ON NEARLY LINEAR RECURRENCE SEQUENCES

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To Professor Robert Tichy on the occasion of his 60th birthday

Abstract. A nearly linear recurrence sequence (nlrs) is a complex sequence (a_n) with the property that there exist complex numbers $A_0,\ldots,$ A_{d-1} such that the sequence $\big(a_{n+d} + A_{d-1}a_{n+d-1} + \cdots + A_0a_n\big)_{n=0}^{\infty}$ is bounded. We give an asymptotic Binet-type formula for such sequences. We compare (a_n) with a natural linear recurrence sequence (lrs) $(\tilde{a}_n)$ associated with it and prove under certain assumptions that the difference sequence $(a_n - \tilde{a}_n)$ tends to infinity. We show that several finiteness results for lrs, in particular the Skolem-Mahler-Lech theorem and results on common terms of two lrs, are not valid anymore for nlrs with integer terms. Our main tool in these investigations is an observation that lrs with transcendental terms may have large fluctuations, quite different from lrs with algebraic terms. On the other hand we show under certain hypotheses, that though there may be infinitely many of them, the common terms of two nlrs are very sparse. The proof of this result combines our Binet-type formula with a Baker type estimate for logarithmic forms.

1. Introduction

This paper was motivated by the investigations on shift radix systems, defined in [1]. For real numbers $S_0,\ldots,S_{d-1}$ and initial values $s_0,\ldots,s_{d-1} \in \mathbb{Z}$, the inequality

$$0 \le s_{n+d} + S_{d-1}s_{n+d-1} + \cdots + S_0s_n < 1,\ n \ge 0, \tag{1}$$

uniquely defines a sequence of integers (s_n). If $S_0,\ldots,S_{d-1} \in \mathbb{Z}$ then (s_n) is a linear recurrence sequence. However, if some of the coefficients are non-integers, then we get sequences of a different nature. In earlier papers [2], [4], [11] the case $d = 2, S_0 = 1$ and $|S_1| < 2$ was investigated, as a model of discretized rotation in the plane. In that case it is conjectured that the sequence (s_n) is always periodic.

In this paper, we largely generalize the sequences given by shift radix systems. Let $A_0,\ldots,A_{d-1} \in \mathbb{C}$. Let (a_n) be a sequence of complex numbers and define the error sequence (e_n) by the initial terms $e_0 = \cdots = e_{d-1} = 0$

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and by the equations

$$e_{n+d} = a_{n+d} + A_{d-1}a_{n+d-1} + \cdots + A_0 a_n \tag{2}$$

for $n \geq 0$. We call (a_n) a *nearly linear recurrence sequence*, in shortcut nlrs, if for some choice of d and $A_0, \ldots, A_{d-1}$, the sequence $(|e_n|)$ is bounded. The sequence (s_n) from (1) is obviously an nlrs because in that case the terms of the error sequence lie in the interval $[0,1)$. An interesting number theoretical example is when a_n lies in the integer ring R of an imaginary quadratic field and e_n is chosen to be in a fundamental region of the lattice associated with R, see [16].

It is easily shown that for a given nlrs (a_n), the set of polynomials $B_t x^t + B_{t-1}x^{t-1} + \cdots + B_0$ with complex coefficients such that the sequence $(\sum_{i=0}^{t} B_i a_{n+i})$ is bounded is an ideal of the polynomial ring $\mathbb{C}[x]$, called the *ideal of* (a_n). There is a unique, monic polynomial generating the ideal of (a_n), called the *characteristic polynomial of* (a_n). This corresponds to the necessarily unique relation (2) of minimal length for which (e_n) is bounded.

We mention here that the characteristic polynomial of a linear recurrence sequence (lrs) (a_n) may be different from the characteristic polynomial of (a_n) when viewed as an nlrs. For instance, the Fibonacci sequence (a_n) given by $a_0 = 0$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$ has characteristic polynomial $x^2 - x - 1$ when viewed as an lrs, but characteristic polynomial $x - \theta$ with $\theta = \frac{1}{2}(1 + \sqrt{5})$ when viewed as an nlrs, since the sequence $(a_{n+1} - \theta a_n)$ is bounded. Indeed we will see in Lemma 2.1 (i) in Section 2, that the characteristic polynomial of an nlrs does not have roots of modulus < 1, and the characteristic polynomial of an lrs and that of the sequence viewed as an nlrs differ only by factors of the form $x - \alpha$ with $|\alpha| < 1$.

Let (a_n) be an nlrs and

$$P(x) = x^d + A_{d-1}x^{d-1} + \cdots + A_0$$

its characteristic polynomial. Further, let (e_n) be the error sequence from (2). Define the generating function

$$c(z) = \sum_{j=1}^{\infty} e_{d+j-1} z^{-j}.$$

Since (e_n) is bounded, $c(z)$ is convergent for all complex z with $|z| > 1$. If, moreover, (e_n) is a sequence of real numbers then we have $c(\overline{z}) = \overline{c(z)}$ for all $z \in \mathbb{C}, |z| > 1$, where $\overline{z}$ denotes the complex conjugate of z.

To (a_n) we associate two lrs $(\hat{a}_n)$ and $(\tilde{a}_n)$, as follows. Let $(\hat{a}_n)$ denote the lrs having the initial terms $\hat{a}_0 = \cdots = \hat{a}_{d-2} = 0, \hat{a}_{d-1} = 1$ and satisfying the recursion

$$\hat{a}_{n+d} + A_{d-1}\hat{a}_{n+d-1} + \cdots + A_0\hat{a}_n = 0. \tag{3}$$

The lrs $(\tilde{a}_n)$ is defined by the same recursion (3) with different initial terms $\tilde{a}_j = a_j$ $(j = 0, \ldots, d-1)$.

For the distinct roots $\alpha_1, \dots, \alpha_h$ of $P(x)$ denote by $m_1, \dots, m_h$ their respective multiplicities. Although in this paper we are mainly interested in the separable case, where all multiplicities are equal to 1, we recall the so called *Binet formula*

$$\hat{a}_n = \hat{g}_1(n)\alpha_1^n + \cdots + \hat{g}_h(n)\alpha_h^n \tag{4}$$

in general form. Here the polynomials $\hat{g}_j(x)$ are of degree at most $m_j - 1$ and with coefficients from the field $\mathbb{Q}(\alpha_1, \dots, \alpha_h)$ for $j = 1, \dots, h$. For $\tilde{a}_n$ we have a similar expression, with polynomials $\tilde{g}_j(x)$ instead of $\hat{g}_j(x)$. In the case that $P(x)$ is separable, i.e., that all its roots are simple, the polynomials $\hat{g}_j(x)$, $\tilde{g}_j(x)$ are just constants and we write $\hat{g}_j$, $\tilde{g}_j$ for them. With these notions we will prove the following theorem, the essential part of which is a Binet-type expression for nlrs.

Theorem 1.1. *Assume that the characteristic polynomial of the nlrs* (a_n) *is separable and its zeros are ordered as*

$$|\alpha_1| \geq \cdots \geq |\alpha_{r_1}| > 1 = |\alpha_{r_1+1}| = \cdots = |\alpha_{r_1+r_2}|,$$

where $r_1 + r_2 = d$. *Denote by* $\tilde{g}_j, \hat{g}_j$ *the (constant) coefficients of* $\alpha_j^n, j = 1, \dots, d$ *in the expression* (4) *of* $\tilde{a}_n$ *and* $\hat{a}_n$ *respectively. Then*

(i) *if* $r_1 > 0$ *and* $r_2 = 0$ *then*

$$a_n = (\tilde{g}_1 + \hat{g}_1 c(\alpha_1))\alpha_1^n + \cdots + (\tilde{g}_{r_1} + \hat{g}_{r_1} c(\alpha_{r_1}))\alpha_{r_1}^n + O(1)$$

and $\tilde{g}_i + \hat{g}_i c(\alpha_i) \neq 0$ *for* $i = 1, \dots, r_1$*;*

(ii) *if* $r_1 > 0$ *and* $r_2 > 0$ *then*

$$a_n = (\tilde{g}_1 + \hat{g}_1 c(\alpha_1))\alpha_1^n + \cdots + (\tilde{g}_{r_1} + \hat{g}_{r_1} c(\alpha_{r_1}))\alpha_{r_1}^n + O(n)$$

and $\tilde{g}_i + \hat{g}_i c(\alpha_i) \neq 0$ *for* $i = 1, \dots, r_1$*;*

(iii) *and if* $r_1 = 0$ *and* $r_2 > 0$ *then*

$$a_n = O(n).$$

We prove this theorem in Section 2. It is easy to show that the converse of Theorem 1.1 (i) is also true, that is, if $a_n = \beta_1\alpha_1^n + \cdots + \beta_{r_1}\alpha_{r_1}^n + O(1)$ for certain constants $\beta_1, \dots, \beta_{r_1}$ then (a_n) is an nlrs. We will present the simple proof in Section 2. Moreover, at the end of Section 2 we give examples showing that the $O(n)$-term in Theorem 1.1 (ii), (iii) can not be improved.

In Section 3 we prove first that the fluctuation of an lrs can be extremely large, then we analyze the distance between an nlrs and a naturally chosen lrs. We also deduce some other consequences for nlrs. First, we show that if (a_n) is an nlrs with separable characteristic polynomial and $\alpha_1, \dots, \alpha_{r_1}$ are as in Theorem 1.1, then the constants $c_1, \dots, c_{r_1}$ such that $a_n = c_1\alpha_1^n + \cdots + c_{r_1}\alpha_{r_1}^n + O(n)$ are unique. Second, we prove that the analogue of the Skolem-Lech-Mahler theorem, see e.g. [10], [19], does not hold generally for nlrs with at least two dominating roots with equal absolute values.

In the last Section 4 we investigate the common terms of nlrs, i.e., the solutions (k, m) in non-negative integers of the equation

$$a_k = b_m \tag{5}$$

for two nlrs (a_n), (b_n). We consider the case that the characteristic polynomials of (a_n), (b_n) have multiplicatively independent, real algebraic dominating roots of modulus larger than 1. For lrs (a_n), (b_n) we know that in that case (5) has only finitely many solutions. We give an example, showing that for nlrs this is no longer true. On the other hand, we show that the solutions of (5) are very sparse. More precisely, we show that if (k_1, m_1), (k_2, m_2) are any two distinct solutions of (5) with $\max(k_2, m_2) \geq \max(k_1, m_1)$, then in fact $\max(k_2, m_2)$ exceeds an exponential function of $\max(k_1, m_1)$.

2. Proof of Theorem 1.1

We start with a lemma which imposes some restrictions on the characteristic polynomial of an nlrs.

Lemma 2.1. *Let (a_n) be an nlrs with characteristic polynomial $P(x)$.*

(i) *The roots of $P(x)$ all have modulus ≥ 1.*
(ii) *Assume that $a_n = O(n)$ holds for all n. Then the roots of $P(x)$ all have modulus equal to 1.*

Proof. Let $Q(x) = \sum_{i=0}^{t} Q_i x^i$ be in the ideal of (a_n). Let α be a zero of $Q(x)$ and write $Q(x) = (x - \alpha)R(x)$, $R(x) = \sum_{i=0}^{t-1} R_i x^i$. Define the sequences (q_n), (r_n) by

$$q_{n+t} = \sum_{i=0}^{t} Q_i a_{n+i}, \quad r_{n+t-1} = \sum_{i=0}^{t-1} R_i a_{n+i} \text{ for } n \geq 0. \tag{6}$$

By assumption, the sequence (q_n) is bounded. Putting $R_{-1} = R_t = 0$, we have $Q_i = R_{i-1} - \alpha R_i$ for $i = 0, \ldots, t$, hence

$$\begin{aligned} q_{n+t} &= \sum_{i=0}^{t} (R_{i-1} - \alpha R_i) a_{n+i} = \sum_{i=1}^{t} R_{i-1} a_{n+i} - \alpha \sum_{i=0}^{t-1} R_i a_{n+i} \\ &= r_{n+t} - \alpha r_{n+t-1}. \end{aligned} \tag{7}$$

(i) We prove that if $|\alpha| < 1$, then the sequence (r_n) is also bounded, i.e., $R(x) = Q(x)/(x - \alpha)$ is in the ideal of (a_n). By repeatedly applying this, we see that the ideal of (a_n) contains a polynomial all whose zeros have modulus ≥ 1. In particular, the characteristic polynomial of (a_n), being a divisor of this polynomial, cannot have zeros of modulus < 1.

Let $C := \max(|r_{t-1}|, |q_t|, |q_{t+1}|, \ldots)$. By (7) we have

$$|r_{n+t}| \leq C + |\alpha| \cdot |r_{n+t-1}| \text{ for all } n \geq 0,$$

implying

$$|r_{n+t}| \leq C \cdot \left(1 + |\alpha| + |\alpha|^2 + \cdots + |\alpha|^{n+1}\right) \text{ for all } n \geq 0.$$

This shows that $|r_{n+t}| \le C/(1-|\alpha|)$ for all $n \ge 0$, i.e., the sequence (r_n) is bounded.

(ii) We now prove that (r_n) is bounded if $|\alpha| > 1$ when $a_n = O(n)$. Then similarly as above we can deduce that the characteristic polynomial of (a_n) has no roots of modulus > 1. Assume that the sequence (r_n) is not bounded. Let $C := \max(|q_t|, |q_{t+1}|, \ldots)$. There is n_0 such that $|r_{n_0+t}| > 1+C/(|\alpha|-1)$. By (7) we have $|r_{n+1+t}| \ge |\alpha| \cdot |r_{n+t}| - C$ for $n = n_0, n_0+1, \ldots$ and this implies, by induction on t,

$$\begin{aligned} |r_{n+t}| &\ge |r_{n_0+t}| \cdot |\alpha|^{n-n_0} - C(1 + |\alpha| + \cdots + |\alpha|^{n-n_0-1}) \\ &= |r_{n_0+t}| \cdot |\alpha|^{n-n_0} - C\tfrac{|\alpha|^{n-n_0}-1}{|\alpha|-1}. \end{aligned}$$

So $|r_{n+t}| \ge |\alpha|^{n-n_0}$ for $n \ge n_0$. This shows that for $n \ge n_0 + t$, the sequence (r_n) grows exponentially. On the other hand, from our assumption $a_n = O(n)$ it follows that $r_n = O(n)$ from (6). Thus, our assumption that the sequence (r_n) is unbounded leads to a contradiction. □

We now turn to the proof of Theorem 1.1. We keep the notation from the statement of that theorem. We need a technical lemma, originally given in the context of shift radix systems, Lemma 2 of [15].

Lemma 2.2. *We have*

$$a_n = \tilde{a}_n + \sum_{j=1}^{n-d+1} \hat{a}_{n-j} e_{d-1+j}.$$

Proof. By the definition of $(\hat{a}_n)$ and $(\tilde{a}_n)$, it is clearly true for $n \le d-1$. Assume that it is true for $n \le m+d-1$ with $m \ge 0$. Then

$$\begin{aligned} a_{m+d} - \tilde{a}_{m+d} &= e_{m+d} - \sum_{j=1}^{d} A_{d-j}(a_{m+d-j} - \tilde{a}_{m+d-j}) \\ &= e_{m+d} - \sum_{j=1}^{d} A_{d-j} \sum_{k=1}^{m-j+1} \hat{a}_{m+d-j-k} e_{d-1+k} \\ &= e_{m+d} - \sum_{k=1}^{m} e_{d-1+k} \sum_{j=1}^{\min(d,m+1-k)} A_{d-j} \hat{a}_{m+d-j-k}. \end{aligned}$$

Using the definition of $(\hat{a}_n)$ we have

$$\begin{aligned} a_{m+d} - \tilde{a}_{m+d} &= e_{m+d} - \sum_{k=1}^{m} e_{d-1+k}(-\hat{a}_{m+d-k}) \\ &= \sum_{k=1}^{m+1} \hat{a}_{m+d-k} e_{d-1+k} \end{aligned}$$

which finishes the induction. □

Proof of Theorem 1.1. Both sequences $(\tilde{a}_n), (\hat{a}_n)$ can be written in the form (4) with $\hat{g}_j(x) = \hat{g}_j$, $\tilde{g}_j(x) = \tilde{g}_j$ constants and $h = d$. Lemma 2.2 implies

$$\begin{aligned} a_n &= \sum_{i=1}^{d} \tilde{g}_i \alpha_i^n + \sum_{j=1}^{n-d+1} \sum_{i=1}^{d} \hat{g}_i e_{d-1+j} \alpha_i^{n-j} \\ &= \sum_{i=1}^{d} \left(\tilde{g}_i \alpha_i^n + \hat{g}_i \sum_{j=1}^{n-d+1} e_{d-1+j} \alpha_i^{n-j} \right) \\ &= \sum_{i=1}^{d} \alpha_i^n \left(\tilde{g}_i + \hat{g}_i \sum_{j=1}^{n-d+1} e_{d-1+j} \alpha_i^{-j} \right). \end{aligned}$$

If $r_1 = 0$ then the bases of all exponential terms lie in the closed unit disk. Thus all summands are bounded. Further the number of summands is bounded by nd. Thus we proved the theorem for $r_1 = 0$.

The function $c(z)$ is well defined outside the closed unit disk, among others for all $\alpha_1, \ldots, \alpha_{r_1}$. Thus if $r_1 > 0$ then put

$$\begin{aligned} b_n &= \sum_{i=1}^{r_1} (\tilde{g}_i + \hat{g}_i c(\alpha_i)) \alpha_i^n + \sum_{i=r_1+1}^{d} \left(\tilde{g}_i \alpha_i^n + \hat{g}_i \sum_{j=1}^{n-d+1} e_{d-1+j} \alpha_i^{n-j} \right) \\ &= \sum_{i=1}^{r_1} (\tilde{g}_i + \hat{g}_i c(\alpha_i)) \alpha_i^n + O(r_2 n + 1). \end{aligned}$$

Using this notation we obtain

$$\begin{aligned} b_n - a_n &= \sum_{i=1}^{r_1} \hat{g}_i \alpha_i^n \left(c(\alpha_i) - \sum_{j=1}^{n-d+1} e_{d-1+j} \alpha_i^{-j} \right) \\ &= \sum_{i=1}^{r_1} \hat{g}_i \alpha_i^n \left(\sum_{j=n-d+2}^{\infty} e_{d-1+j} \alpha_i^{-j} \right) \\ &= O(|\alpha_1|^d) = O(1). \end{aligned}$$

From the above observations we immediately deduce (i)–(iii), except that in (i),(ii) we still have to verify that $\tilde{g}_i + \hat{g}_i c(\alpha_i) \neq 0$ for $i = 1, \ldots, r_1$. Let $I \subseteq \{1, \ldots, r_1\}$ be the set of indices i with $\beta_i := \tilde{g}_i + \hat{g}_i c(\alpha_i) \neq 0$, and put

$$c_n := a_n - \sum_{i \in I} \beta_i \alpha_i^n.$$

Then (c_n) is an nlrs with $c_n = O(n)$ for all n. By Lemma 2.1 (ii), the characteristic polynomial $g(x)$ of (c_n) has only roots of modulus 1. In general, if (u_n), (v_n) are two nlrs with characteristic polynomials $P_1(x)$, $P_2(x)$, then $u_n + v_n$ is an nlrs, and $P_1(x)P_2(x)$ is in the ideal of $(u_n + v_n)$. In particular, $g(x)\prod_{i \in I}(x - \alpha_i)$ is in the ideal of (a_n). But since the characteristic polynomial of (a_n) has zeros $\alpha_1, \ldots, \alpha_{r_1}$, we must have $I = \{1, \ldots, r_1\}$. □

Remark 2.1. The assertion (iii) of Theorem 1.1 remains true with simple modifications for nlrs with inseparable characteristic polynomial, but with remainder term $O(n^\kappa)$, where κ is the maximum of the multiplicities of the roots of the characteristic polynomial of (a_n). As in our Diophantine application we can not deal with this case, we postpone the study of the inseparable case.

Remark 2.2. The error term $O(n)$ in Theorem 1.1 (ii), (iii) is best possible. For instance, let $\alpha_1, \dots, \alpha_{r_1}$, $\beta_1, \dots, \beta_{r_1}$ be as above, let γ be a non-zero complex number, and let (a_n) be a sequence of complex numbers such that

$$a_n = \beta_1 \alpha_1^n + \cdots + \beta_{r_1} \alpha_{r_1}^n + \gamma n + O(1)$$

holds for all $n \geq 1$. Then (a_n) is an nlrs with characteristic polynomial $(x - \alpha_1) \cdots (x - \alpha_{r_1})(x - 1)$. Similarly, if $a_n = \gamma n + O(1)$ holds for all $n \geq 1$ then (a_n) is an nlrs with characteristic polynomial $x - 1$.

Remark 2.3. It is easy to see that the converse of Theorem 1.1 (i) is also true. Indeed, let (a_n) be a sequence of complex numbers such that

$$a_n = \beta_1 \alpha_1^n + \cdots + \beta_{r_1} \alpha_{r_1}^n + O(1)$$

holds for all $n \geq 1$ with non-zero complex numbers $\alpha_1, \dots, \alpha_{r_1}, \beta_1, \dots, \beta_{r_1}$ satisfying $|\alpha_j| > 1$, $j = 1, \dots, r_1$. Then (a_n) is an nlrs with characteristic polynomial $(x - \alpha_1) \cdots (x - \alpha_{r_1})$. In general, we can show that if there exist non-zero polynomials $g_1, \dots, g_h$ and complex numbers $\alpha_1, \dots, \alpha_h$ with $|\alpha_i| \geq 1$ such that

$$a_n = g_1(n) \alpha_1^n + \cdots + g_h(n) \alpha_h^n + O(1) \tag{8}$$

then (a_n) is an nlrs with characteristic polynomial

$$\prod_{i=1}^{h} (x - \alpha_i)^{1 + \deg g_i}.$$

This is not true anymore if in (8) we replace the error term $O(1)$ by $O(n^\kappa)$ with a positive integer κ. We give a counterexample in the simplest case when $a_n = O(n^\kappa)$. Take a sequence (b_n) which is not eventually periodic, taking two values $\{-1, 1\}$. Then the sequence $(n^\kappa b_n)$ can not be an nlrs. Indeed, if there are $A_0, \dots A_{\nu-1}$ such that $a_n = n^\kappa b_n$ satisfies

$$a_{n+\nu} + A_{\nu-1} a_{n+\nu-1} + \cdots + A_0 a_n = O(1),$$

then by non-periodicity, we can find two increasing sequences of integers (N_j) and (M_j) for $j = 1, 2, \dots$ such that

$$(N_j + \nu)^\kappa b_{N_j+\nu} + A_{\nu-1} (N_j + \nu - 1)^\kappa b_{N_j+\nu-1} + \cdots + A_0 N_j^\kappa b_{N_j} = O(1),$$
$$(M_j + \nu)^\kappa b_{M_j+\nu} + A_{\nu-1} (M_j + \nu - 1)^\kappa b_{M_j+\nu-1} + \cdots + A_0 M_j^\kappa b_{M_j} = O(1)$$

with $b_{N_j+k} = b_{M_j+k}$ for $k = 0, \dots \nu - 1$ and $b_{N_j+\nu} + b_{M_j+\nu} = 0$. Dividing by $(N_j + \nu)^\kappa$ and $(M_j + \nu)^\kappa$ respectively and taking the difference gives an impossibility:

$$2 b_{N_j+\nu} = o(1) \quad \text{as } j \to \infty.$$

3. On the growth of nlrs

Combining Theorem 1.1 with some Diophantine approximation arguments we are able to prove lower and upper estimates for the growth of nlrs. Specializing our results for lrs we get surprising facts in this case too. Moreover we can estimate the growth of the difference sequence $(a_n - \tilde{a}_n)$. We start with the analysis of a special case.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that $r \geq 2$.*

(i) *Let $\eta_1, \dots, \eta_r$ be any pairwise distinct complex numbers lying on the unit circle and $\gamma_1, \dots, \gamma_r$ any non-zero complex numbers. Then there exists a constant $d_1 > 0$ such that*

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| > d_1 \tag{9}$$

holds for infinitely many positive integers n.

(ii) *Let $\eta_1, \dots, \eta_r$ be any pairwise distinct complex numbers lying on the unit circle such that at least one of the quotients η_j/η_r, $1 \leq j < r$ is not a root of unity and $\gamma_1, \dots, \gamma_{r-1}$ any non-zero complex numbers. Then for all $d_2 > 1$ there exists γ_r such that the inequality*

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| < d_2^{-n} \tag{10}$$

holds for infinitely many positive integers n.

Remark 3.1. In relation to (ii), we should remark here that as a consequence of the p-adic Subspace Theorem of Schmidt and Schlickewei, if $\gamma_1, \dots, \gamma_r$, $\eta_1, \dots, \eta_r$ are all algebraic and $|\eta_1| = \cdots = |\eta_r| = 1$, then for every $d_2 > 1$ there are only finitely many positive integers n with (10), see [17] or [5].

In fact, one can show that if $\gamma_1, \dots, \gamma_{r-1}$ are any non-zero complex numbers and $\eta_1, \dots, \eta_r$ any complex numbers on the unit circle, then for almost all complex γ_r in the sense of Lebesgue measure, we have that for every $d_2 > 1$, inequality (10) holds for only finitely many positive integers n. To see this, let S be the set of $\gamma_r \in \mathbb{C}$ for which there exists $d_2 > 1$ such that (10) holds for infinitely many n. Then $S = \bigcup_{k=1}^{\infty} S_k$, where S_k is the set of $\gamma_r \in \mathbb{C}$ such that (10) with $d_2 = 1 + k^{-1}$ holds for infinitely n. For fixed n, k, let $B_{n,k}$ be the set of $\gamma_r \in \mathbb{C}$ satisfying (10) with $d_2 = 1 + k^{-1}$. Then $B_{n,k}$ has Lebesgue measure $\lambda(B_{n,k}) = \pi(1 + k^{-1})^{-2n}$, the measure of a ball in $\mathbb{C}$ of radius d_2^{-n}. Thus, S_k is the set of $\gamma_r \in \mathbb{C}$ that are contained in $B_{n,k}$ for infinitely many n. We have $\sum_{n=1}^{\infty} \lambda(B_{n,k}) < \infty$ so by the Borel-Cantelli Lemma, S_k has Lebesgue measure 0. But then, S must have Lebesgue measure 0.

The proof of the second assertion of Theorem 3.1 is based on the following Diophantine approximation result.

Lemma 3.1. *Let $\eta_1, \dots, \eta_r$ be any pairwise distinct complex numbers lying on the unit circle, at least one is not a root of unity. For every $d > 0$ there are infinitely many n such that $|\eta_j^n - 1| < d$ holds for $j = 1, \dots, r$.*

Proof. We use the inequality

$$|e^z - 1| = |z| \cdot |\sum_{n=1}^{\infty} z^{n-1}/n!| < |z| \cdot e^{|z|},$$

which holds for all complex z.

Let $0 < d < 1$. Write $\eta_j = e^{2\pi i u_j}$ with real numbers u_j for $j = 1, \ldots, r$. Since by assumption not all η_j are roots of unity, at least one of the u_j is irrational. By Dirichlet's approximation theorem (see, e.g., [6, Chap. XI, Thm. 200], there are infinitely many integers n for which there exist integers $m_j = m_j(n)$ such that $|nu_j - m_j| < d/c$ for $j = 1, \ldots, r$, where $c = 2\pi \cdot e^{2\pi}$. For these n,

$$|\eta_j^n - 1| = |e^{2\pi i(nu_j - m_j)} - 1| < e^{2\pi|nu_j - m_j|} 2\pi |nu_j - m_j| < d.$$

□

The second lemma holds under more general assumptions. Its proof was inspired by an idea we found in the Hungarian lecture notes of P. Turán [8] pp. 361–362.

Lemma 3.2. *Let $\eta_1, \ldots, \eta_r$ pairwise different and lying on the unit circle and $\gamma_1(x), \ldots, \gamma_r(x) \in \mathbb{C}[x]$ non-zero. Let $g(n) = \gamma_1(n)\eta_1^n + \cdots + \gamma_r(n)\eta_r^n$ for $n \in \mathbb{Z}$. Assume that $|g(n)| \le G$ for all $n \ge n_0$. Then for $j = 1, \ldots, r$, $\gamma_j(n)$ is a constant, say γ_j, satisfying $|\gamma_j| \le G$.*

Proof. For every real $\alpha \ge 0$, complex number $\xi \ne 1$ with $|\xi| = 1$ and integer $n_1 \ge n_0$ we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{n=n_1}^{n_1+T-1} \frac{\xi^n}{n^\alpha} = 0, \tag{11}$$

which follows from Abel summation.

Let n^α be the highest power of n occurring in $\gamma_1(n), \ldots, \gamma_r(n)$. It may occur in various $\gamma_i(n)$. Suppose for instance that it occurs in $\gamma_r(n)$ and that the corresponding coefficient is b. Then for any $n_1 \ge n_0$,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \sum_{n=n_1}^{n_1+T-1} \frac{g(n)}{n^\alpha \eta_r^n} &= \sum_{j=1}^{r-1} \lim_{T\to\infty} \frac{1}{T} \sum_{n=n_1}^{n_1+T-1} \frac{\gamma_j(n)}{n^\alpha} \left(\frac{\eta_j}{\eta_r}\right)^n \\ &+ \sum_{j=1}^{r-1} \lim_{T\to\infty} \frac{1}{T} \sum_{n=n_1}^{n_1+T-1} \frac{\gamma_r(n)}{n^\alpha} = b \end{aligned}$$

by (11). So $|b| \le G/n_1^\alpha$ for all $n_1 \ge n_0$, implying $\alpha = 0$. Hence $\gamma_1(n), \ldots, \gamma_r(n)$ are all constants, say $\gamma_1, \ldots, \gamma_r$ respectively.

Let $1 \le j \le r$. Then applying (11) with $\alpha = 0$ we obtain

$$\lim_{T\to\infty} \frac{1}{T} \sum_{n=n_0}^{n_0+T-1} \frac{g(n)}{\eta_j^n} = \lim_{T\to\infty} \frac{1}{T} \sum_{n=n_0}^{n_0+T-1} \gamma_j = \gamma_j.$$

On the other hand the modulus of the left hand side is clearly not greater than G. □

Now we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. **(i)** Let $\Gamma := \max\{|\gamma_j| \ : \ j = 1, \ldots, r\}$ and let $0 < d_1 < \Gamma$. If (9) holds for only finitely many integers n then there exists an n_0 such that

$$|\gamma_1\eta_1^n + \cdots + \gamma_r\eta_r^n| \le d_1$$

holds for all $n > n_0$. Then by Lemma 3.2 $|\gamma_j| \le d_1 < \Gamma$ for all $j = 1, \ldots, r$, which is a contradiction.

(ii) Dividing $\gamma_1\eta_1^n + \cdots + \gamma_r\eta_r^n$ by η_r^n, we see that without loss of generality we may assume that $\eta_r = 1$ and at least one of $\eta_1, \ldots, \eta_{r-1}$ is not a root of unity. We take any non-zero $\gamma_1, \ldots, \gamma_{r-1}$ and construct γ_r.

Let $u_n := \gamma_1\eta_1^n + \cdots + \gamma_{r-1}\eta_{r-1}^n$. We construct a sequence (n_k). Let $n_1 := 1$. For $k \ge 1$, given n_k, choose $n_{k+1} > n_k$ such that

$$|\eta_j^{n_{k+1}-n_k} - 1| < (2d_2)^{-n_k}/|rB|, \ j = 1, \ldots, r-1,$$

where $B > \max(|\gamma_1|, \ldots, |\gamma_{r-1}|)$. This is possible by Lemma 3.1. Then

$$\begin{aligned} |u_{n_{k+1}} - u_{n_k}| &\le \sum_{j=1}^{r-1} |\gamma_j\eta_j^{n_k}| \cdot |\eta_j^{n_{k+1}-n_k} - 1| \\ &< (2d_2)^{-n_k} \end{aligned}$$

for $k = 1, 2, \ldots$

Now let

$$\gamma_r := -u_{n_1} - \sum_{k\ge 1}(u_{n_{k+1}} - u_{n_k}).$$

This is a convergent series, and for $l \ge 1$,

$$\begin{aligned} |\gamma_1\eta_1^{n_l} + \cdots + \gamma_{r-1}\eta_{r-1}^{n_l} + \gamma_r| &= |u_{n_l} + \gamma_r| \\ &= |\sum_{k\ge l}(u_{n_{k+1}} - u_{n_k})| \\ &< (2d_2)^{-n_l} + (2d_2)^{-n_{l+1}} + \ldots \\ &\le \frac{2d_2}{2d_2 - 1}(2d_2)^{-n_l} < d_2^{-n_l}, \end{aligned}$$

completing the proof. □

Theorem 3.1 implies that general linear recurrence sequences may have surprisingly big fluctuation.

Corollary 3.1. *Let $r \ge 2$ be an integer and $h > 1$ be a real number. There exists a lrs u_n of degree r such that $u_n \ne 0$ for all n, $|u_n| \gg h^n$ for infinitely many n and $|u_n| \ll h^{-n}$ for infinitely many n.*

Note that the non zero assumption expels trivial ‘degenerate’ sequences like $u_n = h^{2n}(1 + \gamma^n + \gamma^{2n} + \cdots + \gamma^{(r-1)n})$ for a primitive r-th root of unity γ.

Proof. Take distinct algebraic numbers $\eta_1, \ldots, \eta_{r-1}$ that lie on the unit circle and are not roots of unity and set $\eta_r = 1$. Let $D \geq h$ be an integer and put $\alpha_j = D\eta_j$ for $j = 1, \ldots, r$. Finally let $\gamma_j, j = 1, \ldots, r-1$ be non-zero integers. Taking $d_2 = D^2$ there exists by Theorem 3.1 (ii) a complex number γ_r such that

$$|\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| \ll D^{-2n}$$

holds for infinitely many n. Let $u_n = \gamma_1 \alpha_1^n + \cdots + \gamma_r \alpha_r^n$. Then u_n satisfies a linear recursive recursion, for which we have

$$|u_n| = D^n \cdot |\gamma_1 \eta_1^n + \cdots + \gamma_r \eta_r^n| \ll D^{-n} \ll h^{-n}$$

for infinitely many n.

We claim that γ_r is transcendental. Indeed, assume that γ_r is algebraic. Then by [17] for every $\varepsilon > 0$ we have $|u_n| \gg D^{n(1-\varepsilon)}$ for sufficiently large n, which is a contradiction. Thus γ_r is transcendental and as $\eta_1, \ldots, \eta_r$ and $\gamma_1, \ldots, \gamma_{r-1}$ are algebraic, we have $u_n \neq 0$ for all n. By using Theorem 3.1 (i), $|u_n| \gg D^n$ for infinitely many n. □

We deduce some consequences for nlrs.

Corollary 3.2. *Let (a_n) be an nlrs with separable characteristic polynomial and assume that $\alpha_1, \ldots, \alpha_{r_1}$ are its zeros of modulus > 1 with $r_1 \geq 1$. Then there are unique complex numbers $\beta_1, \ldots, \beta_{r_1}$ such that*

$$a_n = \beta_1 \alpha_1^n + \cdots + \beta_{r_1} \alpha_{r_1}^n + O(n)$$

holds for all $n \geq 1$.

Proof. Such $\beta_1, \ldots, \beta_{r_1}$ exist by Theorem 1.1. Suppose there is also a tuple of complex numbers $(\gamma_1, \ldots, \gamma_{r_1}) \neq (\beta_1, \ldots, \beta_{r_1})$ such that $a_n = \sum_{i=1}^{r_1} \gamma_i \alpha_i^n + O(n)$ for all n. Let k be an index i for which $\gamma_i \neq \beta_i$ and $|\alpha_i|$ is maximal. Then

$$\sum_{i=1}^{r_1} (\gamma_i - \beta_i)(\alpha_i/\alpha_k)^n = O(n \cdot |\alpha_k|^{-n}) \quad \text{as } n \to \infty.$$

But this clearly contradicts Theorem 3.1 (i). □

Recall that the Skolem-Mahler Lech theorem, see e.g. [10] or [19], asserts that if (a_n) is a lrs, then the set of n with $a_n = 0$ is either finite or contains an infinite arithmetic progression. We show that there is no analogue for nlrs.

Corollary 3.3. *There exists an nlrs with integer terms (a_n) such that $\limsup_{n\to\infty} |a_n| = \infty$, but $a_n = 0$ for infinitely many n and the set of n with $a_n = 0$ does not contain an infinite arithmetic progression.*

Proof. Let $\alpha_1, \dots, \alpha_r$ $(r \geq 2)$ be complex numbers such that

$$|\alpha_1| = \cdots = |\alpha_r| > 1,$$

none of the quotients α_i/α_j $(1 \leq i < j \leq r)$ is a root of unity, and $\overline{\alpha_i} \in \{\alpha_1, \dots, \alpha_r\}$ for $i = 1, \dots, r$. Choose non-zero $\gamma_1, \dots, \gamma_{r-1} \in \mathbb{C}$. Let $C > 1$. By Theorem 3.1 (ii) there exists $\gamma_r \in \mathbb{C}$ such that

$$|\gamma_1\alpha_1^n + \cdots + \gamma_r\alpha_r^n| < C^{-n}$$

for infinitely many n. Let t_n denote the real part of $\sum_{i=1}^r \gamma_i\alpha_i^n$ for all $n \geq 0$ or, in case this is identically 0, $t_n = \frac{1}{2\sqrt{-1}} \cdot \sum_{i=1}^r \gamma_i\alpha_i^n$ for all n. Then t_n is real for all n and $|t_n| < C^{-n}$ for infinitely many n, and by our assumption on the α_i-s, there are $\delta_1, \dots, \delta_r$, not all 0 such that $t_n = \sum_{i=1}^r \delta_i\alpha_i^n$ for all n. Now we take $a_n := \lfloor t_n \rceil$, where $\lfloor x \rceil := [x + 1/2]$ for $x \in \mathbb{R}$. Then clearly, (a_n) is an nlrs in $\mathbb{Z}$ and $a_n = 0$ for infinitely many n.

It remains to prove that the set of n with $a_n = 0$ does not contain an arithmetic progression. Consider the arithmetic progression $u, u+v, u+2v, \dots$. By Theorem 3.1 (i), there are a constant $c > 0$ and infinitely many integers m such that

$$|t_{u+mv}| = |(\delta_1\alpha_1^u)(\alpha_1^v)^m + \cdots + (\delta_r\alpha_r^u)(\alpha_r^v)^m| > c|\alpha_1^v|^m.$$

This implies that $|a_{u+mv}| > c'|\alpha_1^v|^m$ for infinitely many m, where $0 < c' < c$, so in particular, $a_{u+mv} \neq 0$ for infinitely many m. This shows at the same time that $\limsup_{n\to\infty} |a_n| = \infty$. □

In the next corollaries, we compare the nlrs (a_n) and its corresponding lrs analogue $(\tilde{a}_n)$. Although $a_n = \tilde{a}_n$ for $0 \leq n < d$, we can show under a mild condition that the difference $a_n - \tilde{a}_n$ can not be bounded.

Corollary 3.4. *Under the same assumptions as in Theorem 1.1 set*

$$R = \{\alpha_i \mid i = 1, \dots, r_1 \text{ and } c(\alpha_i) \neq 0\}.$$

Assume that $R \neq \emptyset$. If among the elements of R there is exactly one of maximum modulus, then $\lim_{n\to\infty} |a_n - \tilde{a}_n| = \infty$, otherwise

$$\limsup_{n\to\infty} |a_n - \tilde{a}_n| = \infty.$$

Proof. Observe that the coefficients $\hat{g}_j$ in (4) are all non-zero. Indeed, otherwise $\hat{a}_i$ would be a lrs of order less than d and hence identically 0, which it isn't.

(i) Let α_i be the element of R of maximum modulus. Then by Theorem 1.1, we have

$$a_n - \tilde{a}_n = \hat{g}_i c(\alpha_i)\alpha_i^n + o(|\alpha_i|^n).$$

(ii) Let $\alpha_{i_1}, \dots, \alpha_{i_s}$ be the elements of R of maximum modulus. As in case (i) we have

$$\begin{aligned} a_n - \tilde{a}_n &= \hat{g}_{i_1} c(\alpha_{i_1})\alpha_{i_1}^n + \cdots + \hat{g}_{i_s} c(\alpha_{i_s})\alpha_{i_s}^n + o(|\alpha_{i_s}|^n) \\ &= d(n)|\alpha_{i_s}|^n + o(|\alpha_{i_s}|^n), \end{aligned}$$

where

$$d(n) = \hat{g}_{i_1} c(\alpha_{i_1}) \left(\frac{\alpha_{i_1}}{|\alpha_{i_s}|} \right)^n + \cdots + \hat{g}_{i_s} c(\alpha_{i_s}) \left(\frac{\alpha_{i_s}}{|\alpha_{i_s}|} \right)^n .$$

As the assumptions of Theorem 3.1 (i) hold, we can ensure that $|d(n)| > d_0 > 0$ for infinitely many n, and the proof is complete. □

In the next corollary we need stronger assumptions on the nlrs.

Corollary 3.5. *Assume that $A_0, \ldots, A_{d-2}$ are real, the terms of the nlrs (a_n) are integers and $e_n \geq 0$ for all n, where (e_n) denotes the corresponding error sequence. Further assume that the characteristic polynomial has a single root of maximum modulus, which is real, greater than one and not an algebraic integer. Then $\lim_{n\to\infty} |a_n - \tilde{a}_n| = \infty$.*

Proof. Let α_1 be the root of maximum modulus of the characteristic polynomial of (a_n). We prove that $c(\alpha_1) \neq 0$.

Under our assumptions (e_n) is a sequence of real numbers. By definition of $c(z)$, $c(\alpha_1) = 0$ if and only if $e_n = 0$ for all n. This is equivalent to $a_n = \tilde{a}_n$ for all n. If $\tilde{a}_n = a_n$ for all n, then $(\tilde{a}_n)$ is an integer valued lrs. By a result of Fatou (see e.g. [18]), we know that the formal power series

$$\sum_{n=0}^{\infty} \tilde{a}_n x^n$$

with integer coefficients represents a rational function $P(x)/Q(x)$ with $P, Q \in \mathbb{Z}[x]$ and $Q(0) = 1$. Putting $Q(x) = \sum_{i=0}^{m} q_i x^i$ with $q_0 = 1$, the sequence $(\tilde{a}_n)$ satisfies a linear recurrence:

$$\tilde{a}_{n+m} + q_1 \tilde{a}_{n+m-1} + \cdots + q_m \tilde{a}_n = 0$$

for a sufficiently large n, as well as (3), i.e.,

$$\tilde{a}_{n+d} + A_{d-1} \tilde{a}_{n+d-1} + \cdots + A_0 \tilde{a}_n = 0.$$

Considering the characteristic polynomials of these two recursions, we have $Q(1/\alpha_1) = 0$ and hence α_1 is an algebraic integer. This is a contradiction and we know that $c(\alpha_1) \neq 0$. From Corollary 3.4, we get the result. □

Corollary 3.5 has the following immediate consequence.

Corollary 3.6. *If the characteristic polynomial of the nlrs (s_n) from (1) has a single root of maximum modulus, and this is real, greater than one and not an algebraic integer, then $\lim_{n\to\infty} |s_n - \tilde{s}_n| = \infty$.*

4. Common values

Common values of lrs with algebraic terms are quite well investigated. Thanks to the theory of S-unit equations, developed by Evertse [5] and by van der Poorten and Schlickewei [17], M. Laurent [9] characterized those pairs of lrs's (a_n), (b_n) for which there are infinitely many pairs of indices (k, m) with $a_k = b_m$. His result is not effective. A particular case of Laurent's result is that if (a_n), (b_n) have separable characteristic polynomials then the set of (k,m) with $a_k = b_m$ is either finite, or the union of a finite set and of finitely many rational lines. A rational line is a set of the type $\{(k,m) \in \mathbb{Z}^2 : \, k,m \geq K_0, \, Ak + Bm + C = 0\}$, where K_0 is a constant ≥ 0 and A, B, C are rational numbers.

We recall that two non-zero complex numbers α, β are multiplicatively dependent if there are integers m, n, not both zero, with $\alpha^m \beta^n = 1$, and multiplicatively independent otherwise. We say that a root α of a polynomial $P(x)$ with complex coefficients is *dominating* if $|\alpha| > |\beta|$ for every other root β of P.

In the case that the characteristic polynomials of the lrs (a_n), (b_n) both have a dominating root and if these two roots are multiplicatively independent, Mignotte [13] proved that there are only finitely many k, m with $a_k = b_m$ and gave an effective upper bound for them. His result was generalized to sequences with at most three, not necessarily dominating roots by Mignotte, Shorey and Tijdeman [14]. One finds a good overview on effective results concerning common values of lnr's in the book of Shorey and Tijdeman [19]. In the above mentioned results the Binet formula (4) plays a central role.

Theorem 1.1 gives a Binet-type formula for nlrs's, which suggests to study common values of such sequences. The next result implies that the situation for nlrs is quite different from that of lrs.

Theorem 4.1. *Let α, β be two multiplicatively independent real numbers > 1. Then there exist nlrs (a_n), (b_n) with integer terms, having characteristic polynomials with dominating roots α, β, respectively, such that there are infinitely many pairs of non-negative integers (k,m) with $a_k = b_m$. This set of pairs (k,m) has finite intersection with every rational line.*

In the proof we need some lemmas.

Lemma 4.1. *Let a, b be positive real numbers with $a/b \notin \mathbb{Q}$ and let $C > 1$. Then there exists $c \in \mathbb{R}$ such that the inequality*

$$|ak - bm - c| < C^{-(k+m)}$$

has infinitely many solutions in non-negative integers k, m.

Proof. We construct an infinite sequence of triples $(k_n, m_n, \varepsilon_n)$ $(n = 1, 2, \ldots)$ such that $0 < \varepsilon_n < 1$, k_n, m_n are positive integers with $|ak_n - bm_n| < \varepsilon_n$ for all n, and

$$\varepsilon_{n+1} < \min\left(\tfrac{1}{2}\varepsilon_n, (2C)^{-(k_1+\cdots+k_n+m_1+\cdots+m_n)}\right).$$

The existence of such an infinite sequence follows easily from Dirichlet's approximation theorem or the continued fraction expansion of a/b. Now put $s_n := ak_n - bm_n$ and

$$c := \sum_{n=1}^{\infty} s_n.$$

This series is easily seen to be convergent. Further we have, on putting $k'_n := k_1 + \cdots + k_n$, $m'_n := m_1 + \cdots + m_n$,

$$|ak'_n - bm'_n - c| \leq \sum_{l=n+1}^{\infty} |s_l| < 2(2C)^{-(k'_n+m'_n)} < C^{-(k'_n+m'_n)}.$$

This clearly proves our lemma. □

Lemma 4.2. *Let α, β be multiplicatively independent reals > 1 and $C > 1$. Then there exists $\gamma > 1$ such that the inequality*

$$|\alpha^k - \gamma\beta^m| < C^{-(k+m)}$$

has infinitely many solutions in positive integers k, m.

Proof. By the previous lemma, there exist $\gamma > 1$ and infinitely many pairs of positive integers (k, m), such that

$$|k \log \alpha - m \log \beta - \log \gamma| < (2\beta C)^{-(k+m)}.$$

Using the inequality $|e^x - 1| \leq 2|x|$ for real x sufficiently close to 0, we infer that there are infinitely many pairs (k, m) of positive integers such that

$$\begin{aligned} |\alpha^k - \gamma\beta^m| &= \gamma\beta^m \cdot |\alpha^k\beta^{-m}\gamma^{-1} - 1| \\ &\leq 2\gamma\beta^m |k \log \alpha - m \log \beta - \log \gamma| \\ &\leq 2\gamma\beta^m (2\beta C)^{-(k+m)} < C^{-(k+m)}. \end{aligned}$$

□

Proof of Theorem 4.1. The previous lemma implies that there exists $\gamma > 0$ such that $[\alpha^k] - [\gamma\beta^m] \in \{-1, 0, 1\}$ for infinitely many pairs of non-negative integers k, m. This implies that there are $u \in \{-1, 0, 1\}$ and infinitely many pairs of non-negative integers k, m such that $[\alpha^k] - [\gamma\beta^m] = u$. Now define (a_n), (b_n) by $a_n := [\alpha^n]$, $b_n := [\gamma\beta^n + u]$. These are easily seen to be nlrs with dominating roots α, β, respectively, and clearly, $a_k = b_m$ for infinitely many pairs k, m. There is $C > 0$ such that $|k \log \alpha - m \log \beta| \leq C$ for all pairs of non-negative integers k, m with $a_k = b_m$. Since by assumption, $\log \alpha / \log \beta \notin \mathbb{Q}$, only finitely many of these pairs (k, m) can lie on a given rational line. This completes our proof. □

Below, we consider the set of pairs (k, m) satisfying $a_k = b_m$ for two given nlrs (a_n), (b_n) in more detail. One of our results is that if (a_n), (b_n) satisfy the conditions of Theorem 4.1 and if moreover α, β are algebraic, then the set of these pairs (k, m) is very sparse.

The main ingredient of our proof is an effective lower bound for linear forms in logarithms of algebraic numbers. We use here a theorem of Matveev [12]. For our qualitative result below it would be enough to use a less explicit form, but we could save almost nothing with it. Before formulating the theorem we have to define the *absolute logarithmic height* - $h(\beta)$ - of an algebraic number β. Let β be an algebraic number of degree n and denote by b_0 the leading coefficient of its defining polynomial. Further, denote by $\beta = \beta^{(1)}, \dots, \beta^{(n)}$ the (algebraic) conjugates of β. Then

$$h(\beta) = \frac{1}{n}\left(\log|b_0| + \sum_{j=1}^{n} \log\max\{|\beta^{(j)}|, 1\}\right).$$

Theorem 4.2. *Let $\gamma_1, \dots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field K of degree D and $b_1, \dots, b_t$ rational integers such that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \neq 0.$$

Then

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\} \quad \textit{for all } i = 1, \dots, t.$$

Now we are in the position to state and prove our main result of this section.

Theorem 4.3. *Let (a_n) and (b_n) be two nlrs. Assume that the characteristic polynomials of (a_n), (b_n) have dominating roots α, β respectively, and that α, β are real algebraic, have absolute value > 1, and are multiplicatively independent.*

Then there exist effectively computable constants K_0, K_1, K_2 depending only on the characteristic polynomials, the initial values and the sizes of the error terms of (a_n) and (b_n) such that if $(k_1, m_1), (k_2, m_2) \in \mathbb{Z}^2$ are solutions of the diophantine equation

$$a_k = b_m \tag{12}$$

with $K_0 \leq k_1 < k_2$ then $k_2 > k_1 + K_1\exp(K_2 k_1)$.

Proof. As $|\alpha|, |\beta| > 1$ there exist by Theorem 1.1 non-zero constants γ, δ depending only on the starting terms of the sequences (a_n), (b_n) and the coefficients of their characteristic polynomials, and a constant ε depending only on the second largest zeros such that

$$a_k = \gamma\alpha^k + O(\alpha^{k(1-\varepsilon)}), \quad b_m = \delta\beta^m + O(\beta^{m(1-\varepsilon)}) \tag{13}$$

hold for all large enough k, m. Thus if equation (12) holds then we have

$$\gamma\alpha^k - \delta\beta^m = O(|\alpha|^{k(1-\varepsilon)}) + O(|\beta|^{m(1-\varepsilon)}).$$

For fixed m this inequality has finitely many solutions in k. Let K_0 be a large enough constant, which we will specify later, and assume that (12) has at least one solution (m, k) with $k > K_0$.

We may assume $\alpha, \beta > 0$ without loss of generality. Indeed, otherwise we consider the cases k, m odd and even separately. Moreover we may also assume $\alpha^k > \beta^m$ (equality cannot occur as α and β are multiplicatively independent). Then the last inequality implies

$$\left|\frac{\delta}{\gamma}\frac{\beta^m}{\alpha^k} - 1\right| < C_1\alpha^{-k\varepsilon} + C_2\beta^{-m\varepsilon}.$$

Here and in the sequel the constants $C_1, C_2, \dots$ are effectively computable and depend on the parameters of the sequences, i.e. on their initial terms and the heights of the coefficients and their characteristic polynomials, on ε and on the upper bound of the terms of the error sequences only.

We now assume that K_0 is large enough so that the right hand side of the last inequality is less than $1/2$. Then

$$\beta^m > \left|\frac{\gamma}{2\delta}\right| \cdot \alpha^k,$$

thus

$$\left|\frac{\delta}{\gamma}\frac{\beta^m}{\alpha^k} - 1\right| < C_3\alpha^{-k\varepsilon}. \tag{14}$$

This inequality seems to have already the form for which Matveev's theorem 4.2 could be applied. Unfortunately we are not yet so far because we know nothing about the arithmetic nature of γ and δ. They can be (and are probably usually) transcendental numbers. Thus Theorem 4.2 is not applicable and we cannot deduce an upper bound for $\max\{k, m\}$. On the other hand inequality (14) is strong enough to allow us to prove that the sequence of solutions of (12) is growing very fast.

Indeed, let $(k_1, m_1), (k_2, m_2)$ be solutions of (12) such that $K_0 < k_1 < k_2$. Then (14) holds for both solutions and we get

$$\left|\frac{\delta}{\gamma}\frac{\beta^{m_1}}{\alpha^{k_1}} - \frac{\delta}{\gamma}\frac{\beta^{m_2}}{\alpha^{k_2}}\right| < 2C_3\alpha^{-k_1\varepsilon}.$$

Dividing this inequality by the first term, which lies by (14) in the interval $(\frac{1}{2}, \frac{3}{2})$ we get

$$|\Lambda| < C_4\alpha^{-k_1\varepsilon}, \tag{15}$$

where

$$\Lambda = \beta^{m_2-m_1}\alpha^{k_1-k_2} - 1.$$

As α and β are positive real numbers and multiplicatively independent we have $\Lambda \neq 0$, thus we may apply Theorem 4.2 to it, with $t = 2$. In our situation, D is the degree of the number field $\mathbb{Q}(\alpha, \beta)$, A_1, A_2 are constants depending only on the coefficients of the characteristic polynomials of the sequences. Further $b_1 = m_2 - m_1$ and $b_2 = k_1 - k_2$. We proved above that if k, m are integers with $a_k = b_m$ and $k > K_0$ then either

$$\alpha^k > \beta^m > \left|\frac{\gamma}{2\delta}\right|\alpha^k$$

or

$$\beta^m > \alpha^k > \left|\frac{\delta}{2\gamma}\right|\beta^m$$

holds. We have in both cases

$$\left| m - \frac{\log \alpha}{\log \beta} k \right| < C_5.$$

This implies

$$\left| |m_1 - m_2| - \frac{\log \alpha}{\log \beta} |k_1 - k_2| \right| < C_6. \tag{16}$$

Thus $|b_1| < C_7 |b_2| + C_6$ and Theorem 4.2 implies

$$|\Lambda| > \exp(-C_8 D^2 (1 + \log D) A_1 A_2 (2 + \log C_7 + \log(k_2 - k_1)).$$

Comparing this inequality with (15) we obtain

$$C_9 \log(k_2 - k_1) + C_{10} > C_{11} \varepsilon k_1 - \log C_4,$$

which implies

$$k_2 > k_1 + K_2 \exp(K_1 k_1), \tag{17}$$

with $K_1 = C_{11}\varepsilon / C_9$ and $K_2 = \exp(-C_{10}/C_9 - (\log C_4)/C_9)$. □

We now consider the case that the (a_n), (b_n) have characteristic polynomials with multiplicatively dependent dominant roots. We show that in this case, if the number of pairs (k, m) with $a_k = b_m$ is infinite, then apart from at most finitely many exceptions they lie on a rational line.

Theorem 4.4. *Let (a_n) and (b_n) be nlrs's. Assume that the characteristic polynomials of both sequences are separable, and have dominating roots α, β with $|\alpha| > 1$, $|\beta| > 1$ which are multiplicatively dependent. If the equation*

$$a_k = b_m \tag{18}$$

has infinitely many solutions in non-negative integers k, m then there exist integers u, v, w such that for all but finitely many solutions we have $k = um/v + w/v$.

Proof. Like in the proof of Theorem 4.3 we write

$$a_k = \gamma \alpha^k + O(|\alpha|^{k(1-\varepsilon)}) \quad \text{and} \quad b_m = \delta \beta^m + O(|\beta|^{m(1-\varepsilon)}),$$

with $\gamma, \delta \neq 0$. As α and β are multiplicatively dependent, there exist positive integers u, v such that

$$\alpha^u = \beta^v,$$

i.e., there exists a v-th root of unity ζ with

$$\beta = \zeta \alpha^{u/v}.$$

If $a_k = b_m$ then

$$\gamma \alpha^k - \zeta^m \delta \alpha^{um/v} = O(|\alpha|^{k(1-\varepsilon)}) + O(|\alpha|^{um(1-\varepsilon)/v}). \tag{19}$$

Assume that $k - um/v > \ell_1$, where the integer ℓ_1 is so large that

$$|\delta \alpha^{-\ell_1}| < \left| \frac{\gamma}{3} \right|.$$

If, moreover, k is large enough then dividing (19) by α^k would make the absolute value of the right hand side smaller than $\left|\frac{\gamma}{3}\right|$ too, which is impossible. Thus if (18) has infinitely many solutions k, m then $k - um/v \leq \ell_1$. Similarly, if $um/v - k > \ell_2$, where the integer ℓ_2 is so large that

$$|\gamma\alpha^{-\ell_2}| < \left|\frac{\delta}{3}\right|,$$

then repeating the former argument we get again a contradiction. Thus setting $\ell = \max\{\ell_1, \ell_2\}$ we must have $|k - um/v| \leq \ell$ for all but finitely many solutions of (18).

Thus we have shown that for all but finitely many solutions (k, m) of (18) there is $w \in [-v\ell, v\ell] \cap \mathbb{Z}$ such that $k - um/v = w/v$. We have to show that w is independent of the choice of (k, m). Clearly, there is w such that $k - um/v = w/v$ holds for infinitely many solutions (k, m) of (18). Dividing (19) by $|\alpha|^k$ we see that

$$|\gamma - \zeta^m \delta \alpha^{w/v}| = O(|\alpha|^{-k\varepsilon})$$

for these solutions (k, m). Since the left-hand side of this inequality assumes only finitely many values and the right-hand side can become arbitrarily small, there must be an integer r such that

$$\gamma = \zeta^r \delta \alpha^{w/v}.$$

We show that this uniquely determines w. Indeed, suppose we have $\gamma = \zeta^{r_1}\delta\alpha^{w_1/v} = \zeta^{r_2}\delta\alpha^{w_2/v}$ for two tuples of integers (r_1, w_1), (r_2, w_2). Then $\alpha^{(w_1 - w_2)/v}$ is a root of unity, which implies $w_1 = w_2$ since by assumption, α is not a root of unity. This shows that $k = um/v + w/v$ holds for all but finitely many solutions (k, m) of (18). □

A nearly immediate consequence of Theorem 4.4 is the following assertion.

Corollary 4.1. *Let (a_n) be an nlrs. Assume that its characteristic polynomial is separable, and has a dominant root α with $|\alpha| > 1$. Then the equation*

$$a_k = a_m \tag{20}$$

has only finitely many solutions with $k \neq m$.

Proof. We apply Theorem 4.4 in the situation that the sequences under consideration are equal. We have plainly $u = v = 1$, i.e. $k = m + w$ holds with a fixed integer w for all but finitely many solutions of (20). Then

$$\gamma\alpha^k(\alpha^{m-k} - 1) = O(|\alpha|^{k(1-\varepsilon)}),$$

which is absurd, if $m - k = w \neq 0$. □

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