ i=0,\ldots,N_v)$. Then by a straightforward computation, we have for any linear subspace U of V with $U\supseteq U_1$,

$$\begin{aligned} &w_{\overline{\mathcal{L}},\mathbf{c}}(U/U_1)\\ &= \sum_{v\in S}\sum_{i=0}^{N_v} c_{iv}\dim\big((V_{iv}+U_1)\cap U/(V_{i-1,v}+U_1)\cap U\big)\\ &= w_{\mathcal{L},\mathbf{c}}(U) - w_{\mathcal{L},\mathbf{c}}(U_1)\end{aligned} \tag{2.7}$$

and for any linear subspace U of V with $U\supsetneq U_1$,

$$\begin{aligned}\mu_{\overline{\mathcal{L}},\mathbf{c}}(U/U_1) &= \frac{w_{\mathcal{L},\mathbf{c}}(U)-w_{\mathcal{L},\mathbf{c}}(U_1)}{\dim U-\dim U_1}\\ &= \frac{\mu_{\mathcal{L},\mathbf{c}}(U)\dim U-\mu_{\mathcal{L},\mathbf{c}}(U_1)\dim U_1}{\dim U-\dim U_1}.\end{aligned} \tag{2.8}$$

Henceforth we write $w(U)$, $\mu(U)$, $w(U/U_1)$, $\mu(U/U_1)$ for $w_{\mathcal{L},\mathbf{c}}(U)$, $\mu_{\mathcal{L},\mathbf{c}}(U)$, $w_{\overline{\mathcal{L}},\mathbf{c}}(U/U_1)$, $\mu_{\overline{\mathcal{L}},\mathbf{c}}(U/U_1)$.

The vector space V is called *semistable* with respect to $(\mathcal{L},\mathbf{c})$ if

$$\mu(U)\leqslant\mu(V) \text{ for every linear subspace } U\neq(\mathbf{0}) \text{ of } V. \tag{2.9}$$

In case that V is not semistable, we have the following:

Lemma 2.1. *There is a unique filtration* $(\mathbf{0}) = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} V_2 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_r = V$ *of* V *(called the Harder-Narasimhan filtration of* V *with respect to* $(\mathcal{L}, \mathbf{c})$*) with the following properties:*

(i) $\mu(U) \leqslant \mu(V_1)$ *for every non-zero linear subspace* U *of* V*, and* $\mu(U) < \mu(V_1)$ *if* $U \not\subseteq V_1$*.*

(ii) For $i = 2, \ldots, r$ *we have* $\mu(U/V_{i-1}) \leqslant \mu(V_i/V_{i-1})$ *for every non-zero linear subspace* U *of* V *with* $U \underset{\neq}{\supset} V_{i-1}$*, and* $\mu(U/V_{i-1}) < \mu(V_i/V_{i-1})$ *if* $U \not\subseteq V_i$*.*

(iii) $\mu(V_1) > \mu(V_2/V_1) > \cdots > \mu(V/V_{r-1})$*.*

Proof. The filtration is clearly uniquely determined by (i),(ii).

(i) Let μ_0 be the maximum of the numbers $\mu(U)$, for all non-zero linear subspaces U of V. If U_1, U_2 are linear subspaces of V with $\mu(U_1) = \mu(U_2) = \mu_0$, then by (2.5) we have

$$\begin{aligned}\mu(U_1 + U_2) &\geqslant \frac{\mu_0 \dim U_1 + \mu_0 \dim U_2 - \mu(U_1 \cap U_2)\dim(U_1 \cap U_2)}{\dim(U_1 + U_2)} \\ &\geqslant \mu_0\end{aligned}$$

and so $\mu(U_1 + U_2) = \mu_0$ by the maximality of μ_0. Now take for V_1 the sum of all linear subspaces U of V with $\mu(U) = \mu_0$. Then V_1 satisfies (i). Further, by (2.8),

$$\mu(U/V_1) < \mu(V_1) \tag{2.10}$$

for every linear subspace U of V_1 with $U \underset{\neq}{\supset} V_1$.

(ii),(iii) By applying (i) with $\overline{\mathcal{L}}$ instead of $\mathcal{L}$, where $\overline{\mathcal{L}}$ consists of the images of the elements of $\mathcal{L}$ under $V \to V/V_1$, we obtain a linear subspace V_2/V_1 of V/V_1 satisfying the condition of (ii). By (2.10) we have $\mu(V_2/V_1) < \mu(V_1)$. Similarly, we obtain a linear subspace V_3/V_2 of V/V_2 with the properties specified in (ii), etc. □

Let again V_1 be the first non-zero vector space in the Harder-Narasimhan filtration of V with respect to $(\mathcal{L}, \mathbf{c})$. Define the integers $d := \dim V_1$ and

$d_{iv} := \dim(V_1 \cap V_{iv})$ $(v \in S,\ i = 0, \dots, N_v)$. If U is any linear subspace of V with

$$\dim U = d, \quad \dim U \cap V_{iv} = d_{iv}\ (v \in S,\ i = 0, \dots, N_v) \tag{2.11}$$

then $\mu(U) = \mu(V_1)$. From Lemma 2.1 it then follows that $U = V_1$. That is, V_1 is the unique linear subspace U of V satisfying (2.11). The numbers d, d_{iv} can be chosen from a finite collection independent of $\mathbf{c}$ and the spaces V_{iv} can be chosen from a finite collection independent of $\mathbf{c}$ as well. Hence V_1 can be chosen from a finite collection independent of $\mathbf{c}$.

The vector space V_1 can be determined effectively (in principle) by encoding the unknown vector space U in (2.11) in terms of its Plücker coordinates, translating (2.11) into a system of algebraic equations, and determine the unique solution to this system by means of effective elimination theory. Another approach to determine V_1 effectively in principle is by using an argument of VOJTA [13]. More precisely, by applying [13, Theorem 5.10] to (2.6) one can show that V_1 can be obtained by starting with the vector spaces $K \cdot L_{iv}$ $(v \in S,\ i = 0, \dots, N_v)$, and then repeatedly taking the intersection or the sum of two previously obtained subspaces of V. Moreover, the number of intersections or sums that have to be taken to obtain V_1 can be bounded above by an effectively computable number depending only on $\dim V$ and the number of elements of $\mathcal{L}$. Thus, the vector space V_1 can be found by checking a finite, effectively computable list. Needless to say that neither of these approaches to determine V_1 is of any practical use. Except for some special cases allowing an ad hoc approach, we do not know of any really useful general method to determine V_1.

Repeating the above argument for $V_2, V_3, \dots$, it follows that the Harder-Narasimhan filtration of V with respect to $(\mathcal{L}, \mathbf{c})$ can be chosen from a finite collection independent of $\mathbf{c}$ and that it can be determined effectively in principle.

3. A refinement of the Subspace Theorem.

Let k be a number field and K a finite Galois extension of k. Thus, for every place $v \in M_K$ and each $\sigma \in \mathrm{Gal}(K/k)$ there is a place v^σ such that $|\sigma(x)|_v = |x|_{v^\sigma}$ for $x \in K$. For any integer $N > 0$, let

$$V^{(N)} := \{\alpha_0 X_0 + \cdots + \alpha_N X_N \,:\, \alpha_0, \ldots, \alpha_N \in K\}, \tag{3.1}$$

i.e., the $(N+1)$-dimensional K-vector space of linear forms in $N+1$ variables. For $L = \sum_{i=0}^{N} \alpha_i X_i \in V^{(N)}$, $\sigma \in \mathrm{Gal}(K/k)$, we put $\sigma(L) := \sum_{i=0}^{N} \sigma(\alpha_i) X_i$.

Let $N > 0$. We consider systems of inequalities slightly more general than (1.2). Let S be a finite set of places of K and

$$\mathcal{L} = (L_{iv} \,:\, v \in S,\ i = 0, \ldots, N_v), \quad \mathbf{c} = (c_{iv} \,:\, v \in S,\ i = 0, \ldots, N_v)$$

a tuple of linear forms in $V^{(N)}$ and a tuple of reals with

$$c_{0v} \geqslant c_{1v} \geqslant \cdots \geqslant c_{N_v,v} > 0 \text{ for } v \in S. \tag{3.2}$$

In addition we impose some Galois symmetry conditions:

$$v^\sigma \in S,\ N_{v^\sigma} = N_v \text{ for } v \in S,\ \sigma \in \mathrm{Gal}(K/k) \tag{3.3}$$

and for each $v \in S$, $\sigma \in \mathrm{Gal}(K/k)$ there is a permutation $\pi_{v\sigma}$ of $0, \ldots, N_v$ such that

$$\begin{gathered} L_{iv} = \sigma(L_{\pi_{v\sigma}(i), v^\sigma}), \quad c_{iv} = c_{\pi_{v\sigma}(i), v^\sigma} \\ \text{for } \sigma \in \mathrm{Gal}(K/k),\ v \in S,\ i = 0, \ldots, N_v. \end{gathered} \tag{3.4}$$

We deal with the system of inequalities

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ \ (v \in S,\ i = 0, \ldots, N_v) \quad \text{in } \mathbf{x} \in \mathbb{P}^N(k). \tag{3.5}$$

Any finite set of places S and tuples $\mathcal{L}$, $\mathbf{c}$ with (3.2) can be enlarged so as to satisfy (3.3), (3.4) without affecting the set of solutions of (3.5). We have not included inequalities with $c_{iv} \leqslant 0$ since (provided the norms of the involved linear forms L_{iv} are sufficiently small) these impose no restriction. In contrast to other formulations of the Subspace Theorem, we do not require a priori that the systems $\{L_{iv} \,:\, i = 0, \ldots, N_v\}$ $(v \in S)$ have rank $N+1$.

We use the theory discussed in Section 2, taking for V the vector space $V^{(N)}$ defined by (3.1). We start with a simple case.

Theorem 3.1. *Let* S, $\mathcal{L} = (L_{iv} : v \in S, i = 0, \dots, N_v)$, $\mathbf{c} = (c_{iv} : v \in S, i = 0, \dots, N_v)$ *satisfy (3.2)-(3.4). Assume that* $V^{(N)}$ *is semistable with respect to* $(\mathcal{L}, \mathbf{c})$.
(i) If $\mu(V^{(N)}) > 1$ *or there is* $v \in S$ *such that*

$$\text{rank}\{L_{iv} : i = 0, \dots, N_v\} = N + 1, \tag{3.6}$$

then (3.5) has only finitely many solutions.
(ii) If $\mu(V^{(N)}) < 1$ *and there is no* $v \in S$ *for which (3.6) holds, then the solutions of (3.5) are Zariski dense in* $\mathbb{P}^N(k)$.

Part (i) of this result is in the paper by Faltings and Wüstholz [4, Theorem 9.1]. Faltings and Wüstholz observed that if $V^{(N)}$ is semistable and $\mu(V^{(N)}) < 1$, then the set of solutions of (3.5) is infinite. We have proved (ii) by using some arguments from Schmidt [12].

We now consider the case that $V^{(N)}$ is not semistable with respect to $(\mathcal{L}, \mathbf{c})$. We keep our assumptions that S, $\mathcal{L}$, $\mathbf{c}$ satisfy (3.2)-(3.4), and denote by $(\mathbf{0}) = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V^{(N)}$ the Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$ as defined by Lemma 2.1. Recall that by part (iii) of that lemma, $\mu(V_1) > \mu(V_2/V_1) > \cdots > \mu(V^{(N)}/V_{r-1})$.

If there is $i \in \{1, \dots, r\}$ with $\mu(V_i/V_{i-1}) > 1$ then let i_0 be the largest such i; if there is no such i put $i_0 := 0$. That is, if $1 \leqslant i_0 \leqslant r - 1$ then $\mu(V_{i_0}/V_{i_0-1}) > 1 \geqslant \mu(V_{i_0+1}/V_{i_0})$, $i_0 = r$ means that $\mu(V^{(N)}/V_{r-1}) > 1$, while $i_0 = 0$ means that $\mu(V_1) \leqslant 1$.

For any linear subspace U of $V^{(N)}$, we put

$$T(U) := \{\mathbf{x} \in \mathbb{P}^N(K) : L(\mathbf{x}) = 0 \text{ for every } L \in U\}.$$

Further, we write $L|_T$ for the restriction of a linear form L to a linear subspace T of $\mathbb{P}^N(K)$.

The next result is implicit in the paper of Faltings and Wüstholz [4].

Theorem 3.2. *Assume that* $i_0 \geqslant 1$. *Then the set of solutions of (3.5) is contained in* $T(V_{i_0}) \cup F$ *where* F *is a finite set.*

In particular, if $i_0 = r$, i.e., if $\mu(V^{(N)}/V_{r-1}) > 1$ then (3.5) has only finitely many solutions.

In the next result we consider in more detail the set of solutions of (3.5) lying in $T(V_{i_0})$.

Theorem 3.3. *Assume that* $i_0 < r$.
(i) Assume there is $v \in S$ *such that*

$$\text{rank}\{L_{iv}|_{T(V_{i_0})} : i = 0, \ldots, N_v\} = \dim T(V_{i_0}) + 1. \tag{3.7}$$

Then (3.5) has only finitely many solutions.
(ii) Assume that $\mu(V_{i_0+1}/V_{i_0}) < 1$ *and that there is no* $v \in S$ *for which (3.7) holds. Then the set of solutions* $\mathbf{x} \in T(V_{i_0}) \cap \mathbb{P}^N(k)$ *of (3.5) is Zariski dense in* $T(V_{i_0})$.

The Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$ can be determined effectively. From this, one can determine effectively the quantities $\mu(V_1)$, $\mu(V_2/V_1), \ldots$ and from these, the index i_0. Further, knowing i_0, one can determine effectively whether (3.7) holds for some $v \in S$ or not. Thus, Theorems 3.2 and 3.3 give an effective procedure to decide whether the number of solutions of (3.5) is finite or infinite, except in the case not covered by these theorems, that is when there is $i_0 \in \{0, \ldots, r-1\}$ such that $\mu(V_{i_0+1}/V_{i_0}) = 1$ and there is no $v \in S$ with (3.7).

4. A result on twisted heights.

We state a result on twisted heights implying the results from the previous section. Let K be a number field, S a finite set of places of K, and $V^{(N)}$ the K-vector space of linear forms in $K[X_0, \ldots, X_N]$. Further, let $\mathcal{L} = (L_{iv} :$

$v \in S,\ i = 0, \dots, N_v)$ be a tuple in $V^{(N)}$ and $\mathbf{c} = \{c_{iv} :\ v \in S,\ i = 0, \dots, N_v\}$ a tuple of reals such that

$$\text{(4.1)} \qquad \begin{aligned} &\operatorname{rank}(L_{0v}, \dots, L_{N_v,v}) = N + 1 \text{ for } v \in S, \\ &c_{0v} \geqslant \cdots \geqslant c_{N_v,v} \text{ for } v \in S. \end{aligned}$$

In contrast to Section 3, we assume that for each $v \in S$ the linear forms L_{iv} have rank $N + 1$, but we do not require that the c_{iv} be positive.

For $Q \in \mathbb{R}_{\geqslant 1}$ we define a twisted height on $\mathbb{P}^N(K)$:

$$\text{(4.2)} \quad H_Q(\mathbf{x}) = H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) := \left(\prod_{v \in S} \max_{i=0,\dots,N_v} \left(|L_{iv}(\mathbf{x})|_v Q^{c_{iv}} \right) \right) \cdot \left(\prod_{v \notin S} \|\mathbf{x}\|_v \right).$$

This is a well-defined height on $\mathbb{P}^N(K)$ in view of the product formula.

For $\lambda \in \mathbb{R}_{>0}$, let $T_Q(\lambda) = T_{Q,\mathcal{L},\mathbf{c}}(\lambda)$ denote the smallest linear subspace of $\mathbb{P}^N(K)$ containing

$$\{\mathbf{x} \in \mathbb{P}^N(K) :\ H_Q(\mathbf{x}) \leqslant \lambda\}.$$

For $j = 1, \dots, N + 1$, denote by $\lambda_j(Q) = \lambda_j(Q, \mathcal{L}, \mathbf{c})$ the minimum of all values λ such that $T_Q(\lambda)$ has (projective) dimension at least $j - 1$.

We use again the notation from Section 2 and denote by $(\mathbf{0}) = V_0 \underset{\neq}{\subsetneq} V_1 \underset{\neq}{\subsetneq} V_2 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} V_r = V^{(N)}$ the Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$. Put

$$\text{(4.3)} \qquad \begin{aligned} &T_i = T(V_{r-i}) \ (i = 0, \dots, r), \\ &d_0 := 0,\ d_i := \dim T_i + 1,\ \ \nu_i := \mu(V_{r+1-i}/V_{r-i}) \ (i = 1, \dots, r), \end{aligned}$$

where $T(V_{r-i}) = \{\mathbf{x} \in \mathbb{P}^N(K) :\ L(\mathbf{x}) = 0 \text{ for } L \in V_{r-i}\}$. Then

$$\text{(4.4)} \qquad \begin{cases} \emptyset = T_0 \underset{\neq}{\subsetneq} T_1 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} T_{r-1} \underset{\neq}{\subsetneq} T_r = \mathbb{P}^N(K), \\ 0 = d_0 < d_1 < \cdots < d_r = N + 1, \\ \mu(V^{(N)}/V_{r-1}) = \nu_1 < \nu_2 < \cdots < \nu_r = \mu(V_1). \end{cases}$$

Theorem 4.1. *Let $\mathcal{L}$ be a tuple in $V^{(N)}$ with (4.1) and* $\mathbf{c}$ *a tuple of reals. Then for every $\delta > 0$ there is $Q_0 > 0$ such that for every $Q \geqslant Q_0$,*

$$Q^{\nu_i - \delta} < \lambda_{d_{i-1}+1}(Q) \leqslant \cdots \leqslant \lambda_{d_i}(Q) < Q^{\nu_i + \delta} \quad \text{for } i = 1, \ldots, r, \tag{4.5}$$

$$T_Q(\lambda_{d_i}(Q)) = T_i \quad \text{for } i = 1, \ldots, r. \tag{4.6}$$

The following corollary was basically proved by Fujimori [5, Theorem 2.8].

Corollary 4.2. *Assume that $V^{(N)}$ is semistable with respect to $(\mathcal{L}, \mathbf{c})$. Then for every $\delta > 0$ there is $Q_0 > 0$ such that for every $Q \geqslant Q_0$,*

$$Q^{\mu - \delta} < \lambda_1(Q) \leqslant \cdots \leqslant \lambda_{N+1}(Q) < Q^{\mu + \delta}, \tag{4.7}$$

where $\mu := \mu_{\mathcal{L}, \mathbf{c}}(V^{(N)})$.

Proof. The Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$ is $(\mathbf{0}) \subsetneq V^{(N)}$. So one has to apply Theorem 4.1 with $r = 1$, $d_0 = 0$, $d_1 = N + 1$. □

We remark that Q_0 in Theorem 4.1 and Corollary 4.2 cannot be determined effectively from our arguments.

We deduce two further corollaries of Theorem 4.1.

Corollary 4.3. *We have*

$$\lim_{Q \to \infty} \frac{\log \lambda_j(Q)}{\log Q} = \nu_i \quad \text{for } i = 1, \ldots, r,\ j = d_{i-1} + 1, \ldots, d_i.$$

Proof. Obvious. □

Corollary 4.4. *There is a finite, effectively determinable collection of proper linear subspaces $\{U_1, \ldots, U_m\}$ of $\mathbb{P}^N(K)$, depending on the linear forms in $\mathcal{L}$ but independent of* $\mathbf{c}$ *with the following property:*

for every $\delta > 0$ there is $Q_0 > 1$ such that for every $Q \geqslant Q_0$, there is $T \in \{U_1, \dots, U_m\}$ with

$$\{\mathbf{x} \in \mathbb{P}^N(k) : H_Q(\mathbf{x}) \leqslant Q^{\mu(V_1)-\delta}\} \subset T.$$

Proof. Let $\delta > 0$. By Theorem 4.1, we have for $Q \geqslant Q_0$ that $Q^{\mu(V_1)-\delta} < \lambda_{d_{r-1}+1}(Q)$. Therefore, again by Theorem 4.1,

$$\{\mathbf{x} \in \mathbb{P}^N(K) : H_Q(\mathbf{x}) \leqslant Q^{\mu(V_1)-\delta}\} \subset T(\lambda_{d_{r-1}}(Q)) = T_{r-1} = T(V_1).$$

As we observed in Section 2, the space V_1, and so the space $T(V_1)$ can be chosen from an effectively computable finite collection depending only on $\mathcal{L}$. □

5. Geometry of numbers

We have collected some results on the geometry of numbers over number fields.

Let K be a number field. Put

$$\varepsilon_v := \frac{[K_v : \mathbb{R}]}{[K : \mathbb{Q}]} \text{ if } v \text{ is archimedean, } \varepsilon_v := 0 \text{ if } v \text{ is non-archimedean.}$$

We will use frequently that $\sum_{v \in M_K} \varepsilon_v = 1$ and

$$|x_1 + \cdots + x_n|_v \leqslant n^{\varepsilon_v} \max_{1 \leqslant i \leqslant n} |x_i|_v$$

for $v \in M_K$, $x_1, \dots, x_n \in K$.

Let $M_{iv} \in K[X_0, \dots, X_N]$ ($v \in M_K$, $i = 0, \dots, N$) be linear forms and A_{iv} ($v \in M_K$, $i = 0, \dots, N$) positive reals such that

$$\begin{aligned} &\mathrm{rank}\{M_{0v}, \dots, M_{Nv}\} = N+1 \text{ for } v \in M_K, \\ &M_{0v} = X_0, \dots, M_{Nv} = X_N \text{ for all but finitely many } v \in M_K \\ &A_{0v} =, \dots, = A_{Nv} = 1 \text{ for all but finitely many } v \in M_K. \end{aligned} \tag{5.1}$$

Define the 'parallelepiped'

$$\Pi := \{\mathbf{x} \in K^{N+1} : |M_{iv}(\mathbf{x})|_v \leqslant A_{iv} \text{ for } v \in M_K, i = 0, \dots, N\} \tag{5.2}$$

and its dilation

$$(5.3)\quad \mu * \Pi := \{\mathbf{x} \in K^{N+1} : |M_{iv}(\mathbf{x})|_v \leqslant \mu^{\varepsilon_v} A_{iv} \text{ for } v \in M_K,\ i = 0, \dots, N\}$$

for $\mu \in \mathbb{R}_{>0}$. For $j = 1, \dots, N+1$ we define the j-th successive minimum μ_j of Π to be the minimum of all $\mu \in \mathbb{R}_{>0}$ such that $\mu * \Pi$ contains j linearly independent points. This minimum exists since the set of values assumed by $\mu(\mathbf{x}) := \inf\{\mu : \mu * \Pi \ni \mathbf{x}\}$ ($\mathbf{x} \in K^{N+1}$) is discrete. The following result is a very special case of an adèlic version of Minkowski's convex body theorem, due to McFeat [6] (and proved much later independently by Bombieri and Vaaler [1]). For $v \in M_K$ we denote by G_v the value group of $|\cdot|_v$. This is the group of positive reals if v is an infinite place and a discrete cyclic group if v is a finite place.

Lemma 5.1. *There is an effectively computable constant $C_1 > 0$ depending only on K, N, the linear forms M_{iv}, and the set of places $T = \{v \in M_K : \exists i : A_{iv} \notin G_v\}$ such that*

$$\prod_{j=1}^{N+1} \mu_j \leqslant C_1 \Big(\prod_{v \in M_K} \prod_{i=0}^{N} A_{iv} \Big)^{-1}.$$

Proof. In the case that $T = \emptyset$, McFeat's result implies at once that $\prod_{j=1}^{N+1} \mu_j \leqslant C_1' \Big(\prod_{v \in K} \prod_{i=0}^{N} A_{iv} \Big)^{-1}$ with a constant C_1' depending on K and the linear forms M_{iv} only. In case that $T \neq \emptyset$, for $v \in T$, $i = 0, \dots, N$ let A_{iv}' be the largest value in G_v which is $\leqslant A_{iv}$. Then there is a constant C_1'' depending only on K and T such that $\prod_{v \in T} \prod_{i=0}^{N+1} (A_{iv}/A_{iv}') \leqslant C_1''$. Now Lemma 5.1 holds with $C_1 := C_1' C_1''$. □

We deduce some consequences.

Lemma 5.2. *Let T be a finite set of places of K. Then there is an effectively computable constant $C_2 > 0$ depending only on K, T, with the following property: Let $\mathbf{B} = (B_v : v \in M_K)$ be any tuple of positive reals such that $B_v = 1$ for all but finitely $v \in M_K$, $B_v \in G_v$ for $v \notin T$, and $\prod_{v \in K} B_v > C_2$.*

Then there is $x \in K^*$ *with*

$$|x|_v \leqslant B_v \quad \text{for } v \in M_K.$$

Proof. Apply Lemma 5.1 with $N = 0$. □

Let S be a finite set of places of K, $V^{(N)}$ the K-vector space of linear forms in $K[X_0, \ldots, X_N]$ and $\mathcal{L} = (L_{iv} : v \in S,\ i = 0, \ldots, N_v)$ a tuple of linear forms in $V^{(N)}$ and $\mathbf{c} = (c_{iv} : v \in S,\ i = 0, \ldots, N_v)$ a tuple of reals, satisfying (4.1). For convenience, we put

$$N_v := N,\ L_{iv} := X_i,\ c_{iv} := 0 \text{ for } v \in M_K \setminus S,\ i = 0, \ldots, N. \tag{5.4}$$

Thus,

$$\mathrm{rank}(L_{0v}, \ldots, L_{N_v,v}) = N + 1 \text{ for } v \in M_K. \tag{5.5}$$

We use again the notation from Section 2.

We apply the above results to the twisted height introduced in (4.2),

$$\begin{aligned} H_Q(\mathbf{x}) = H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) &= \Big(\prod_{v \in S} \max_{0 \leqslant i \leqslant N_v} \big(|L_{iv}(\mathbf{x})|_v Q^{c_{iv}}\big)\Big) \cdot \Big(\prod_{v \notin S} \|\mathbf{x}\|_v\Big) \\ &= \prod_{v \in M_K} \max_{0 \leqslant i \leqslant N_v} \big(|L_{iv}(\mathbf{x})|_v Q^{c_{iv}}\big). \end{aligned}$$

We may view H_Q both as a height on K^{N+1} and on $\mathbb{P}^N(K)$.

For $j = 1, \ldots, N+1$, the j-th minimum $\lambda_j(Q)$ of H_Q is equal to the minimum of all $\lambda \in \mathbb{R}_{>0}$ such that $\{\mathbf{x} \in K^{N+1} : H_Q(\mathbf{x}) \leqslant \lambda\}$ contains j linearly independent points.

Lemma 5.3. *Let* $\mathbf{a} \in K^{N+1} \setminus \{\mathbf{0}\}$ *and let* D_v $(v \in M_K)$ *be positive numbers such that* $D_v = 1$ *for all but finitely many* v. *Then there is a constant* $C_3 > 0$ *depending only on* K, S *and* $\{v \in M_K : D_v \neq 1\}$, *such that if*

$$\prod_{v \in S} D_v > C_3 H_Q(\mathbf{a}),$$

then there is $\alpha \in K^*$ *such that* $\mathbf{b} := \alpha\mathbf{a}$ *satisfies*

$$|L_{iv}(\mathbf{b})|_v \leqslant D_v Q^{-c_{iv}} \text{ for } v \in M_K,\ i = 0, \ldots, N_v. \tag{5.6}$$

Proof. Take for C_3 the constant C_2 from Lemma 5.2 where for T we take $S \cup \{v : D_v \neq 1\}$ minus the archimedean places. Then there is $\alpha \in K^*$ such that

$$|\alpha|_v \leqslant D_v \cdot \left(\max_{0 \leqslant i \leqslant N_v} |L_{iv}(\mathbf{a})|_v Q^{c_{iv}}\right)^{-1}$$

for $v \in M_K$. Now the vector $\mathbf{b} := \alpha\mathbf{a}$ satisfies (5.6). □

Lemma 5.4. *There is an effectively computable positive constant* C_4 *depending on* K, S, N *and* $\mathcal{L}$ *such that*

$$\prod_{j=1}^{N+1} \lambda_j(Q) \leqslant C_4 Q^{w(V^{(N)})}. \tag{5.7}$$

Proof. Write λ_j for $\lambda_j(Q)$. For $v \in S$, $l = 0, \ldots, N$, let i_{lv} be the smallest index i such that $\operatorname{rank}\{L_{0v}, \ldots, L_{iv}\} = l + 1$, and put $M_{lv} := L_{i_{lv},v}$, $d_{lv} := c_{i_{lv},v}$. Further, for $v \in M_K \setminus S$, $l = 0, \ldots, N$, put $M_{lv} := X_l$, $d_{lv} := 0$. By $C_5, C_6, \ldots$ we denote effectively computable constants depending only on K, S, N and $\mathcal{L}$.

Denote by $\mu_1, \ldots, \mu_{N+1}$ the successive minima of the parallelepiped

$$\Pi := \{\mathbf{x} \in K^{N+1} : |M_{lv}(\mathbf{x})|_v \leqslant Q^{-d_{lv}}\ (v \in M_K,\ l = 0, \ldots, N)\}$$

Notice that by (5.4), (2.3), (2.4) we have

$$\sum_{v \in M_K} \sum_{l=0}^{N} d_{lv} = \sum_{v \in S} \sum_{l=0}^{N} d_{lv} = w(V^{(N)}).$$

So by Lemma 5.1,

$$\prod_{j=1}^{N+1} \mu_j \leqslant C_5 Q^{w(V^{(N)})}. \tag{5.8}$$

Let $\mathbf{a}_1, \dots, \mathbf{a}_{N+1}$ be a basis of K^{N+1} such that $\mathbf{a}_j \in \mu_j * \Pi$. By our definition of the linear forms M_{lv} and the reals d_{lv}, we have for $v \in S$, $i = 0, \dots, N_v$ that L_{iv} is a linear combination of M_{lv} with $i_{lv} \leqslant i$. Therefore,

$$|L_{iv}(\mathbf{a}_j)|_v \leqslant C_6 \max_{l:\, i_l \leqslant i} |M_{lv}(\mathbf{a}_j)|_v \leqslant C_7 \mu_j^{\varepsilon_v} Q^{-c_{iv}}$$

for $v \in S$, $i = 0, \dots, N_v$, $j = 1, \dots, N+1$. Clearly also $\|\mathbf{a}_j\|_v \leqslant \mu_j^{\varepsilon_v}$ for $v \in M_K \setminus S$. It follows that $H_Q(\mathbf{a}_j) \leqslant C_7^{\# S} \mu_j$ for $j = 1, \dots, N+1$. Consequently, $\lambda_j \leqslant C_7^{\# S} \mu_j$ for $j = 1, \dots, N+1$. Together with (5.8) this implies our lemma. □

Now assume that K is a finite Galois extension of a number field k. In the two lemmata below we assume that $\mathcal{L}$ satisfies (5.5), and that S, $\mathcal{L}$, $\mathbf{c}$ satisfy the Galois symmetry conditions (3.3), (3.4) from Section 3. Incorporating (5.4), our requirement can be stated as

$$(5.9) \qquad \begin{aligned} L_{iv} = \sigma(L_{\pi_{v\sigma}(i), v^\sigma}), \quad c_{iv} = c_{\pi_{v\sigma}(i), v^\sigma} \\ \text{for } \sigma \in \mathrm{Gal}(K/k),\ v \in M_K,\ i = 0, \dots, N_v \end{aligned}$$

where $\pi_{v\sigma}$ is a permutation of $0, \dots, N_v$ for $v \in S$ and $\pi_{v\sigma}$ is the identity if $v \in M_K \setminus S$.

Lemma 5.5. *There are an effectively computable constant $C_8 > 0$ depending only on k, K, S, N and $\mathcal{L}$ and a basis $\mathbf{g}_1, \dots, \mathbf{g}_{N+1}$ of k^{N+1} such that for $j = 1, \dots, N+1$,*

$$(5.10) \qquad |L_{iv}(\mathbf{g}_j)|_v \leqslant Q^{-c_{iv}} (C_8 \lambda_j(Q))^{\varepsilon_v} \text{ for } v \in M_K,\ i = 0, \dots, N_v.$$

Proof. For $\mathbf{a} \in K^{N+1}$, $\sigma \in \mathrm{Gal}(K/k)$, denote by $\sigma(\mathbf{a})$ the vector obtained by applying σ to the coordinates of $\mathbf{a}$. We write λ_j for $\lambda_j(Q)$. Fix a basis $\{\omega_1, \dots, \omega_D\}$ of K over k. Let E_v $(v \in M_K)$ be constants specified later, depending only on k, K and N, such that $E_v = 1$ for all but finitely many v, $E_v = E_{v^\sigma}$ for $v \in M_K$, $\sigma \in \mathrm{Gal}(K/k)$ and $\prod_{v \in M_K} E_v = 1$.

The vector space K^{N+1} has a basis $\{\mathbf{a}_1, \dots, \mathbf{a}_{N+1}\}$ such that $H_Q(\mathbf{a}_j) = \lambda_j$ for $i = 1, \dots, N+1$. By Lemma 5.3, there are a constant $C_9 > 0$ depending only on $k, K, S, N, \mathcal{L}$ and the set of places v with $E_v \neq 1$, and non-zero scalar multiples $\mathbf{b}_1, \dots, \mathbf{b}_{N+1} \in K^{N+1}$ of $\mathbf{a}_1, \dots, \mathbf{a}_{N+1}$, respectively, such that for $j = 1, \dots, N+1$,

$$|L_{iv}(\mathbf{b}_j)|_v \leqslant E_v Q^{-c_{iv}} (C_9 \lambda_j)^{\varepsilon_v} \text{ for } v \in M_K,\ i = 0, \dots, N_v. \tag{5.11}$$

The constants E_v are chosen to depend only on k, K and N, so in fact C_9 depends only on k, K, S, N and $\mathcal{L}$.

By (5.9) we have for $\sigma \in \mathrm{Gal}(K/k)$, $v \in M_K$, $i = 0, \dots, N_v$, $j = 1, \dots, N+1$,

$$\begin{aligned} |L_{iv}(\sigma(\mathbf{b}_j))|_v &= |\sigma(L_{\pi_{v\sigma}(i), v^\sigma}(\mathbf{b}_j))|_v = |L_{\pi_{v\sigma}(i), v^\sigma}(\mathbf{b}_j)|_{v^\sigma} \\ &\leqslant E_{v^\sigma} Q^{-c_{\pi_{v\sigma}(i), v^\sigma}} (C_9 \lambda_j)^{\varepsilon_{v^\sigma}} = E_v Q^{-c_{iv}} (C_9 \lambda_j)^{\varepsilon_v}. \end{aligned} \tag{5.12}$$

We have $\mathbf{b}_j = \sum_{l=1}^{D} \omega_l \mathbf{g}_{jl}$ with $\mathbf{g}_{jl} \in k^{N+1}$ for $j = 1, \dots, N+1$, $l = 1, \dots, D$. This implies $\sigma(\mathbf{b}_j) = \sum_{l=1}^{D} \sigma(\omega_l) \mathbf{g}_{jl}$ for $\sigma \in \mathrm{Gal}(K/k)$. So, since the matrix with entries $\sigma(\omega_l)$ is invertible, $\mathbf{g}_{jl} = \sum_{\sigma \in \mathrm{Gal}(K/k)} \beta_{l\sigma} \sigma(\mathbf{b}_j)$ with $\beta_{l\sigma} \in K$ depending only on the chosen basis $\omega_1, \dots, \omega_D$ and on N. We can select linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_{N+1}$ with $\mathbf{g}_j \in \{\mathbf{g}_{jl} : l = 1, \dots, D\}$. Then (5.11), (5.12) imply

$$|L_{iv}(\mathbf{g}_j)|_v \leqslant D_v E_v Q^{-c_{iv}} (C_9 \lambda_j)^{\varepsilon_v} \text{ for } v \in M_K,\ i = 0, \dots, N_v,$$

for certain numbers D_v depending only on S, N and on the chosen basis of K over k, such that at most finitely of the D_v are $\neq 1$ and $D_v = D_{v^\sigma}$ for $v \in M_K$, $\sigma \in \mathrm{Gal}(K/k)$. Now choose $E_v := D_v^{-1} \Big(\prod_{w \in M_K} D_w \Big)^{\varepsilon_v}$ for $v \in M_K$; these numbers do indeed depend only on k, K, S and N. Then we obtain (5.10) with $C_8 := C_9 \prod_{w \in M_K} D_w$. □

6. Proof of Theorem 4.1.

We keep the notation introduced before. Thus, K is a number field, S a finite set of places of K and N a positive integer. Our starting point is the Parametric Subspace Theorem by Schlickewei and the author [3, Theorem 2.1] which we recall here.

Proposition 6.1. *For $v \in S$, $i = 0, \dots, N$, let L_{iv} be a linear form in $K[X_0, \dots, X_N]$ and d_{iv} a real such that* $\mathrm{rank}\{L_{0v}, \dots, L_{Nv}\} = N+1$ *for $v \in S$ and $\sum_{v \in S} \sum_{i=0}^{N} d_{iv} = 0$. Then for every $\delta > 0$ there are $Q_1 > 1$ and* *a finite collection* $\{T_1, \dots, T_t\}$ *of proper linear subspaces of* $\mathbb{P}^N(K)$*, with the property that for every* $Q > Q_1$ *there is* $T \in \{T_1, \dots, T_t\}$ *such that*

$$\{\mathbf{x} \in \mathbb{P}^N(K) :\, H_Q(\mathbf{x}) \leqslant Q^{-\delta}\} \subset T,$$

where

$$H_Q(\mathbf{x}) := \prod_{v \in S} \Big(\max_{i=0,\dots,N} |L_{iv}(\mathbf{x})|_v Q^{d_{iv}} \Big) \cdot \prod_{v \in M_K \setminus S} \|\mathbf{x}\|_v.$$

This result holds true also for $N = 0$ if we interpret $\mathbb{P}^0(K)$ as a point, and agree that the empty set is the only linear subspace of $\mathbb{P}^0(K)$.

As before, denote by $V^{(N)}$ the vector space of linear forms in $K[X_0, \dots, X_N]$, let N_v $(v \in S)$ be integers with $N_v \geqslant N$, and let

$$\mathcal{L} = (L_{iv};\, v \in S,\, i = 0, \dots, N_v), \quad \mathbf{c} = (c_{iv} :\, v \in S,\, i = 0, \dots, N_v)$$

be a tuple from $V^{(N)}$ and a tuple of reals satisfying the conditions (4.1). We denote by $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$ the twisted height defined by (4.2). For a linear subspace T of $\mathbb{P}^N(K)$, denote by $V(T)$ the vector space of linear forms in $V^{(N)}$ vanishing identically on T. As before, the slope μ is taken with respect to $(\mathcal{L}, \mathbf{c})$, as defined in Section 2.

We prove a generalization of Proposition 6.1.

Lemma 6.2. *Let* U_1, U_2 *be two linear subspaces of* $V^{(N)}$ *with* $(\mathbf{0}) \subseteq U_1 \subsetneq U_2$*. Then for every* $\delta > 0$ *there are* $Q_2 = Q_2(U_1, U_2, \delta) > 1$ *and a finite collection of proper linear subspaces* $\{T_1, \dots, T_t\}$ *of* $T(U_1)$ *containing* $T(U_2)$ *with the property that for every* $Q > Q_2$ *there is* $T \in \{T_1, \dots, T_t\}$ *such that*

$$\{\mathbf{x} \in T(U_1) :\, H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \leqslant Q^{\mu(U_2/U_1) - \delta}\} \subset T.$$

Proof. Given $L \in V^{(N)}$, we denote by $\overline{L}$ the image of L under the canonical map $V^{(N)} \to V^{(N)}/U_1$. Choose $M_0, \dots, M_R \in U_2$ such that $\overline{M_0}, \dots, \overline{M_R}$ form a basis of U_2/U_1 and define the map $\varphi : \mathbf{x} \to (M_0(\mathbf{x}), \dots, M_R(\mathbf{x}))$ from $T(U_1) \setminus T(U_2)$ to $\mathbb{P}^R(K)$. It induces a vector space isomorphism $\overline{\varphi}$ from U_2/U_1 to $V^{(R)}$ (the linear forms in $K[X_0, \dots, X_R]$) given by $\overline{\varphi}(M_i) = X_i$ for $i = 0, \dots, R$.

For $v \in S$, $i = 0, \dots, N_v$, let $\overline{V}_{iv} := \mathrm{span}(\overline{L}_{0v}, \dots, \overline{L}_{iv})$. For $v \in S$, $l = 0, \dots, R$, let i_{lv} be the smallest i such that $\dim(\overline{V}_{iv} \cap (U_2/U_1)) = l + 1$. Then by (2.3), (2.4) we have

$$\mu(U_2/U_1) = \frac{1}{R+1}\Big(\sum_{v\in S}\sum_{h=0}^{R} c_{i_{hv},v}\Big). \tag{6.1}$$

For $v \in S$, $l = 0, \dots, R$, choose $M_{lv} \in U_2$ such that $\overline{M}_{lv} \in \overline{V}_{i_{lv},v} \cap (U_2/U_1)$ and $\overline{M}_{lv} \notin \overline{V}_{i_{lv}-1,v} \cap (U_2/U_1)$, and write $M'_{lv} := \overline{\varphi}(\overline{M}_{lv})$ for $v \in S$, $l = 0, \dots, R$, $\mathcal{M}' := (M'_{lv} : v \in S, l = 0, \dots, R)$. Further, put $d_{lv} := c_{i_{lv},v} - \frac{1}{R+1}\Big(\sum_{h=0}^{R} c_{i_{hv},v}\Big)$ for $v \in S$, $l = 0, \dots, N$, and $\mathbf{d} := (d_{lv} : v \in S, l = 0, \dots, R)$. Notice that

$$\mathrm{rank}(M'_{0v}, \dots, M'_{Rv}) = R + 1 \text{ for } v \in S, \quad \sum_{v\in S}\sum_{l=0}^{R} d_{lv} = 0. \tag{6.2}$$

For $\mathbf{x} \in T(U_1) \setminus T(U_2)$ we have

$$\begin{aligned} H_{Q,\mathcal{M}',\mathbf{d}}(\varphi(\mathbf{x})) &= \Big(\prod_{v\in S}\max_{0\leqslant l\leqslant R}|M'_{lv}(\varphi(\mathbf{x}))|_v Q^{d_{lv}}\Big)\cdot\Big(\prod_{v\notin S}\|\varphi(\mathbf{x})\|_v\Big) \\ &= \Big(\prod_{v\in S}\max_{0\leqslant l\leqslant R}|M_{lv}(\mathbf{x})|_v Q^{d_{lv}}\Big)\cdot\Big(\prod_{v\notin S}\max_{0\leqslant l\leqslant r}|M_l(\mathbf{x})|_v\Big). \end{aligned}$$

For $v \in S$, M_{lv} is a linear combination modulo U_1 of the linear forms L_{iv} with $i \leqslant i_{lv}$ and $d_{lv} \leqslant c_{iv} - \frac{1}{R+1}\Big(\sum_{h=0}^{R} c_{i_{hv},v}\Big)$ for $i \leqslant i_{lv}$. Together with (6.1) this implies that there is a constant C depending only on K, S, $\mathcal{L}$, the choices of the M_{lv} and $M_0, \dots, M_R$, such that

$$H_{Q,\mathcal{M}',\mathbf{d}}(\varphi(\mathbf{x})) \leqslant C \cdot Q^{-\mu(U_2/U_1)} H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \text{ for } \mathbf{x} \in T(U_1) \setminus T(U_2). \tag{6.3}$$

Let $\delta > 0$. Then by (6.3), we have for every sufficiently large Q and for every $\mathbf{x} \in T(U_1) \setminus T(U_2)$ with $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \leqslant Q^{\mu(U_2/U_1)-\delta}$,

$$H_{Q,\mathcal{M}',\mathbf{d}}(\varphi(\mathbf{x})) \leqslant Q^{-\delta/2}.$$

In view of (6.2), we can apply Proposition 6.1 to the latter, and thus conclude that there is a finite collection $\{T'_1, \dots, T'_t\}$ of proper linear subspaces of $\mathbb{P}^R(K)$ with the property that for every sufficiently large Q, there is

$T' \in \{T_1', \dots, T_t'\}$ such that for every $\mathbf{x} \in T(U_1) \setminus T(U_2)$ with (6.3) we have $\varphi(\mathbf{x}) \in T'$.

Now clearly, our lemma is satisfied with $T_i = \varphi^{-1}(T_i') \cup T(U_2)$ for $i = 1, \dots, t$. $\square$

We denote by $(\mathbf{0}) = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V^{(N)}$ the Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$. Henceforth we write H_Q for $H_{Q,\mathcal{L},\mathbf{c}}$.

Lemma 6.3. *For every $\delta > 0$ there is $Q_3 > 1$ such that for every $Q > Q_3$,*

$$H_Q(\mathbf{x}) > Q^{\mu(V_{i+1}/V_i) - \delta} \text{ for } \mathbf{x} \in T(V_i) \setminus T(V_{i+1}),\ i \in \{0, \dots, r-1\}. \tag{6.4}$$

Remark. The lower bounds Q_1, Q_2 for Q in Proposition 6.1 and Lemma 6.2 can be computed effectively, but in the proof of Lemma 6.3 the effectivity is lost.

Proof. Fix $\delta > 0$, $i \in \{0, \dots, r-1\}$. We prove by induction on the dimension that for every linear subspace T of $T(V_i)$ with $T \supsetneq T(V_{i+1})$ there is $Q_T > 0$ such that for every $Q > Q_T$, $\mathbf{x} \in T \setminus T(V_{i+1})$ we have (6.4).

If $\dim T = \dim T(V_{i+1}) + 1$ this follows at once from Lemma 6.2. Suppose that $\dim T = m$ with $m > \dim T(V_{i+1}) + 1$ and that our assertion holds for every linear subspace of $T(V_i)$ of dimension $< m$ strictly containing $T(V_{i+1})$. Let as before $V(T)$ denote the vector space of $L \in V^{(N)}$ vanishing identically on T. By (2.7), (2.8) and Lemma 2.1 we have

$$\begin{aligned} \mu(V_{i+1}/V(T)) &= \frac{\mu(V_{i+1}/V_i)\dim(V_{i+1}/V_i) - \mu(V(T)/V_i)\dim(V(T)/V_i)}{\dim(V_{i+1}/V_i) - \dim(V(T)/V_i)} \\ &\geqslant \mu(V_{i+1}/V_i). \end{aligned}$$

Applying Lemma 6.2 with $U_1 = V(T)$, $U_2 = V_{i+1}$ and using the above inequality, we infer that there are proper linear subspaces $T_1, \dots, T_t$ of T containing $T(V_{i+1})$ such that for every $Q > Q'_T$, say, there is $T_j \in \{T_1, \dots, T_t\}$ with the property that every $\mathbf{x} \in T$ with $H_Q(\mathbf{x}) \leqslant Q^{\mu(V_{i+1}/V_i) - \delta}$ lies in T_j.

By the induction hypothesis, for every $Q > Q_{T_j}$, $\mathbf{x} \in T_j \setminus T(V_{i+1})$ we have (6.4). (This Q_{T_j} cannot be determined effectively since T_j cannot be determined effectively). Hence for every $Q > Q_T := \max(Q'_T, Q_{T_1}, \ldots, Q_{T_t})$ and $\mathbf{x} \in T \setminus T(V_{i+1})$ we have (6.4). This completes the induction step. □

Proof of Theorem 4.1. Let $\delta > 0$. Choose $\delta' > 0$ sufficiently small compared with δ. Rewriting (6.4) in accordance with (4.3), we obtain that for every sufficiently large Q,

$$H_Q(\mathbf{x}) > Q^{\nu_i - \delta'} \text{ for } \mathbf{x} \in T_i \setminus T_{i-1},\ i = 1, \ldots, r. \tag{6.5}$$

This implies $\lambda_{d_{i-1}+1}(Q) > Q^{\nu_i - \delta'}$ for $i = 1, \ldots, r$. Put $\xi_j := \nu_i$ for $j = d_{i-1} + 1, \ldots, d_i$, $i = 1, \ldots, r$. Thus, for every sufficiently large Q,

$$\lambda_j(Q) > Q^{\xi_j - \delta'} \text{ for } j = 1, \ldots, N + 1. \tag{6.6}$$

Now on the one hand, by Lemma 5.4 we have for every sufficiently large Q,

$$\prod_{j=1}^{N+1} \lambda_j(Q) < Q^{w(V^{(N)}) + \delta'},$$

on the other hand, using (4.3) and (2.7),

$$\begin{aligned} \sum_{j=1}^{N+1} \xi_j &= \sum_{i=1}^{r} (d_i - d_{i-1}) \nu_i = \sum_{i=1}^{r} (\dim V_i - \dim V_{i-1}) \mu(V_i / V_{i-1}) \\ &= \sum_{i=1}^{r} \big(w(V_i) - w(V_{i-1}) \big) = w(V^{(N)}). \end{aligned}$$

Together with (6.6), this implies that for every sufficiently large Q,

$$\lambda_j(Q) < Q^{w(V^{(N)}) + \delta' - \sum_{l \neq j} (\xi_l - \delta')} = Q^{\xi_j + (N+1)\delta'}.$$

Again combining this with (6.5) and taking $\delta' < \delta/(N+1)$ we conclude that for every sufficiently large Q we have $Q^{\xi_j - \delta} < \lambda_j(Q) < Q^{\xi_j + \delta}$ for $j = 1, \ldots, N + 1$, which is precisely (4.5).

It remains to prove (4.6). This is certainly correct if $i = r$. Take $i \in \{1, \ldots, r - 1\}$. Observe that by (4.5), for δ sufficiently small and Q sufficiently large we have $\lambda_{d_i}(Q) < Q^{\nu_i + \delta} < Q^{\nu_{i+1} - \delta}$. Invoking (6.5) we obtain that if Q is sufficiently large then every $\mathbf{x} \in \mathbb{P}^N(K)$ with $H_Q(\mathbf{x}) \leqslant \lambda_{d_i}(Q)$

lies in T_i, which means that $T_Q(\lambda_{d_i}(Q)) \subseteq T_i$. Since these spaces have the same dimension they must be equal. $\square$

7. Proof of Theorem 3.1.

Let k be a number field. K be a finite Galois extension of k, S a finite set of places of K, $\mathcal{L} = (L_{iv} : v \in S,\, i = 0, \ldots, N_v)$ a tuple of linear forms and $\mathbf{c} = (c_{iv} : v \in S,\, i = 0, \ldots, N_v)$ and a tuple of reals satisfying (3.2), (3.3), (3.4). We suppose that K is contained in a given algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

We introduce some notation. Put

$$\begin{aligned} &L_{N_v+i+1} := X_i,\ c_{N_v+i+1,v} := 0,\ \text{for } v \in S,\ i = 0, \ldots, N, \\ &N'_v := N_v + N + 1 \text{ for } v \in S, \\ &\mathcal{L}' := (L_{iv} : v \in S,\, i = 0, \ldots, N'_v), \\ &\mathbf{c}' := (c_{iv} : v \in S,\, i = 0, \ldots, N'_v) \end{aligned} \tag{7.1}$$

and define the twisted height

$$H_Q(\mathbf{x}) = H_{Q,\mathcal{L}',\mathbf{c}'}(\mathbf{x}) := \left(\prod_{v\in S} \max_{i=0,\ldots,N'_v} \left(|L_{iv}(\mathbf{x})|_v Q^{c_{iv}}\right)\right)\cdot\left(\prod_{v\notin S} \|\mathbf{x}\|_v\right). \tag{7.2}$$

It is clear that $\text{rank}\{L_{iv} : i = 0, \ldots, N'_v\} = N + 1$ for $v \in S$. Further, by (3.2) we have $c_{0v} \geqslant \cdots \geqslant c_{N_v,v} > 0 = c_{N_v+1,v} = \cdots = c_{N'_v,v}$. Hence (4.1) is satisfied with $\mathcal{L}', \mathbf{c}'$ replacing $\mathcal{L}$, $\mathbf{c}$. So the theory of Section 4 is applicable. Further, from (2.3) it follows that the weights, slopes and the Harder-Narasimhan filtration with respect to $(\mathcal{L}', \mathbf{c}')$ are equal to those with respect to $(\mathcal{L}, \mathbf{c})$. We need the following fact.

Lemma 7.1. *Let $\mathbf{x} \in \mathbb{P}^N(k)$ be a solution of (3.5) and put $Q := H(\mathbf{x})$. Then*

$$H_Q(\mathbf{x}) \leqslant Q\,.$$

Proof. Straightforward computation. $\square$

Proof of Theorem 3.1. Assume that $V^{(N)}$ is semistable with respect to $(\mathcal{L}, \mathbf{c})$. Put $\mu := \mu_{\mathcal{L},\mathbf{c}}(V^{(N)}) = \mu_{\mathcal{L}',\mathbf{c}'}(V^{(N)})$. Denote by $\lambda_1(Q), \ldots, \lambda_{N+1}(Q)$ the successive minima of H_Q.

We first prove (i). Assume that $\mu > 1$. In (4.7) we take $\delta > 0$ such that $\mu - \delta > 1$. Then for $Q > Q_0$ we have $\lambda_1(Q) > Q$. Let $\mathbf{x} \in \mathbb{P}^N(k)$ be a solution of (3.5) and put $Q := H(\mathbf{x})$. By Lemma 7.1 we have $\lambda_1(Q) \leqslant Q$, so $H(\mathbf{x}) = Q \leqslant Q_0$. By Northcott's Theorem, (3.5) has only finitely many solutions.

Now assume that there is $v \in S$ such that $\operatorname{rank}\{L_{iv} : i = 0, \dots, N_v\} = N + 1$. Then $c_v := \min(c_{0v}, \dots, c_{N_v,v}) > 0$. The linear forms $X_0, \dots, X_N$ can be expressed as linear combinations of $L_{0v}, \dots, L_{N_v,v}$. Hence there is a constant $A_v > 0$ such that $\|\mathbf{x}\|_v \leqslant A_v \max_i |L_{iv}(\mathbf{x})|_v$ for $\mathbf{x} \in K^{N+1}$, where the maximum is over $i = 0, \dots, N_v$. So if $\mathbf{x} \in \mathbb{P}^N(k)$ is a solution of (3.5) then

$$A_v^{-1} \leqslant \max_i \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_v},$$

i.e., $H(\mathbf{x}) \leqslant A_v^{1/c_v}$. Hence (3.5) has only finitely many solutions. This proves part (i).

We prove part (ii). Assume that $\mu < 1$ and that there is no $v \in S$ such that $\operatorname{rank}\{L_{iv} : i = 0, \dots, N_v\} = N + 1$. It suffices to prove the following. Let $F \in \overline{\mathbb{Q}}[X_0, \dots, X_N]$ be a non-zero homogeneous polynomial. Then for every $\delta > 0$ and every sufficiently large Q, there is $\mathbf{x}_Q \in k^{N+1}$ with the following properties:

(7.3) $\quad |L_{iv}(\mathbf{x}_Q)|_v \leqslant Q^{(\mu+\delta)\varepsilon_v - c_{iv}}$ for $v \in S,\ i = 0, \dots, N_v$;

(7.4) $\quad \|\mathbf{x}_Q\|_v \geqslant Q^{\mu\varepsilon_v - \delta}$ for $v \in S,\ i = 0, \dots, N_v$;

(7.5) $\quad H(\mathbf{x}_Q) \leqslant Q^{\mu+\delta}$;

(7.6) $\quad F(\mathbf{x}_Q) \neq 0$.

Suppose we have proved this. Choose $\delta > 0$ such that $\frac{c_{iv} - (\varepsilon_v + 1)\delta}{\mu + \delta} > c_{iv}$ for $v \in S$, $i = 0, \dots, N_v$. This is possible by our assumptions that $c_{iv} > 0$ for $v \in S$, $i = 0, \dots, N_v$ and $\mu < 1$. Then for every non-zero homogeneous polynomial $F \in \overline{\mathbb{Q}}[X_0, \dots, X_N]$ and every sufficiently large Q, there is $\mathbf{x} = \mathbf{x}_Q \in \mathbb{P}^N(k)$ such that $F(\mathbf{x}) \neq 0$ and

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant Q^{\varepsilon_v(\mu+\delta) - c_{iv} - \mu\varepsilon_v + \delta} \leqslant H(\mathbf{x})^{\frac{(\varepsilon_v+1)\delta - c_{iv}}{\mu+\delta}} \leqslant H(\mathbf{x})^{-c_{iv}}$$

for $v \in S$, $i = 0, \dots, N_v$. This shows that indeed the set of solutions of (3.5) is Zariski dense in $\mathbb{P}^N(k)$. So it remains to prove that for every sufficiently large Q there is $\mathbf{x}_Q \in \mathbb{P}^N(k)$ with (7.3)–(7.6).

According to Corollary 4.4 there is a finite collection $\{W_1, \dots, W_t\}$ of proper linear subspaces of $\mathbb{P}^N(K)$ depending only on $\mathcal{L}'$ with the following property: for every tuple of reals $\mathbf{d} = (d_{iv} : v \in S, i = 0, \dots, N_v)$ and every $\kappa < \mu_{\mathcal{L}',\mathbf{d}}(V^{(N)})$, we have that for every sufficiently large Q there is $W_i \in \{W_1, \dots, W_t\}$ with

$$\{\mathbf{x} \in \mathbb{P}^N(K) : H_{Q,\mathcal{L}',\mathbf{d}}(\mathbf{x}) \leqslant Q^\kappa\} \subset W_i. \tag{7.7}$$

For $i = 1, \dots, t$, choose a non-zero linear form $F_i \in K[X_0, \dots, X_N]$ vanishing identically on W_i and define

$$G := F \cdot \prod_{i=1}^{t} F_i.$$

We will construct $\mathbf{x}_Q \in \mathbb{P}^N(k)$ satisfying (7.3), (7.4), (7.5) but instead of (7.6) the stronger requirement

$$G(\mathbf{x}_Q) \neq 0. \tag{7.8}$$

We make some further preparations. For $v \in M_K \setminus S$, let $N'_v := N$, $L_{iv} := X_i$, $c_{iv} := 0$ for $i = 0, \dots, N$. Further, let $\pi_{v\sigma}$ ($v \in S$, $\sigma \in \mathrm{Gal}(K/k)$) be the permutations from (3.3). Put $\pi_{v\sigma}(i) = i$ for $v \in S$, $i = N_v+1, \dots, N'_v$ and $\pi_{v\sigma}(i) = i$ for $v \in M_K \setminus S$, $i = 0, \dots, N$. Then condition (5.9) is satisfied, and so Lemma 5.5 is applicable. By applying this lemma with $\delta/5$ instead of δ, say, and invoking (4.7), it follows that for every sufficiently large Q there exists a basis $\mathbf{g}_1, \dots, \mathbf{g}_{N+1}$ of k^{N+1} such that

$$|L_{iv}(\mathbf{g}_j)|_v \leqslant Q^{\varepsilon_v(\mu+\delta/4)-c_{iv}} \quad \text{for } v \in M_K,\ i = 0, \dots, N'_v,\ j = 1, \dots, N+1.$$

Fix such a sufficiently large Q and a basis $\mathbf{g}_1, \dots, \mathbf{g}_{N+1}$. Consider vectors

$$\mathbf{x} = \sum_{j=1}^{N+1} u_j \mathbf{g}_j \text{ with } u_j \in \mathbb{Z},\ |u_j| \leqslant Q^{\delta/4} \quad \text{for } j = 1, \dots, N+1. \tag{7.9}$$

The number of vectors $\mathbf{x}$ with (7.9) is at least $Q^{(N+1)\delta/4}$ while the number of vectors $\mathbf{x}$ with (7.9) and with $G(\mathbf{x}) = 0$ is at most $cQ^{N\delta/4}$ where c depends on N and the degree of G only. So assuming Q is sufficiently large, there is a vector $\mathbf{x} = \mathbf{x}_Q \in k^{N+1}$ with (7.8). Further, assuming again that Q is sufficiently large, we have

$$|L_{iv}(\mathbf{x}_Q)|_v \leqslant Q^{\varepsilon_v(\mu+3\delta/4)-c_{iv}} \quad \text{for } v \in M_K,\ i = 0, \ldots, N_v'. \tag{7.10}$$

This implies (7.3). By our definitions, the set $\{L_{iv} : i = 0, \ldots, N_v'\}$ contains $X_0, \ldots, X_N$ for every $v \in M_K$. So by (7.10),

$$\|\mathbf{x}_Q\|_v \leqslant \max_{0 \leqslant i \leqslant N_v'} |L_{iv}(\mathbf{x}_Q)|_v \leqslant Q^{\varepsilon_v(\mu+3\delta/4)}$$

for $v \in M_K$. This implies (7.5).

It remains to prove (7.4). Assume that $0 < \delta < \min\{c_{iv} : v \in S, i = 0, \ldots, N_v\}$. This is possible by (3.2). Let $v_0 \in S$ and consider the system of inequalities in $\mathbf{x} = (x_0, \ldots, x_N) \in K^{N+1}$:

$$\begin{aligned} &|L_{iv}(\mathbf{x})|_v \leqslant Q^{\varepsilon_v(\mu+3\delta/4)-c_{iv}} \quad (v \in M_K,\ i = 0, \ldots, N_v'), \\ &|x_i|_{v_0} \leqslant Q^{\varepsilon_v \mu - \delta} \quad (i = 0, \ldots, N). \end{aligned} \tag{7.11}$$

Put $d_{iv} := c_{iv}$ for $v \in M_K \setminus \{v_0\}$, $i = 0, \ldots, N_v'$, $d_{i,v_0} := c_{i,v_0}$ for $i = 0, \ldots, N_{v_0}$, $d_{i,v_0} := \delta$ for $i = N_{v_0} + 1, \ldots, N_{v_0} + N + 1 = N_{v_0}'$ and $\mathbf{d} := (d_{iv} : v \in S, i = 0, \ldots, N_v')$, where we have used the notation (7.1). Then $d_{0v} \geqslant \cdots \geqslant d_{N_v'}$ for $v \in S$, $v \neq v_0$ and $d_{0,v_0} \geqslant \cdots \geqslant d_{N_{v_0},v_0} > \delta = d_{N_{v_0}+1,v_0} = \cdots = d_{N_{v_0}',v_0}$. From (2.3) with $U = V^{(N)}$ and our assumption $\mathrm{rank}\{L_{i,v_0} : i = 0, \ldots, N_{v_0}\} < N + 1$ it follows that $\mu_{\mathcal{L}',\mathbf{d}}(V^{(N)}) \geqslant \mu + \delta$. Further, every non-zero $\mathbf{x} \in K^{N+1}$ with (7.11) satisfies also

$$H_{Q,\mathcal{L}',\mathbf{d}}(\mathbf{x}) \leqslant Q^{\mu+3\delta/4}.$$

Now an application of (7.7) yields that if Q is sufficiently large, then every $\mathbf{x} \in K^{N+1}$ with (7.11) lies in one of the subspaces $W_1, \ldots, W_t$, i.e., has $G(\mathbf{x}) = 0$. Our vector $\mathbf{x}_Q$ satisfies $G(\mathbf{x}_Q) \neq 0$ so it cannot satisfy (7.11) for any $v_0 \in S$. Since $\mathbf{x}_Q$ satisfies (7.10) it necessarily has to satisfy (7.4). This completes our proof of Theorem 3.1. □

8. Proofs of Theorems 3.2, 3.3.

As before, k is a number field, K a finite Galois extension of k, S a finite set of places of K, $\mathcal{L} = (L_{iv} : v \in S, i = 0, \ldots, N_v)$ a tuple of linear forms and

$\mathbf{c} = (c_{iv} : v \in S, i = 0, \dots, N_v)$ a tuple of reals satisfying (3.2), (3.3), (3.4). We denote by $(\mathbf{0}) = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V^{(N)}$ the Harder-Narasimhan filtration of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$.

We start with some observations. First, we show that the spaces $V_1, \dots, V_r$ in the Harder-Narasimhan filtration are defined over k. For a linear subspace U of $V^{(N)}$ and $\sigma \in \mathrm{Gal}(K/k)$ we define $\sigma(U) := \{\sigma(L) : L \in U\}$. This is again a linear subspace of $V^{(N)}$, and from (3.3), (3.4), (2.3), (2.4) it follows that $\mu_{\mathcal{L},\mathbf{c}}(\sigma(U)) = \mu_{\mathcal{L},\mathbf{c}}(U)$ for every non-zero linear subspace U of $V^{(N)}$ and each $\sigma \in \mathrm{Gal}(K/k)$. This implies that for each $\sigma \in \mathrm{Gal}(K/k)$, the filtration $(\mathbf{0}) \subsetneq \sigma(V_1) \subsetneq \cdots \subsetneq \sigma(V_{r-1}) \subsetneq V^{(N)}$ satisfies the conditions of Lemma 2.1. But these conditions determine the filtration uniquely. Therefore, $\sigma(V_i) = V_i$ for $\sigma \in \mathrm{Gal}(K/k)$, i.e., V_i is defined over k for $i = 1, \dots, r-1$. As a consequence, the Harder-Narasimhan filtration is already characterized uniquely if in Lemma 2.1 we restrict ourselves to subspaces U of $V^{(N)}$ defined over k. Another consequence is that $V^{(N)}$ is semistable with respect to $(\mathcal{L}, \mathbf{c})$ if and only if $\mu(U) \leqslant \mu(V^{(N)})$ for every non-zero linear subspace U of $V^{(N)}$ defined over k.

Second, in our proof we will need the technical condition that $[K : k] > N + 1$. We show that this is no restriction. Let K' be any finite extension of K which is Galois over k. Let $V^{(N)\prime}$ denote the K'-vector space of linear forms in $K'[X_0, \dots, X_N]$. On K' we define normalized absolute values $|\cdot|_{v'}$ ($v' \in M_{K'}$) similarly as on K. Thus, if v' lies above $v \in M_K$ then the restriction of $|\cdot|_{v'}$ to K is $|\cdot|^{d(v'|v)}$, where $d(v'|v) := [K'_{v'} : K_v]/[K' : K]$, i.e., the quotient of the local degree and the global degree. Now define $N_{v'} := N_v$, $L_{i,v'} := L_{iv}$, $c'_{i,v'} := d(v'|v)c_{iv}$ for $v' \in S'$, $i = 0, \dots, N_{v'}$, where v is the place in S lying below v'. For $v' \in S'$ and $\sigma' \in \mathrm{Gal}(K'/k)$ we define the permutation $\pi_{v',\sigma'} := \pi_{v,\sigma}$ where again v is the place of S lying below v' and σ is the restriction of σ' to K. Thus, the tuples $\mathcal{L}' = (L_{i,v'} : v' \in S',\ i = 0, \dots, N_{v'})$, $\mathbf{c}' = (c_{i,v'} : v' \in S',\ i = 0, \dots, N_{v'})$ satisfy the analogues of (3.2)–(3.4). Further, system (3.5) can be translated into

$$\frac{|L_{i,v'}(\mathbf{x})|_{v'}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{i,v'}} \quad (v' \in S',\ i = 0, \dots, N_{v'}) \quad \text{in } \mathbf{x} \in \mathbb{P}^N(k).$$

Lastly, from (2.3), (2.4) and the fact that $\sum_{v'|v} d(v'|v) = 1$ for $v \in S$ where the sum is taken over the places $v' \in S'$ lying above v, it follows that if U is a linear subspace of $V^{(N)}$ and $U' := U \otimes_K K'$ the tensor product of U with K', i.e., the K'-linear subspace of $V^{(N)\prime}$ generated by U, then $\mu_{\mathcal{L}',\mathbf{c}'}(U') = \mu_{\mathcal{L},\mathbf{c}}(U)$. Since as observed above, the Harder-Narasimhan filtration is already determined by considering in Lemma 2.1 only vector spaces U of $V^{(N)\prime}$ defined over k, it follows that the Harder-Narasimhan filtration of $V^{(N)\prime}$ with respect to $(\mathcal{L}', \mathbf{c}')$ is obtained from that of $V^{(N)}$ with respect to $(\mathcal{L}, \mathbf{c})$ by tensoring the vector spaces in the latter with K'.

As a consequence of all this, the theorems in Section 3 remain unaffected if we replace K by any finite extension K' of K which is Galois over k. In particular, we may replace K by a finite Galois extension of degree $> N+1$ over k.

In what follows, $\mathcal{L}', \mathbf{c}'$ will be as in (7.1) and the twisted height $H_Q = H_{Q,\mathcal{L}',\mathbf{c}'}$ as in (7.2).

Proof of Theorem 3.2. Choose $\delta > 0$ such that $\mu_{i_0} := \mu(V_{i_0}/V_{i_0-1}) - \delta > 1$ and let Q_0 be the number from Theorem 4.1 (applied to H_Q). Let $\mathbf{x} \in \mathbb{P}^N(k)$ be a solution of (3.5). Assume that $Q := H(\mathbf{x}) > Q_0$. By (4.5) we have $\lambda_{d_{r-i_0}+1}(Q) \geqslant Q^{\mu_{i_0} - \delta}$, while on the other hand, by Lemma 7.1 we have $H_Q(\mathbf{x}) \leqslant Q$. Hence $\mathbf{x}$ belongs to the vector space $T(\lambda_{d_{r-i_0}}(Q))$ i.e., spanned by all points of H_Q-height at most $\lambda_{d_{r-i_0}}(Q)$. By (4.6) this space is equal to $T(V_{i_0})$. So if $\mathbf{x} \notin T(V_{i_0})$ then $H(\mathbf{x}) \leqslant Q_0$. This proves Theorem 3.2. □

Proof of Theorem 3.3. First assume that there is $v \in S$ with (3.7). Notice that $c_v := \min(c_{0v}, \ldots, c_{N_v,v}) > 0$. The restrictions $X_0|_{T(V_{i_0})}, \ldots, X_N|_{T(V_{i_0})}$ can be expressed as linear combinations of $L_{0v}|_{T(V_{i_0})}, \ldots, L_{N_v,v}|_{T(V_{i_0})}$. Hence there is a constant $A_v > 0$ such that $\|\mathbf{x}\|_v \leqslant A_v \max_i |L_{iv}(\mathbf{x})|_v$ for $\mathbf{x} \in T(V_{i_0})$, where the maximum is over $i = 0, \ldots, N_v$. So if $\mathbf{x} \in T(V_{i_0}) \cap \mathbb{P}^N(k)$ is a solution of (3.5) then

$$A_v^{-1} \leqslant \max_i \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_v},$$

i.e., $H(\mathbf{x}) \leqslant A_v^{1/c_v}$. Hence (3.5) has only finitely many solutions in $T(V_{i_0}) \cap \mathbb{P}^N(k)$.

To prove the second part of Theorem 3.3, we first consider the case $i_0 = 0$. Then our assumptions become $\mu(V_1) < 1$, and there is no $v \in S$ such that $\mathrm{rank}\{L_{iv} : i = 0, \dots, N_v\} = N + 1$. We make a reduction to part (ii) of Theorem 3.1, using a clever idea of SCHMIDT [12]. Here we assume that $[K : k] > N + 1$. Further we may assume that $\mu(V^{(N)}) < \mu(V_1)$ since otherwise we can apply part (ii) of Theorem 3.1 directly. Put $\Delta := \mu(V_1) - \mu(V^{(N)})$.

Choose $\alpha_0, \dots, \alpha_N \in K$ such that $K = k(\alpha_0, \dots, \alpha_N)$ and $\alpha_0, \dots, \alpha_N$ are linearly independent over k; this is possible by our assumption on K. Choose $v_1 \in M_K \setminus S$, and let $v_1, \dots, v_t$ be the different places among v_1^σ ($\sigma \in \mathrm{Gal}(K/k)$). For $l = 1, \dots, t$, let $\sigma_l \in \mathrm{Gal}(K/k)$ be such that $v_l = v_1^{\sigma_l}$ and put $L_{0,v_l} = \sum_{j=0}^N \sigma_l^{-1}(\alpha_j) X_j$. We add other inequalities to (3.5) and show that the set of solutions of the augmented system is Zariski dense in $\mathbb{P}^N(k)$. More precisely, we consider

$$(8.1) \qquad \begin{cases} \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} & (v \in S,\ i = 0, \dots, N_v), \\ \frac{|L_{0,v_l}(\mathbf{x})|_{v_l}}{\|\mathbf{x}\|_{v_l}} \leqslant H(\mathbf{x})^{-(N+1)\Delta/t} & (l = 1, \dots, t) \end{cases}$$

in $\mathbf{x} \in \mathbb{P}^N(k)$.

Let $S' = S \cup \{v_1, \dots, v_t\}$, $N_v = 0$ for $v = v_1, \dots, v_t$, $c_{0v} = (N+1)\Delta/t$ ($v = v_1, \dots, v_t$). Let $\mathcal{L}' = (L_{iv} : v \in S',\ i = 0, \dots, N_v)$, $\mathbf{c}' = (c_{iv} : v \in S',\ i = 0, \dots, N_v)$. Further, $\mathcal{L}'$, $\mathbf{c}'$ satisfy the analogues of (3.2)–(3.4), taking $\pi_{v_l\sigma}(0) = 0$ for $l = 1, \dots, t$.

We show that $V^{(N)}$ is semistable with respect to $(\mathcal{L}', \mathbf{c}')$. Denote the slopes with respect to $(\mathcal{L}', \mathbf{c}')$ by μ'. As observed above, it suffices to prove that $\mu'(U) \leqslant \mu'(V^{(N)})$ for every non-trivial linear subspace U of $V^{(N)}$ defined over k.

Applying (2.3) we obtain $\mu'(V^{(N)}) = \mu(V^{(N)}) + \Delta = \mu(V_1)$. Let U be a proper, non-trivial linear subspace of $V^{(N)}$ defined over k. If $L_{0,v_l} \in U$ for some $l \in \{1, \dots, t\}$ then so are $\sigma(L_{0,v_l})$ for each $\sigma \in \mathrm{Gal}(K/k)$. But this is impossible, for since $\alpha_0, \dots, \alpha_N$ are linearly independent over k, the linear forms $\sigma(L_{0,v_l})$ have rank $N+1$. So the linear forms L_{0,v_l} cannot lie in U, and therefore, $\mu'(U) = \mu(U) \leqslant \mu(V_1) = \mu'(V^{(N)})$. This establishes the semistability. Now by part (ii) of Theorem 3.1, the solutions of (8.1), and hence (3.5), are Zariski dense in $\mathbb{P}^N(k)$.

We now consider the case $1 \leqslant i_0 < r$. So assume that $\mu(V_{i_0+1}/V_{i_0}) < 1$ and that there is no $v \in S$ with (3.7). Let $R := \dim T(V_{i_0})$. If $R = 0$, then all linear forms L_{iv} ($v \in S$, $i = 0, \ldots, N_v$) must vanish identically on $T(V_{i_0})$ and so every $\mathbf{x} \in T(V_{i_0}) \cap \mathbb{P}^N(k)$ is a solution. Assume henceforth that $R > 0$. Since V_{i_0} is defined over k, there is a linear isomorphism φ defined over k from $\mathbb{P}^R$ to $T(V_{i_0})$. This induces an isomorphism φ^* from $V^{(N)}/V_{i_0}$ to $V^{(R)}$ (the linear forms in $R+1$ variables with coefficients in K). Define the tuple of linear forms in $V^{(R)}$ $\mathcal{M} = (M_{iv} : v \in S, i = 0, \ldots, N_v)$, where $M_{iv} := \varphi^*(L_{iv}|_{T(V_{i_0})})$. Thus, if $\mathbf{y} \in \mathbb{P}^R(k)$ then $\varphi(\mathbf{y}) \in T(V_{i_0}) \cap \mathbb{P}^N(k)$ and $M_{iv}(\mathbf{y}) = L_{iv}(\varphi(\mathbf{y}))$. Choose $\delta > 0$ such that $\mu(V_{i_0+1}/V_{i_0}) + s\delta < 1$ where $s := \#S$ and define $\mathbf{d} = (c_{iv} + \delta : v \in S, i = 0, \ldots, N_v)$. Notice that $(\mathcal{M}, \mathbf{d})$ satisfies the analogues of (3.2)–(3.4). Further, for any non-trivial linear subspace U of $V^{(N)}$ strictly containing V_{i_0} we have $\mu_{\mathcal{M},\mathbf{d}}(\varphi^*(U/V_{i_0})) = \mu(U/V_{i_0}) + s\delta$ and the Harder-Narasimhan filtration of $V^{(R)}$ with respect to $(\mathcal{M}, \mathbf{d})$ is $(\mathbf{0}) \subsetneq \varphi^*(V_{i_0+1}/V_{i_0}) \subsetneq \cdots \subsetneq V^{(R)}$. Now $\mu_{\mathcal{M},\mathbf{d}}(\varphi^*(V_{i_0+1}/V_{i_0})) < 1$ and so by what has been established above, the set of solutions of

$$\frac{|M_{iv}(\mathbf{y})|_v}{\|\mathbf{y}\|_v} \leqslant H(\mathbf{y})^{-c_{iv}-\delta} \quad (v \in S,\ i = 0, \ldots, N_v) \tag{8.2}$$

in $\mathbf{y} \in \mathbb{P}^R(k)$ is Zariski dense in $\mathbb{P}^R(k)$. Using that the heights and norms of $\mathbf{y}$ and $\mathbf{x} = \varphi(\mathbf{y})$ are equal up to a bounded factor, we infer that there is a constant $A > 0$ such that if $\mathbf{y} \in \mathbb{P}^R(k)$ satisfies (8.2), then $\mathbf{x} := \varphi(\mathbf{y})$ lies in $T(V_{i_0}) \cap \mathbb{P}^N(k)$ and satisfies

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant AH(\mathbf{x})^{-c_{iv}-\delta} \quad (v \in S,\ i = 0, \ldots, N_v).$$

If $H(\mathbf{y})$ is sufficiently large, then $\mathbf{x}$ satisfies (3.5). This shows that the solutions $\mathbf{x} \in T(V_{i_0}) \cap \mathbb{P}^N(k)$ of (3.5) are Zariski dense in $T(V_{i_0})$. $\square$

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