THE NUMBER OF SOLUTIONS OF THE THUE-MAHLER EQUATION.

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Abstract. Let K be an algebraic number field and S a set of places on K of finite cardinality s, containing all infinite places. We deal with the Thue-Mahler equation over K, (*) $F(x,y) \in \mathcal{O}_S^*$ in $x, y \in \mathcal{O}_S$, where $\mathcal{O}_S$ is the ring of S-integers, $\mathcal{O}_S^*$ is the group of S-units, and $F(X,Y)$ is a binary form with coefficients in $\mathcal{O}_S$. Bombieri [2] showed that if F has degree $r \geq 6$ and F is irreducible over K, then (*) has at most $(12r)^{12s}$ solutions; here two solutions (x_1, y_1), (x_2, y_2) are considered equal if $x_1/y_1 = x_2/y_2$. In this paper, we improve Bombieri's upper bound to $(5\times 10^6\, r)^s$. Our method of proof is not a refinement of Bombieri's. Instead, we apply the method of [5] to Thue-Mahler equations and work out the improvements which are possible in this special case.

§1. Introduction.

Let $F(X,Y) = a_r X^r + a_{r-1} X^{r-1} Y + \cdots + a_0 Y^r$ be a binary form of degree $r \geq 3$ with coefficients in $\mathbb{Z}$ which is irreducible over $\mathbb{Q}$ and $\{p_1, ..., p_t\}$ a (possibly empty) set of prime numbers. Extending a result of Thue [10], Mahler [8] proved that the equation

$$(1.1) \qquad |F(x,y)| = p_1^{z_1} \cdots p_t^{z_t} \quad \text{in } x, y, z_1, \ldots, z_t \in \mathbb{Z} \text{ with } \gcd(x,y) = 1$$

has only finitely many solutions.

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Mahler's result has been generalised to number fields. Let K be an algebraic number field and denote its ring of integers by $\mathcal{O}_K$. Further, denote by M_K the set of places of K. The elements of M_K are the embeddings $\sigma: K \hookrightarrow \mathbb{R}$ which are called *real infinite places*; the pairs of complex conjugate embeddings $\{\sigma, \overline{\sigma}: K \hookrightarrow \mathbb{C}\}$ which are called *complex infinite places*; and the prime ideals of $\mathcal{O}_K$ which are also called *finite places*. For every $v \in M_K$ we define a normalised absolute value $|\cdot|_v$ as follows:

$|\cdot|_v := |\sigma(\cdot)|^{1/[K:\mathbb{Q}]}$ if v is a real infinite place $\sigma : K \hookrightarrow \mathbb{R}$;
$|\cdot|_v := |\sigma(\cdot)|^{2/[K:\mathbb{Q}]} = |\overline{\sigma}(\cdot)|^{2/[K:\mathbb{Q}]}$ if v is a complex infinite place $\{\sigma, \overline{\sigma} : K \hookrightarrow \mathbb{C}\}$;
$|\cdot|_v := (N\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(\cdot)/[K:\mathbb{Q}]}$ if v is a finite place, i.e. prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$;

here $N\mathfrak{p}$ is the norm of $\mathfrak{p}$, i.e. the cardinality of $\mathcal{O}_K/\mathfrak{p}$, and $\mathrm{ord}_{\mathfrak{p}}(x)$ is the exponent of $\mathfrak{p}$ in the prime ideal decomposition of (x).

Let S be a finite set of places of K, containing all infinite places. We define the ring of S-integers and the group of S-units as usual by

$$\mathcal{O}_S = \{x \in K : \ |x|_v \leq 1 \ \text{ for } v \notin S\},$$
$$\mathcal{O}_S^* = \{x \in K : \ |x|_v = 1 \ \text{ for } v \notin S\},$$

respectively, where '$v \notin S$' means '$v \in M_K \backslash S$.' Instead of (1.1) one may consider the equation

$$F(x, y) \in \mathcal{O}_S^* \ \text{ in } (x, y) \in \mathcal{O}_S^2 \ , \tag{1.2}$$

where $F(X, Y)$ is a binary form of degree $r \geq 3$ with coefficients in $\mathcal{O}_S$ which is irreducible over K. An $\mathcal{O}_S^*$*-coset* of solutions of (1.2) is a set $\{\varepsilon(x, y) : \varepsilon \in \mathcal{O}_S^*\}$, where (x, y) is a fixed solution of (1.2). Clearly, every element of such a coset is a solution of (1.2). Now the generalisation of Mahler's result mentioned above states that the set of solutions of (1.2) is the union of finitely many $\mathcal{O}_S^*$-cosets. [1)]

[1)] This follows from Lang's generalisation [6] of Siegel's theorem that an algebraic curve over K of genus at least 1 has only finitely many S-integral points, but was probably known before.

It is easily verified that this implies that (1.1) has only finitely many solutions, by observing that with $S=\{\infty, p_1, \ldots, p_t\}$, ($\infty$ being the infinite place of $\mathbb{Q}$) we have $\mathcal{O}_S^* = \{\pm p_1^{z_1}\cdots p_t^{z_t} : \ z_1, \ldots, z_t \in \mathbb{Z}\}$ and that any coset contains precisely two pairs $(x,y)\in\mathbb{Z}^2$ with $\gcd(x,y)=1$.

There are several papers in which explicit upper bounds for the number of ($\mathcal{O}_S^*$-cosets of) solutions of (1.1) and (1.2) are given, e.g. [7], [4], [2], and the last two papers give bounds independent of the coefficients of the form F. The most recent result among these, due to Bombieri [2], states that if F has degree $r\geq 6$ and S has cardinality s, then (1.2) has at most $(12r)^{12s}$ $\mathcal{O}_S^*$-cosets of solutions. A better bound was obtained earlier in a special case by Bombieri and Schmidt [3], who showed that the Thue equation $F(x,y)=\pm 1$ in $x,y\in\mathbb{Z}$ (which is eq. (1.2) with $K=\mathbb{Q}, S=\{\infty\}$) has at most constant$\times r$ solutions, where the constant can be taken equal to 430 if r is sufficiently large. In this paper we prove:

Theorem 1. *Let K be an algebraic number field and S a finite set of places on K of cardinality s, containing all infinite places. Further, let $F(X,Y)$ be a binary form of degree $r\geq 3$ with coefficients in $\mathcal{O}_S$ which is irreducible over K. Then the set of solutions of*

$$F(x,y)\in\mathcal{O}_S^* \quad \text{in } (x,y)\in\mathcal{O}_S^2 \tag{1.2}$$

is the union of at most

$$\left(5{\times}10^6\, r\right)^s$$

$\mathcal{O}_S^*$-cosets.

Like Bombieri, we distinguish between "large" and "not large" $\mathcal{O}_S^*$-cosets of solutions of (1.2) and treat the large cosets by applying the "Thue principle" (cf. [1]). Our treatment of the not large cosets is not a refinement of Bombieri's, but is based on rather different ideas. Bombieri (similarly as Bombieri and Schmidt in [3]) heavily uses that the number of $\mathcal{O}_S^*$-cosets of solutions of (1.2) does not change when F is replaced by an equivalent form, where equivalence is defined by means of transformations from $GL_2(\mathcal{O}_S)$, and in his proof he uses some complicated notion of reduction of binary forms. Instead, we apply the method of [5] to Thue-Mahler equations. We will see that there is no loss of generality to assume that $F(X,Y) = (X + c^{(1)}Y) \cdots (X + c^{(r)}Y)$ where $c^{(1)}, \ldots, c^{(r)}$ are the conjugates over K of some algebraic number c. The substance of our method is, that we do not apply the Diophantine approximation techniques to a solution (x, y) of (1.2) but to the number $u := x + cy$ and that we work with the absolute Weil height $H(\mathbf{u})$ of the vector $\mathbf{u} = (u^{(1)}, \ldots, u^{(r)})$ consisting of all conjugates of u. In particular, we will reduce eq. (1.2) to certain Diophantine inequalities in terms of u and $H(\mathbf{u})$ and prove a gap principle for these inequalities.

§2. Reduction to another theorem.

Let K, S, F be as in §1. In the proof of Theorem 1 it is no restriction to assume that $F(1,0) = 1$. Namely, suppose that $F(1,0) \neq 1$ and let $(x_0, y_0) \in \mathcal{O}_S^2$ be a solution of (1.2). The ideal in $\mathcal{O}_S$ generated by x_0, y_0 is (1), hence there are $a, b \in \mathcal{O}_S$ such that $ax_0 - by_0 = 1$. Put $\varepsilon := F(x_0, y_0)$ and define

$$G(X,Y) = \varepsilon^{-1} F(x_0 X + bY, y_0 X + aY).$$

Note that G has its coefficients in $\mathcal{O}_S$ and that $G(1,0) = \varepsilon^{-1} F(x_0, y_0) = 1$. Moreover, since $(x, y) \mapsto (x_0 x + by, y_0 x + ay)$ is an invertible transformation from $\mathcal{O}_S^2$ to itself, the number of cosets of solutions of (1.2) does not change when F is replaced by G.

Assuming, as we may, that $F(1,0) = 1$, we have

$$F(X,Y) = (X + c^{(1)}Y) \cdots (X + c^{(r)}Y),$$

where c is algebraic of degree r over K and $c^{(1)}, \ldots, c^{(r)}$ are the conjugates of c over K. Put $L = K(c)$ and let $\mathcal{O}_{L,S}$ denote the integral closure of $\mathcal{O}_S$ in L and

$\mathcal{O}_{L,S}^*$ the unit group of $\mathcal{O}_{L,S}$. Thus, $c \in \mathcal{O}_{L,S}$. Define the K-vector space

$$V = \{x + cy : \; x, y \in K\} \; .$$

V has the following two properties which will be essential in our investigations:

(2.1) V is a two-dimensional K-linear subspace of L;

(2.2) for every basis $\{a, b\}$ of V we have $L = K(b/a)$.

Namely, (2.1) is obvious. Further, if $\{a, b\}$ is a basis of V then $\{a = \alpha + \beta c, \; b = \gamma + \delta c\}$ with $\alpha, \; \beta, \; \gamma, \; \delta \in K$ and $\alpha\delta - \beta\gamma \neq 0$ and therefore $K(b/a) = K(c) = L$.

An $\mathcal{O}_S^*$*-coset* in L is a set $\{\varepsilon u : \varepsilon \in \mathcal{O}_S^*\}$ where u is a fixed element of L. We need:

Lemma 1. *(x, y) is a solution of (1.2) if and only if $x + cy \in V \cap \mathcal{O}_{L,S}^*$. Further, two solutions $(x_1, y_1), \; (x_2, y_2)$ of (1.2) belong to the same $\mathcal{O}_S^*$-coset if and only if $x_1 + cy_1, \; x_2 + cy_2$ belong to the same $\mathcal{O}_S^*$-coset.*

Proof. For $x, y \in \mathcal{O}_S$ we have that $F(x, y)$ is equal to the norm $N_{L/K}(x + cy)$ and that $x + cy \in V \cap \mathcal{O}_{L,S}$. Now the first assertion follows at once from the fact that for $u \in \mathcal{O}_{L,S}$ we have $N_{L/K}(u) \in \mathcal{O}_S^* \iff u \in \mathcal{O}_{L,S}^*$. As for the second assertion, we have for $x_1, y_1, \; x_2, y_2 \in \mathcal{O}_S, \; \varepsilon \in \mathcal{O}_S^*$ that $x_2 + cy_2 = \varepsilon(x_1 + cy_1) \iff (x_2, y_2) = \varepsilon(x_1, y_1)$ since $\{1, c\}$ is linearly independent over K. □

Now Theorem 1 follows at once from Lemma 1 and

Theorem 2. *Let K be an algebraic number field, L a finite extension of K of degree $r \geq 3$, S a set of places on K of finite cardinality s containing all infinite places, and V a K-vector space satisfying (2.1), (2.2). Then the set*

$$V \cap \mathcal{O}_{L,S}^*$$

is the union of at most

$$\left(5{\times}10^6 \, r\right)^s$$

$\mathcal{O}_S^*$-cosets.

§3. Preliminaries.

We need some basic facts about the normalised absolute values introduced in §1 and about heights. Let again K be an algebraic number field and M_K its set of places. For every normalised absolute value $|\cdot|_v$ ($v \in M_K$) we fix a continuation to the algebraic closure $\overline{K}$ of K which we denote also by $|\cdot|_v$. We define the v-adic norm

$$|\mathbf{x}|_v := \max(|x_1|_v, \dots, |x_n|_v) \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \overline{K}^n,\ v \in M_K.$$

We shall frequently use the

Product formula $$\prod_{v \in M_K} |x|_v = 1 \quad \text{for } x \in K^* \ ;$$

we mention that for $x \in \overline{K} \backslash K$ we have in general that $\prod_{v \in M_K} |x|_v \neq 1$. To be able to deal with archimedean and non-archimedean absolute values simultaneously, we introduce the quantities

$$s(v) := \frac{1}{[K:\mathbb{Q}]} \quad \text{if } v \text{ is a real infinite place,}$$
$$s(v) := \frac{2}{[K:\mathbb{Q}]} \quad \text{if } v \text{ is a complex infinite place,}$$
$$s(v) := 0 \quad \text{if } v \text{ is a finite place.}$$

Thus,

$$\text{(3.1)} \qquad \sum_{v \in S} s(v) = 1 \quad \text{for every set of places } S \text{ containing all infinite places,}$$

and

$$|x_1 + \cdots + x_n|_v \leq n^{s(v)} \max(|x_1|_v, \dots, |x_n|_v) \ ,$$
$$\text{(3.2)} \quad |x_1y_1 + \cdots + x_ny_n|_v \leq n^{s(v)} \max(|x_1|_v, \dots, |x_n|_v) \cdot \max(|y_1|_v, \dots, |y_n|_v)$$
$$\text{for } x_1, \dots, x_n,\ y_1, \dots, y_n \in \overline{K},\ v \in M_K \ .$$

Now let L be a finite extension of K of degree r. Denote the K-isomorphic embeddings of L into $\overline{K}$ by $u \mapsto u^{(1)}, \dots, u \mapsto u^{(r)}$, respectively. To every $u \in L$ we associate the vector

$$\mathbf{u} = (u^{(1)}, \dots, u^{(r)}) \ .$$

(Throughout this paper, we adopt the convention that if we use any slanted character to denote an element of L, then we use the corresponding bold face character to denote the r-dimensional vector consisting of the conjugates over K of this element, e.g. if $a \in L$ then $\mathbf{a} = (a^{(1)}, \dots, a^{(r)})$ etc.) We define the *height* of $\mathbf{u}$ by

$$H(\mathbf{u}) := \prod_{v \in M_K} |\mathbf{u}|_v = \prod_{v \in M_K} \max(|u^{(1)}|_v, \dots, |u^{(r)}|_v) \quad \text{for } u \in L \tag{3.3}$$

(in fact, since the coordinates of $\mathbf{u}$ are the conjugates of u this is the usual absolute Weil height of $\mathbf{u}$; later, we will define another height $H(u)$). If $u' = \lambda u$ for some $\lambda \in K^*$ then from the Product formula it follows that

$$H(\mathbf{u}') = \prod_{v \in M_K} |\lambda|_v \cdot H(\mathbf{u}) = H(\mathbf{u}) \ . \tag{3.4}$$

Further, the Product formula implies

$$H(\mathbf{u}) \geq \Big(\prod_{v \in M_K} |u^{(1)} \cdots u^{(r)}|_v \Big)^{1/r} = 1 \quad \text{for } u \in L^* \ , \tag{3.5}$$

since $u^{(1)} \cdots u^{(r)} = N_{L/K}(u) \in K^*$.

Let S be a finite set of places on K, containing all infinite places. The integral closure $\mathcal{O}_{L,S}$ of $\mathcal{O}_S$ in L is equal to $\{u \in L : \ |u^{(i)}|_v \leq 1 \text{ for } i = 1, \dots, r, \ v \notin S\}$. This implies

$$|u^{(1)}|_v = \cdots = |u^{(r)}|_v = |\mathbf{u}|_v = 1 \quad \text{for } u \in \mathcal{O}_{L,S}^*, \ v \notin S \ . \tag{3.6}$$

Insertion of this into (3.3) gives

$$H(\mathbf{u}) = \prod_{v \in S} |\mathbf{u}|_v \quad \text{for } u \in \mathcal{O}_{L,S}^* \ . \tag{3.7}$$

Now let V be a K-vector space satisfying (2.1) and (2.2). Below we define the height of V. Let $\{a, b\}$ be any basis of V. Define the determinants

$$\Delta_{ij}(a, b) := a^{(i)} b^{(j)} - a^{(j)} b^{(i)} \quad \text{for } 1 \leq i, j \leq r.$$

Note that $\Delta_{ij}(a,b) = -\Delta_{ji}(a,b)$ and that $\Delta_{ij}(a,b) = 0$ if $i = j$. According to our convention, we put $\mathbf{a} = (a^{(1)}, \dots, a^{(r)})$, $\mathbf{b} = (b^{(1)}, \dots, b^{(r)})$. Thus, the exterior product of $\mathbf{a}$, $\mathbf{b}$ is the $\binom{r}{2}$-dimensional vector

$$\mathbf{a} \wedge \mathbf{b} := (\Delta_{12}(a,b), \Delta_{13}(a,b), \dots, \Delta_{r-2,r-1}(a,b), \Delta_{r-2,r}(a,b), \Delta_{r-1,r}(a,b)).$$

Now the height of V is defined by

$$H(V) := \prod_{v \in M_K} |\mathbf{a} \wedge \mathbf{b}|_v = \prod_{v \in M_K} \max_{1 \le i < j \le r} |\Delta_{ij}(a,b)|_v \ . \tag{3.8}$$

This is independent of the choice of the basis $\{a, b\}$: namely, if
$\{a' = \xi_{11}a + \xi_{12}b,\ b' = \xi_{21}a + \xi_{22}b\}$ with $\xi_{ij} \in K$ is another basis, then

$$\Delta_{ij}(a', b') = (\xi_{11}\xi_{22} - \xi_{12}\xi_{21})\Delta_{ij}(a,b) \ \text{ for } 1 \le i, j \le r, \tag{3.9}$$

so

$$\mathbf{a}' \wedge \mathbf{b}' = (\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) \cdot \mathbf{a} \wedge \mathbf{b} \ , \tag{3.10}$$

and this implies, together with the Product formula, that

$$H(\mathbf{a}' \wedge \mathbf{b}') = \Big(\prod_{v \in M_K} |\xi_{11}\xi_{22} - \xi_{12}\xi_{21}|_v \Big) H(\mathbf{a} \wedge \mathbf{b}) = H(\mathbf{a} \wedge \mathbf{b}) \ .$$

We will use that by (3.2) we have

$$|\Delta_{ij}(a,b)|_v \le 2^{s(v)} \max(|a^{(i)}|_v, |a^{(j)}|_v) \max(|b^{(i)}|_v, |b^{(j)}|_v),$$

whence

$$|\mathbf{a} \wedge \mathbf{b}|_v \le 2^{s(v)} |\mathbf{a}|_v |\mathbf{b}|_v \ \text{ for } v \in M_K \ . \tag{3.11}$$

We need some other properties of V:

Lemma 2. *Let $\{a, b\}$ be any basis of V. Then*
(i) $\Delta_{ij}(a,b) \ne 0$ for $1 \le i, j \le r$ with $i \ne j$;
(ii) the discriminant $D(a,b) := \big(\prod_{1 \le i < j \le r} \Delta_{ij}(a,b) \big)^2$ belongs to K^*;

(iii) $H(V) \geq 1$, *and* $H(V) = 1$ *if and only if for every* $v \in M_K$, *the numbers* $|\Delta_{ij}(a,b)|_v$ $(1 \leq i,j \leq r,\ i \neq j)$ *are equal one to another;*
(iv) for every $u \in V$ *and for each* $i,j,k \in \{1,\dots,r\}$ *we have Siegel's identity*

$$\Delta_{jk}(a,b)u^{(i)} + \Delta_{ki}(a,b)u^{(j)} + \Delta_{ij}(a,b)u^{(k)} = 0.$$

Proof. (i). Put $c := b/a$. Then

$$\Delta_{ij}(a,b) = a^{(i)}a^{(j)}(c^{(i)} - c^{(j)})\ . \tag{3.12}$$

Further, by (2.2) we have $L = K(c)$ and therefore $c^{(1)},\dots,c^{(r)}$ are distinct. Together with (3.12) this proves (i).

(ii). We have $D(a,b) \neq 0$ by (i) and $D(a,b) \in K$ since each K-automorphism of $\overline{K}$ permutes, up to sign, the numbers $\Delta_{ij}(a,b)$.

(iii). By (ii) and the Product formula we have

$$H(V) = \prod_{v \in M_K} \frac{|\mathbf{a} \wedge \mathbf{b}|_v}{|D(a,b)|_v^{1/r(r-1)}} = \prod_{v \in M_K} \frac{\max_{1 \leq i < j \leq r} |\Delta_{ij}(a,b)|_v}{(\prod_{1 \leq i < j \leq r} |\Delta_{ij}(a,b)|_v)^{2/r(r-1)}}\ .$$

Each factor in the product is ≥ 1, hence $H(V) \geq 1$. If $H(V) = 1$, then each factor is equal to 1 and this implies that for every $v \in M_K$, the numbers $|\Delta_{ij}(a,b)|_v$ $(1 \leq i,j \leq r,\ i \neq j)$ are equal one to another.

(iv). Write $u = xa + yb$ with $x,y \in K$. Put again $c := b/a$. Then (3.12) implies

$$\begin{aligned}
&\Delta_{jk}(a,b)u^{(i)} + \Delta_{ki}(a,b)u^{(j)} + \Delta_{ij}(a,b)u^{(k)} \\
&\quad = a^{(i)}a^{(j)}a^{(k)}\Big\{(c^{(j)} - c^{(k)})(x + yc^{(i)}) + \\
&\qquad\qquad + (c^{(k)} - c^{(i)})(x + yc^{(j)}) + (c^{(i)} - c^{(j)})(x + yc^{(k)})\Big\} \\
&\quad = 0.
\end{aligned}$$

□

§4. Reduction to Diophantine inequalities.

As before, let K be a number field, L a finite extension of K of degree r, S a finite set of places on K of cardinality s, containing all infinite places, and V a K-vector space satisfying (2.1) and (2.2). Further, let $\mathcal{I}$ be the collection of tuples

$$\mathbf{i} = (i_v : v \in S) \quad \text{with } i_v \in \{1, \dots, r\} \text{ for } v \in S \; .$$

For each $\mathbf{i} \in \mathcal{I}$ we define the quantity

$$\Delta(\mathbf{i}, V) = \left(\prod_{v \in S} \max_{j \neq i_v} |\Delta_{i_v,j}(a,b)|_v \right) \cdot \left(\prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v \right) , \tag{4.1}$$

where $\{a, b\}$ is any basis of V, and where by $j \neq i_v$ we indicate that we let j run through the set of indices $\{1, \dots, r\} \backslash \{i_v\}$. From (3.9), (3.10) and the Product formula, it follows that $\Delta(\mathbf{i}, V)$ is independent of the choice of the basis, i.e. does not change when $\{a, b\}$ is replaced by any other basis $\{a', b'\}$ of V. The quantity $\Delta(\mathbf{i}, V)$ will appear in certain Diophantine inequalities arising from the set $V \cap \mathcal{O}_{L,S}^*$ and in a gap principle related to these inequalities. We also need the quantities $\theta(\mathbf{i})$ $(\mathbf{i} \in \mathcal{I})$ defined by

$$H(V)^{\theta(\mathbf{i})} = \prod_{v \in S} \left\{ \frac{|\mathbf{a} \wedge \mathbf{b}|_v}{\left(\prod_{j \neq i_v} |\Delta_{i_v,j}(a,b)|_v \right)^{\frac{1}{r-1}}} \right\} \tag{4.2}$$

if $H(V) > 1$ and $\theta(\mathbf{i}) := 0$ if $H(V) = 1$.

(3.9) and (3.10) imply that also $\theta(\mathbf{i})$ is independent of the choice of the basis $\{a, b\}$. Note that (4.2) holds true also if $H(V) = 1$: namely, Lemma 2 (iii) implies that in that case the right-hand side of (4.2) is also equal to 1. We need the following inequalities:

Lemma 3. *(i)* $H(V)^{1-\theta(\mathbf{i})} \leq \Delta(\mathbf{i}, V) \leq H(V)$ *for* $\mathbf{i} \in \mathcal{I}$*;*
(ii) $\theta(\mathbf{i}) \geq 0$ *for* $\mathbf{i} \in \mathcal{I}$ *and* $\sum_{\mathbf{i} \in \mathcal{I}} \theta(\mathbf{i}) \leq r^s$.

Proof. Fix a basis $\{a, b\}$ of V and write Δ_{ij} for $\Delta_{ij}(a,b)$. Put $H_v := |\mathbf{a} \wedge \mathbf{b}|_v = \max_{i,j} |\Delta_{ij}|_v$.

(i). Since $\prod_{j\neq i_v}|\Delta_{i_v,j}|_v^{\frac{1}{r-1}}\leq \max_{j\neq i_v}|\Delta_{i_v,j}|_v\leq H_v$ for $v\in S$ we have

$$\Delta(\mathbf{i},V)\leq \prod_{v\in S}H_v\prod_{v\notin S}H_v=H(V),\quad\text{and}$$

$$\begin{aligned}\Delta(\mathbf{i},V)&\geq \prod_{v\in S}\Big(\prod_{j\neq i_v}|\Delta_{i_v,j}|_v\Big)^{\frac{1}{r-1}}\cdot\prod_{v\notin S}H_v=\prod_{v\in S}\left\{\frac{\Big(\prod_{j\neq i_v}|\Delta_{i_v,j}|_v\Big)^{\frac{1}{r-1}}}{H_v}\right\}\cdot H(V)\\&=H(V)^{1-\theta(\mathbf{i})}\ .\end{aligned}$$

(ii). We assume that $H(V)>1$ which is no restriction. We recall that by Lemma 2 (ii) we have that $D:=\big(\prod_{1\leq i<j\leq r}\Delta_{ij}\big)^2\in K^*$. (i) implies that $\theta(\mathbf{i})\geq 0$ for $\mathbf{i}\in\mathcal{I}$. To prove the other assertion, we observe that $\mathcal{I}$ consists of exactly r^s tuples $\mathbf{i}=(i_v:v\in S)$ and that

$$\prod_{\mathbf{i}\in\mathcal{I}}\prod_{j\neq i_v}|\Delta_{i_v,j}|_v=\prod_{i\neq j}|\Delta_{ij}|_v^{r^{s-1}}=|D|_v^{r^{s-1}}\quad\text{for } v\in S\ .$$

Further, we have $|D|_v\leq\max_{1\leq i<j\leq r}|\Delta_{ij}|_v^{r(r-1)}=H_v^{r(r-1)}$ for $v\notin S$. Together with (3.8) and the Product formula applied to D this gives

$$\begin{aligned}H(V)^{\sum_{\mathbf{i}\in\mathcal{I}}\theta(\mathbf{i})}&=\prod_{\mathbf{i}\in\mathcal{I}}\left(\prod_{v\in S}\frac{H_v}{\prod_{j\neq i_v}|\Delta_{i_v,j}|_v^{1/(r-1)}}\right)\\&=\prod_{v\in S}\frac{H_v^{r^s}}{|D|_v^{r^{s-1}/(r-1)}}\leq\prod_{v\in M_K}\frac{H_v^{r^s}}{|D|_v^{r^{s-1}/(r-1)}}\\&=H(V)^{r^s}\end{aligned}$$

which implies (ii). □

Suppose that $V\cap\mathcal{O}_{L,S}^*$ is non-empty. For $u_0\in V\cap\mathcal{O}_{L,S}^*$, define the space

$$u_0^{-1}V=\{u_0^{-1}u:u\in V\}.$$

Let u_0 be an element u of $V\cap\mathcal{O}_{L,S}^*$ for which $H(u^{-1}V)$ is minimal; such an u_0 exists since for each $u\in V\cap\mathcal{O}_{L,S}^*$, $H(u^{-1}V)$ is the absolute Weil height of a vector of given dimension with coordinates in some given finite extension of K (cf. [5] §3), and since the set of values of absolute Weil heights of such vectors is discrete.

Put $V' := u_0^{-1}V$. Then $1 \in V'$ and $H(u^{-1}V') \geq H(V')$ for every $u \in V' \cap \mathcal{O}_{L,S}^*$. Further, V' also satisfies (2.1) and (2.2) and the number of $\mathcal{O}_S^*$-cosets in $V' \cap \mathcal{O}_{L,S}^*$ is the same as that in $V \cap \mathcal{O}_{L,S}^*$. Therefore, in what follows, we may replace V by V'. Thus, we may assume that $1 \in V$ and $H(u^{-1}V) \geq H(V)$ for every $u \in V \cap \mathcal{O}_{L,S}^*$. In the remainder of this paper, we assume that V satisfies these conditions and also (2.1) and (2.2), i.e.

$$\text{(4.3)} \qquad \begin{cases} V \text{ is a two-dimensional } K\text{-linear subspace of } V; \\ \text{for every basis } \{a,b\} \text{ of } V \text{ we have } L = K(b/a); \\ 1 \in V, \quad H(u^{-1}V) \geq H(V) \text{ for every } u \in V \cap \mathcal{O}_{L,S}^* \ . \end{cases}$$

Lemma 4. *For every $u \in V \cap \mathcal{O}_{L,S}^*$ there is a tuple $\mathbf{i} = (i_v : v \in S) \in \mathcal{I}$ such that each of the three inequalities below is satisfied:*

$$\text{(4.4.a)} \qquad \prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \leq \Delta(\mathbf{i}, V) \cdot \frac{2}{H(\mathbf{u})^2 H(V)} \ ,$$

$$\text{(4.4.b)} \qquad \prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \leq \Delta(\mathbf{i}, V) \cdot \frac{4H(V)^{7/2}}{H(\mathbf{u})^3} \ ,$$

$$\text{(4.4.c)} \qquad \prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \leq \Delta(\mathbf{i}, V) \cdot \frac{2^{r-1} H(V)^{r\theta(\mathbf{i})-1}}{H(\mathbf{u})^r} \ .$$

Remark. Inequalities (4.4.a), (4.4.b), (4.4.c) will be used to deal with the "small," "medium" and "large" $\mathcal{O}_S^*$-cosets, respectively.

Proof. Let $u \in V \cap \mathcal{O}_{L,S}^*$. Take any basis $\{a,b\}$ of V and put $\Delta_{ij} := \Delta_{ij}(a,b)$. For each of the inequalities (4.4.a), (4.4.b), (4.4.c) we shall construct a tuple $\mathbf{i} \in \mathcal{I}$ for which that inequality is satisfied. The three tuples we obtain in this way are a priori different, so we must do some effort to show that (4.4.a)-(4.4.c) can be satisfied with the same tuple $\mathbf{i}$.

We first show that there is a tuple $\mathbf{i}$ with (4.4.a). Note that $\{u^{-1}a, u^{-1}b\}$ is a basis of $u^{-1}V$. Further,

$$\Delta_{ij}(u^{-1}a, u^{-1}b) = (u^{(i)}u^{(j)})^{-1}(a^{(i)}b^{(j)} - a^{(j)}b^{(i)}) = (u^{(i)}u^{(j)})^{-1}\Delta_{ij}.$$

By (3.6) we have $|u^{(i)}u^{(j)}|_v = 1$ for $v \notin S$. Hence

$$\begin{aligned} H(u^{-1}V) &= \prod_{v \in M_K} \left\{ \max_{1 \le i < j \le r} \frac{|\Delta_{ij}|_v}{|u^{(i)}u^{(j)}|_v} \right\} \\ &= \prod_{v \in S} \left\{ \max_{i,j} \frac{|\Delta_{ij}|_v}{|u^{(i)}u^{(j)}|_v} \right\} \cdot \prod_{v \notin S} \max_{i,j} |\Delta_{ij}|_v \\ &= \prod_{v \in S} \left\{ \max_{i,j} \frac{|\Delta_{ij}|_v}{|u^{(i)}u^{(j)}|_v} \right\} \cdot \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v \ . \end{aligned}$$

Together with (4.3) this implies

$$H(V) \le \prod_{v \in S} \left\{ \max_{i,j} \frac{|\Delta_{ij}|_v}{|u^{(i)}u^{(j)}|_v} \right\} \cdot \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v \ . \tag{4.5}$$

Fix $v \in S$. Choose p from $\{1, \dots, r\}$ such that $|u^{(p)}|_v = \max_{i=1,\dots,r} |u^{(i)}|_v = |\mathbf{u}|_v$. Further, choose i_v, j_v from $\{1, \dots, r\}$ such that

$$\begin{gathered} \frac{|\Delta_{i_v,j_v}|_v}{|u^{(i_v)}u^{(j_v)}|_v} = \max_{i,j} \frac{|\Delta_{ij}|_v}{|u^{(i)}u^{(j)}|_v}, \\ |\Delta_{j_v,p}u^{(i_v)}|_v \le |\Delta_{i_v,p}u^{(j_v)}|_v; \end{gathered}$$

the inequality can be achieved after interchanging i_v, j_v if necessary. From Lemma 2 (iv) and (3.2) it follows that

$$|\Delta_{i_v,j_v}u^{(p)}|_v = |\Delta_{j_v,p}u^{(i_v)} + \Delta_{p,i_v}u^{(j_v)}|_v \le 2^{s(v)}|\Delta_{p,i_v}u^{(j_v)}|_v \ .$$

Dividing this by $|u^{(i_v)}u^{(j_v)}u^{(p)}|_v$ and using $|u^{(p)}|_v = |\mathbf{u}|_v$ gives

$$\frac{|\Delta_{i_v,j_v}|_v}{|u^{(i_v)}u^{(j_v)}|_v} \le 2^{s(v)} \frac{|\Delta_{p,i_v}|_v}{|u^{(i_v)}u^{(p)}|_v} \le 2^{s(v)} \left(\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \right)^{-1} |\mathbf{u}|_v^{-2} \max_{j \ne i_v} |\Delta_{i_v,j}|_v \ .$$

By inserting this into (4.5), using (3.1), (4.1) and (3.7), we obtain

$$\begin{aligned} H(V) &\le 2 \prod_{v \in S} \left\{ \Big(\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \Big)^{-1} |\mathbf{u}|_v^{-2} \right\} \cdot \left(\prod_{v \in S} \max_{j \ne i_v} |\Delta_{i_v,j}|_v \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v \right) \\ &= 2\Delta(\mathbf{i}, V) \left(\prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \right)^{-1} H(\mathbf{u})^{-2} \end{aligned}$$

with $\mathbf{i} = (i_v : v \in S)$ and this implies (4.4.a).

We now show that there is a tuple $\mathbf{i}$ with (4.4.b). We assume, without loss of generality, that

$$\prod_{v\in M_K} \frac{|u^{(1)}u^{(2)}u^{(3)}|_v}{|\Delta_{12}\Delta_{23}\Delta_{31}|_v^{3/2}} \le \prod_{v\in M_K} \frac{|u^{(i)}u^{(j)}u^{(k)}|_v}{|\Delta_{ij}\Delta_{jk}\Delta_{ki}|_v^{3/2}}$$

for every subset $\{i,j,k\}$ of $\{1,\dots,r\}$. Note that $u^{(1)}\cdots u^{(r)} = N_{L/K}(u) \in K^*$ and that $\prod_{1\le i<j\le r} \Delta_{ij}^2 \in K^*$ by Lemma 2 (ii). Now the Product formula applied to these quantities gives

$$\text{(4.6)} \quad \prod_{v\in M_K} \frac{|u^{(1)}u^{(2)}u^{(3)}|_v}{|\Delta_{12}\Delta_{23}\Delta_{31}|_v^{3/2}} \le \left\{ \prod_{\{i,j,k\}\subseteq\{1,\dots,r\}} \prod_{v\in M_K} \frac{|u^{(i)}u^{(j)}u^{(k)}|_v}{|\Delta_{ij}\Delta_{jk}\Delta_{ki}|_v^{3/2}} \right\}^{1/\binom{r}{3}}$$

$$= \prod_{v\in M_K} \frac{|u^{(1)}\cdots u^{(r)}|_v^{\binom{r-1}{2}/\binom{r}{3}}}{|\prod_{1\le i<j\le r}\Delta_{ij}^2|_v^{3\binom{r-2}{1}/4\binom{r}{3}}}$$

$$= 1.$$

Now let $v \in M_K$. Choose i_v from $\{1,2,3\}$ such that

$$|u^{(i_v)}|_v = \min\left(|u^{(1)}|_v, |u^{(2)}|_v, |u^{(3)}|_v\right) .$$

Further, let again $p \in \{1,\dots,r\}$ be such that $|u^{(p)}|_v = |\mathbf{u}|_v$. Then for $k \in \{1,2,3\}$, $k \ne i_v$ we have, by Lemma 2 (iv) and (3.2),

$$|\mathbf{u}|_v = |u^{(p)}|_v = |\Delta_{i_v,k}|_v^{-1}|\Delta_{kp}u^{(i_v)} + \Delta_{p,i_v}u^{(k)}|_v$$
$$\le 2^{s(v)}|\Delta_{i_v,k}|_v^{-1} \max\left(|\Delta_{kp}|_v, |\Delta_{i_v,p}|_v\right)\cdot \max\left(|u^{(i_v)}|_v, |u^{(k)}|_v\right)$$
$$\le 2^{s(v)}|\Delta_{i_v,k}|_v^{-1}|\mathbf{a}\wedge\mathbf{b}|_v \cdot |u^{(k)}|_v .$$

Together with $|\Delta_{i_v,k}|_v \le \max_{j\ne i_v} |\Delta_{i_v,j}|_v$ this implies

$$\text{(4.7)} \quad |\mathbf{u}|_v \le 2^{s(v)}|\Delta_{i_v,k}|_v^{-3/2}|\mathbf{a}\wedge\mathbf{b}|_v \cdot \max_{j\ne i_v} |\Delta_{i_v,j}|_v^{1/2}\cdot |u^{(k)}|_v$$
$$\text{for } k \in \{1,2,3\},\ k \ne i_v .$$

Let $\{j_v, k_v\} = \{1,2,3\}\backslash\{i_v\}$. From (4.7) with $k = j_v, k_v$ and $|\Delta_{j_v,k_v}|_v \le |\mathbf{a}\wedge\mathbf{b}|_v$ we infer

$$\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le \frac{|u^{(1)}u^{(2)}u^{(3)}|_v}{|\mathbf{u}|_v^3} \cdot 4^{s(v)}|\Delta_{i_v,j_v}\Delta_{i_v,k_v}|_v^{-3/2}|\mathbf{a}\wedge\mathbf{b}|_v^2 \cdot \max_{j\ne i_v}|\Delta_{i_v,j}|_v$$
$$\le \max_{j\ne i_v}|\Delta_{i_v,j}|_v \cdot 4^{s(v)}\frac{|u^{(1)}u^{(2)}u^{(3)}|_v}{|\Delta_{12}\Delta_{23}\Delta_{31}|_v^{3/2}} \cdot \frac{|\mathbf{a}\wedge\mathbf{b}|_v^{7/2}}{|\mathbf{u}|_v^3} .$$

By taking the product over $v \in M_K$, using (4.6), (3.1), (3.3) and (3.8), we get

$$(4.8) \qquad \prod_{v \in M_K} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le \Big(\prod_{v \in M_K} \max_{j \ne i_v} |\Delta_{i_v,j}|_v \Big) \cdot \frac{4H(V)^{7/2}}{H(\mathbf{u})^3} \ .$$

By (3.6) we have $|u^{(i_v)}|_v = |\mathbf{u}|_v = 1$ for $v \notin S$. Further, it is obvious that

$$\prod_{v \in M_K} \max_{j \ne i_v} |\Delta_{i_v,j}|_v \le \prod_{v \in S} \max_{j \ne i_v} |\Delta_{i_v,j}|_v \cdot \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v = \Delta(\mathbf{i}, V) \ ,$$

with $\mathbf{i} = (i_v : \ v \in S)$. By inserting this into (4.8) we obtain (4.4.b).

It is obvious that (4.4.a), (4.4.b) hold true simultaneously for a tuple $\mathbf{i}$ for which $\prod_{v \in S} \Big(|u^{(i_v)}|_v / |\mathbf{u}|_v \Big) \cdot \Delta(\mathbf{i}, V)^{-1}$ is minimal. We remark that $\mathbf{i} = (i_v : \ v \in S)$ with $i_v \in \{1, \dots, r\}$ given by

$$(4.9) \qquad \frac{|u^{(i_v)}|_v}{\max_{k \ne i_v} |\Delta_{i_v,k}|_v} = \min_{j=1,\dots,r} \frac{|u^{(j)}|_v}{\max_{k \ne j} |\Delta_{jk}|_v} \quad \text{for } v \in S$$

(where k is the only running index in the maxima) is such a tuple: namely, for each tuple $\mathbf{j} = (j_v : \ v \in S)$ with $j_v \in \{1, \dots, r\}$ for $v \in S$ we have

$$\begin{aligned} \prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \cdot \Delta(\mathbf{i}, V)^{-1} &= \Big(\prod_{v \in S} \frac{|u^{(i_v)}|_v}{\max_{k \ne i_v} |\Delta_{i_v,k}|_v} \Big) \Big(\prod_{v \in S} |\mathbf{u}|_v^{-1} \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v^{-1} \Big) \\ &\le \Big(\prod_{v \in S} \frac{|u^{(j_v)}|_v}{\max_{k \ne j_v} |\Delta_{j_v,k}|_v} \Big) \Big(\prod_{v \in S} |\mathbf{u}|_v^{-1} \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v^{-1} \Big) = \prod_{v \in S} \frac{|u^{(j_v)}|_v}{|\mathbf{u}|_v} \cdot \Delta(\mathbf{j}, V)^{-1} \ . \end{aligned}$$

We now prove that also (4.4.c) holds true for the tuple $\mathbf{i}$ defined by (4.9). Fix $v \in S$. We show that $|u^{(j)}|_v$ is close to $|\mathbf{u}|_v$ for each $j \ne i_v$. Choose p with $|u^{(p)}|_v = |\mathbf{u}|_v$. Fix $j \ne i_v$. From Lemma 2 (iv), (3.2) and from

$$|\Delta_{jp} u^{(i_v)}|_v \le \max_{k \ne j} |\Delta_{jk}|_v \cdot |u^{(i_v)}|_v \le \max_{k \ne i_v} |\Delta_{i_v,k}|_v \cdot |u^{(j)}|_v \le |\mathbf{a} \wedge \mathbf{b}|_v |u^{(j)}|_v$$

which is a consequence of (4.9) it follows that

$$\begin{aligned} |\mathbf{u}|_v = |u^{(p)}|_v &= |\Delta_{i_v,j}|_v^{-1} |\Delta_{jp} u^{(i_v)} + \Delta_{p,i_v} u^{(j)}|_v \\ &\le 2^{s(v)} |\Delta_{i_v,j}|_v^{-1} |\mathbf{a} \wedge \mathbf{b}|_v |u^{(j)}|_v \ . \end{aligned}$$

Hence

$$\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le 2^{(r-1)s(v)} \cdot \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \prod_{j \ne i_v} \left(\frac{|\mathbf{a} \wedge \mathbf{b}|_v}{|\Delta_{i_v,j}|_v} \cdot \frac{|u^{(j)}|_v}{|\mathbf{u}|_v} \right)$$
$$= 2^{(r-1)s(v)} \cdot \frac{|\mathbf{a} \wedge \mathbf{b}|_v^{r-1}}{\prod_{j \ne i_v} |\Delta_{i_v,j}|_v} \cdot \frac{|u^{(1)} \cdots u^{(r)}|_v}{|\mathbf{u}|_v^r} \, .$$

We take the product over $v \in S$. Note that since $u^{(1)} \cdots u^{(r)} \in \mathcal{O}_{L,S}^* \cap K = \mathcal{O}_S^*$ we have

$$\prod_{v \in S} |u^{(1)} \cdots u^{(r)}|_v = 1 \, . \tag{4.10}$$

Therefore,

$$\prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le 2^{r-1} \cdot \Big(\prod_{v \in S} \frac{|\mathbf{a} \wedge \mathbf{b}|_v^{r-1}}{\prod_{j \ne i_v} |\Delta_{i_v,j}|_v} \Big) H(\mathbf{u})^{-r} \quad \text{by (3.1), (3.7), (4.10)}$$
$$= 2^{r-1} \cdot H(V)^{(r-1)\theta(\mathbf{i})} H(\mathbf{u})^{-r} \quad \text{by (4.2)}$$
$$\le \Delta(\mathbf{i}, V) \cdot 2^{r-1} H(V)^{r\theta(\mathbf{i})-1} H(\mathbf{u})^{-r} \quad \text{by Lemma 3 (i)}$$

which is (4.4.c). This completes the proof of Lemma 4. □

§5. A gap principle.

As before, let K be a number field, L a finite extension of K of degree r, S a set of places on K of finite cardinality s, containing all infinite places, and V a K-vector space satisfying (4.3). Further, we put $d := [K : \mathbb{Q}]$.

The following lemma is needed to derive a gap principle that can deal also with "very small" solutions.

Lemma 5. *Let F be a real > 1 and let $\mathcal{C}$ be a subset of $V \cap \mathcal{O}_{L,S}^*$ that can not be contained in the union of fewer than*

$$\max(2F^{2d}, 4 \times 7^{d+2s})$$

$\mathcal{O}_S^*$-cosets. *Then there are* $u_1, u_2 \in \mathcal{C}$ *such that* $\{u_1, u_2\}$ *is a basis of* V *and*

$$\prod_{v \notin S} |\mathbf{u_1} \wedge \mathbf{u_2}|_v \leq F^{-1} \ , \tag{5.1}$$

where $\mathbf{u}_j = (u_j^{(1)}, \dots, u_j^{(r)})$ *for* $j = 1, 2$.

Proof. The proof is similar to that of Lemma 6 of [5]. We assume, with no loss of generality, that any two distinct elements of $\mathcal{C}$ belong to different $\mathcal{O}_S^*$-cosets, and that $\mathcal{C}$ has cardinality at least $\max(2F^{2d}, 4{\times}7^{d+2s})$. Using that $\mathcal{O}_{L,S}^* \cap K = \mathcal{O}_S^*$, it follows easily that any two K-linearly dependent elements of $V \cap \mathcal{O}_{L,S}^*$ belong to the same $\mathcal{O}_S^*$-coset. Hence any two distinct elements of $\mathcal{C}$ form a basis of V. For every $v \notin S$, choose $u_{1v}, u_{2v} \in \mathcal{C}$ such that

$$|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v = \max_{u_1, u_2 \in \mathcal{C}} |\mathbf{u_1} \wedge \mathbf{u_2}|_v \ , \tag{5.2}$$

where $\mathbf{u}_{iv} = (u_{iv}^{(1)}, \dots, u_{iv}^{(r)})$ for $i = 1, 2$. The coordinates of $\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}$ belong to $\mathcal{O}_{L,S}$, hence $|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v \leq 1$ for $v \notin S$. Therefore, it suffices to show that there are distinct $u_1, u_2 \in \mathcal{C}$ with

$$\prod_{v \notin S} \frac{|\mathbf{u_1} \wedge \mathbf{u_2}|_v}{|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v} \leq F^{-1}.$$

(5.2) implies that each factor in the product in the left-hand side is ≤ 1. Therefore, it suffices to show that there are $u_1, u_2 \in \mathcal{C}$, $v \notin S$, such that

$$\frac{|\mathbf{u_1} \wedge \mathbf{u_2}|_v}{|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v} \leq F^{-1}, \quad u_1 \neq u_2 \ . \tag{5.3}$$

Among all prime ideals outside S, we choose one with minimal norm, $\mathfrak{p}$ say; let $N\mathfrak{p}$ denote the norm of this prime ideal. Since by assumption $F > 1$, there is an integer $m \geq 1$ with

$$N\mathfrak{p}^{(m-1)/d} < F \leq N\mathfrak{p}^{m/d} \ . \tag{5.4}$$

We distinguish between the cases $m = 1$ and $m \geq 2$.

The case $m = 1$.

First assume that

$$(5.5) \qquad |\mathbf{u_1} \wedge \mathbf{u_2}|_v = |\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v$$
$$\text{for every } v \notin S \text{ and every } u_1, u_2 \in \mathcal{C} \text{ with } u_1 \neq u_2 \ .$$

By assumption, $\mathcal{C}$ has cardinality ≥ 3. Fix $u_1, u_2, u_3 \in \mathcal{C}$. We have $u_3 = \alpha u_1 + \beta u_2$ with $\alpha, \beta \in K$, since $\{u_1, u_2\}$ is a basis of V. Now (5.5) implies that

$$|\alpha|_v = \frac{|\mathbf{u_3} \wedge \mathbf{u_2}|_v}{|\mathbf{u_1} \wedge \mathbf{u_2}|_v} = 1, \quad |\beta|_v = \frac{|\mathbf{u_1} \wedge \mathbf{u_3}|_v}{|\mathbf{u_1} \wedge \mathbf{u_2}|_v} = 1 \ \text{ for } v \notin S \ ,$$

hence $\alpha, \beta \in \mathcal{O}_S^*$. Let $u \in \mathcal{C}$, $u \neq u_1, u_2, u_3$. We have $u = xu_1 + yu_2$ with $x, y \in K$. Similarly as above, we have $x, y \in \mathcal{O}_S^*$. Moreover, (5.5) implies that

$$|\beta x - \alpha y|_v = \frac{|\mathbf{u} \wedge \mathbf{u_3}|_v}{|\mathbf{u_1} \wedge \mathbf{u_2}|_v} = 1 \ \text{ for } v \notin S \ ,$$

whence $\beta x - \alpha y \in \mathcal{O}_S^*$. Since any two distinct elements of $\mathcal{C}$ form a basis of V, we have that $u \in \mathcal{C}$ is uniquely determined by the quotient x/y. Further, by Theorem 1 of [4] there are at most $3 \times 7^{d+2s}$ quotients $x/y \in \mathcal{O}_S^*$ for which $(\beta x/\alpha y) - 1 \in \mathcal{O}_S^*$. Since we have considered only $u \in \mathcal{C}$ distinct from u_1, u_2, u_3, this implies that $\mathcal{C}$ has cardinality at most $3 + 3 \times 7^{d+2s} < 4 \times 7^{d+2s}$. But this is against our assumption. Therefore, (5.5) can not be true.

Hence there are distinct $u_1, u_2 \in \mathcal{C}$ and $v \notin S$ such that $|\mathbf{u_1} \wedge \mathbf{u_2}|_v < |\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v$. Recall that $v = \mathfrak{q}$ is a prime ideal of $\mathcal{O}_K$ outside S. For $i = 1, 2$ we have $u_i = x_i u_{1v} + y_i u_{2v}$ with $x_i, y_i \in K$. Thus,

$$\frac{|\mathbf{u_1} \wedge \mathbf{u_2}|_v}{|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v} = |x_1 y_2 - x_2 y_1|_v = N\mathfrak{q}^{-n/d}$$

for some positive integer n. Now by our choice of $\mathfrak{p}$ and by (5.4) and $m = 1$ we have $N\mathfrak{q}^{-n/d} \leq N\mathfrak{p}^{-1/d} \leq F^{-1}$. Hence v and u_1, u_2 satisfy (5.3).

The case $m \geq 2$.

Let $v = \mathfrak{p}$. Every $u \in \mathcal{C}$ can be expressed uniquely as $u = xu_{1v} + yu_{2v}$ with $x, y \in K$. We have $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, with

$$\mathcal{C}_1 = \{u \in \mathcal{C} : \ |x|_v \leq |y|_v\}, \quad \mathcal{C}_2 = \{u \in \mathcal{C} : \ |y|_v \leq |x|_v\} \ .$$

We assume, without loss of generality, that $\mathcal{C}_1$ has cardinality $\geq \frac{1}{2}\mathrm{Card}\,\mathcal{C}$. Thus, by our assumption on $\mathcal{C}$, and by (5.4) and $m \geq 2$,

$$\text{(5.6)} \qquad \mathrm{Card}\,\mathcal{C}_1 \geq F^{2d} > N\mathfrak{p}^{2m-2} \geq N\mathfrak{p}^{m} \; .$$

Define the local ring $\mathcal{O} = \{z \in K : |z|_v \leq 1\}$ and the ideal of $\mathcal{O}$, $\mathfrak{a} = \{z \in K : |z|_v \leq N\mathfrak{p}^{-m/d}\}$. The residue class ring $\mathcal{O}/\mathfrak{a}$ is isomorphic to $\mathcal{O}_K/\mathfrak{p}^m$. Therefore, $\mathcal{O}/\mathfrak{a}$ has cardinality $N\mathfrak{p}^m$. Since any two distinct elements of $\mathcal{C}$ form a basis of V, $u \in \mathcal{C}$ is uniquely determined by x/y. So (5.6) implies that there are distinct $u_1, u_2 \in \mathcal{C}_1$ with $u_i = x_i u_{1v} + y_i u_{2v}$ for $i = 1, 2$, where $x_i, y_i \in K$ and $x_1/y_1 \equiv x_2/y_2 \bmod \mathfrak{a}$, i.e. $|(x_1/y_1) - (x_2/y_2)|_v \leq N\mathfrak{p}^{-m/d}$. By (5.2) we have $|y_i|_v = |\mathbf{u}_{1v} \wedge \mathbf{u}_i|_v / |\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v \leq 1$ for $i = 1, 2$. These inequalities imply, together with (5.4),

$$\frac{|\mathbf{u_1} \wedge \mathbf{u_2}|_v}{|\mathbf{u}_{1v} \wedge \mathbf{u}_{2v}|_v} = |x_1 y_2 - x_2 y_1|_v = |y_1 y_2|_v \left| \frac{x_1}{y_1} - \frac{x_2}{y_2} \right|_v \leq N\mathfrak{p}^{-m/d} \leq F^{-1} \; ,$$

which is (5.3). This completes the proof of Lemma 5. $\square$

The next combinatorial lemma is a special case of Lemma 4 of [4] . It is a formalisation of an idea of Mahler.

Lemma 6. *Let q be an integer ≥ 1 and λ a real with $0 < \lambda \leq \frac{1}{2}$. Then there exists a set Γ of q-tuples $(\gamma_1, \ldots, \gamma_q)$ of real numbers with*

$$\gamma_i \geq 0 \text{ for } i = 1, \ldots, q, \qquad \sum_{i=1}^{q} \gamma_i = 1 - \lambda,$$

such that

$$\mathrm{Card}(\Gamma) \leq \left(\frac{e}{\lambda}\right)^{q-1} \qquad (e = 2.7182\ldots)$$

and such that for every set of reals $F_1, \ldots, F_q, \Lambda$ with

$$0 < F_j \leq 1 \text{ for } j = 1, \ldots, q, \qquad \prod_{j=1}^{q} F_j \leq \Lambda$$

there is a tuple $(\gamma_1, \ldots, \gamma_q) \in \Gamma$ with

$$F_j \leq \Lambda^{\gamma_j} \quad \text{for } j = 1, \ldots, q.$$

$\square$

The gap principle which we prove below is of a similar type as a gap principle for the Subspace theorem proved by Schmidt (cf. [9], Lemma 3.1). Fix $\mathbf{i} = (i_v : v \in S) \in \mathcal{I}$ and let $\Delta(\mathbf{i}, V)$ be the quantity defined by (4.1).

Lemma 7. *(Gap principle.) Let C, P, B be reals with*

$$C \geq 1, \quad B \geq P > 1. \tag{5.7}$$

Then the set of $u \in V \cap \mathcal{O}_{L,S}^*$ satisfying

$$\prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \leq \Delta(\mathbf{i}, V) \cdot \frac{7C/2}{H(\mathbf{u})^2 P} \ , \qquad H(\mathbf{u}) < B \tag{5.8}$$

is the union of at most

$$C^{2d}\Big(14000 \cdot \big\{1 + 2\frac{\log B}{\log P}\big\}\Big)^s$$

$\mathcal{O}_S^*$-cosets.

Proof. Put

$$\kappa := \frac{\log B}{\log P} \ , \qquad \lambda := \frac{1}{2(2\kappa+1)} \ ,$$

$$C_v := \frac{\max_{j \neq i_v} |\Delta_{i_v,j}(a,b)|_v}{|\mathbf{a} \wedge \mathbf{b}|_v} \quad \text{for } v \in S \ ,$$

where $\{a, b\}$ is any basis of V. Note that by (3.9), C_v does not depend on the choice of the basis. Let $u \in V \cap \mathcal{O}_{L,S}^*$ satisfy (5.8) and put

$$F_v(u) := \min\Big(1, \ \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} C_v^{-1} \{(7C/2) \cdot H(V)\}^{-1/s}\Big) \ \text{ for } v \in S \ .$$

From (5.8) and from

$$\prod_{v \in S} C_v = \frac{\prod_{v \in S} \max_{j \neq i_v} |\Delta_{i_v,j}(a,b)|_v \cdot \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v}{\prod_{v \in S} |\mathbf{a} \wedge \mathbf{b}|_v \cdot \prod_{v \notin S} |\mathbf{a} \wedge \mathbf{b}|_v} = \frac{\Delta(\mathbf{i}, V)}{H(V)}$$

which is a consequence of (4.1) and (3.8), it follows that

$$\begin{aligned} \prod_{v \in S} F_v(u) &\leq \Big(\prod_{v \in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v}\Big)\Big(\prod_{v \in S} C_v\Big)^{-1} \big((7C/2) \cdot H(V)\big)^{-1} \\ &= \frac{1}{H(\mathbf{u})^2 P} \ . \end{aligned}$$

By Lemma 6, there is an s-tuple $(\gamma_v : v \in S)$ with $\gamma_v \geq 0$ for $v \in S$ and $\sum_{v\in S} \gamma_v = 1-\lambda$, such that

$$F_v(u) \leq \Big(\frac{1}{H(\mathbf{u})^2 P}\Big)^{\gamma_v} \quad \text{for } v \in S \tag{5.9}$$

and such that $(\gamma_v : v \in S)$ belongs to a set Γ independent of u of cardinality at most $(e/\lambda)^{s-1}$. The condition $H(\mathbf{u}) < B$ implies that there is an integer k with $0 \leq k < 2\kappa$ and

$$P^{k/2} \leq H(\mathbf{u}) < P^{(k+1)/2}\ . \tag{5.10}$$

Now let k be any integer with $0 \leq k \leq 2\kappa$ and $(\gamma_v : v \in S)$ any tuple of non-negative reals with $\sum_{v\in S} \gamma_v = 1-\lambda$ and let $\mathcal{C}$ be the set of elements $u \in V \cap \mathcal{O}_{L,S}^*$ satisfying (5.8), (5.9) and (5.10). We claim that

(5.11) $\mathcal{C}$ is contained in the union of fewer than $4C^{2d} \cdot 7^{4s}$ $\mathcal{O}_S^*$-cosets.

Taking into consideration the number of possibilities for k and the cardinality of Γ, (5.11) implies that the set of $u \in V \cap \mathcal{O}_{L,S}^*$ with (5.8) is the union of fewer than

$$\begin{aligned} 4C^{2d} \cdot 7^{4s} \cdot (2\kappa+1) \cdot \Big(\frac{e}{\lambda}\Big)^{s-1} \\ &\leq C^{2d} \cdot 4\times 7^{4s} \cdot (2\kappa+1) \cdot \big(2e\{2\kappa+1\}\big)^{s-1} \\ &< C^{2d}\big(14000\{2\kappa+1\}\big)^{s} \end{aligned}$$

$\mathcal{O}_S^*$-cosets. Thus, (5.11) implies Lemma 7.

It remains to prove (5.11). Assume the contrary, i.e. that $\mathcal{C}$ can not be contained in the union of fewer than $4C^{2d} \cdot 7^{4s}$ $\mathcal{O}_S^*$-cosets. This quantity is at least $\max(2\times (7C)^{2d}, 4\times 7^{d+2s})$, since d is at most two times the number of infinite places of K, hence at most $2s$. Therefore, from Lemma 5 with $F = 7C$ it follows that there are $u_1, u_2 \in \mathcal{C}$ such that $\{u_1, u_2\}$ is a basis of V and such that

$$\prod_{v\notin S} |\mathbf{u_1} \wedge \mathbf{u_2}|_v \leq (7C)^{-1}\ . \tag{5.12}$$

Without loss of generality we assume that

$$H(\mathbf{u_1}) \le H(\mathbf{u_2}). \tag{5.13}$$

Let

$$S' := \{v \in S: \ \gamma_v > 0\}, \qquad s' := \text{Card } S' \ ,$$

and put

$$\Delta'(\mathbf{i}, V) := \Big(\prod_{v \in S'} \max_{j \ne i_v} |\Delta_{i_v,j}(a,b)|_v \Big) \Big(\prod_{v \in M_K \setminus S'} |\mathbf{a} \wedge \mathbf{b}|_v \Big) \ .$$

S' is non-empty since $\sum_{v \in S} \gamma_v = 1 - \lambda > 0$. From (3.8) it follows that

$$\prod_{v \in S'} C_v = \frac{\Delta'(\mathbf{i}, V)}{H(V)} \ . \tag{5.14}$$

Hence $\Delta'(\mathbf{i}, V)$ is independent of the choice of the basis $\{a, b\}$. Below, we will estimate $\Delta'(\mathbf{i}, V)$ from above by computing it with respect to the basis $\{u_1, u_2\}$ instead of $\{a, b\}$. For convenience, we introduce the quantities

$$c' := \sum_{v \in S'} s(v), \quad c'' := \sum_{v \in S \setminus S'} s(v),$$
$$H_j' := \prod_{v \in S'} |\mathbf{u}_j|_v, \quad H_j'' := \prod_{v \in S \setminus S'} |\mathbf{u}_j|_v \quad \text{for } j = 1, 2.$$

Note that by (3.1) and (3.7) we have

$$c' + c'' = 1, \quad H_j' H_j'' = H(\mathbf{u}_j) \quad \text{for } j = 1, 2. \tag{5.15}$$

Let $v \in S'$. Choose j_v from $\{1, \dots, r\} \backslash \{i_v\}$ such that $|\Delta_{i_v, j_v}(u_1, u_2)|_v = \max_{j \ne i_v} |\Delta_{i_v,j}(u_1, u_2)|_v$. (5.9), (3.4) and $P > 1$ imply that $F_v(u_j) < 1$ for $j = 1, 2$. Hence

$$\frac{|u_j^{(i_v)}|_v}{|\mathbf{u}_j|_v} \le C_v \big((7C/2) H(V) \big)^{1/s} \Big(H(\mathbf{u})^2 P \Big)^{-\gamma_v} \quad \text{for } j = 1, 2.$$

Together with (3.2) and (5.13) this implies that

$$\begin{aligned}
\max_{j \ne i_v} |\Delta_{i_v,j}(u_1, u_2)|_v &= |u_1^{(i_v)} u_2^{(j_v)} - u_2^{(i_v)} u_1^{(j_v)}|_v \\
&\le 2^{s(v)} \max(|u_1^{(i_v)} u_2^{(j_v)}|_v, |u_2^{(i_v)} u_1^{(j_v)}|_v) \\
&\le 2^{s(v)} |\mathbf{u_1}|_v |\mathbf{u_2}|_v \max \Big(\frac{|u_1^{(i_v)}|_v}{|\mathbf{u_1}|_v}, \frac{|u_2^{(i_v)}|_v}{|\mathbf{u_2}|_v} \Big) \\
&\le 2^{s(v)} |\mathbf{u_1}|_v |\mathbf{u_2}|_v \cdot C_v \cdot \Big((7C/2) H(V) \Big)^{1/s} \Big\{ \frac{1}{H(\mathbf{u_1})^2 P} \Big\}^{\gamma_v} ,
\end{aligned}$$

and by taking the product over $v \in S'$, using (5.14) and $\sum_{v\in S'} \gamma_v = \sum_{v\in S} \gamma_v = 1-\lambda$ we obtain

$$\prod_{v\in S'} \max_{j\neq i_v} |\Delta_{i_v,j}(u_1,u_2)|_v \le 2^{c'} H_1' H_2' \frac{\Delta'(\mathbf{i},V)}{H(V)} \Big((7C/2)H(V)\Big)^{s'/s} \Big\{H(\mathbf{u_1})^2 P\Big\}^{\lambda-1}$$

$$\le \Delta'(\mathbf{i},V)\cdot 2^{c'}(7C/2)\cdot H_1'H_2' \Big\{H(\mathbf{u_1})^2 P\Big\}^{\lambda-1} . \tag{5.16}$$

By (3.11) we have

$$\prod_{v\in S\setminus S'} |\mathbf{u_1}\wedge \mathbf{u_2}|_v \le 2^{c''} H_1'' H_2'' . \tag{5.17}$$

Now, by combining (5.16), (5.17) and (5.12) and using (5.15) we get

$$\begin{aligned}\Delta'(\mathbf{i},V) &= \prod_{v\in S'} \max_{j\neq i_v} |\Delta_{i_v,j}(u_1,u_2)|_v \cdot \prod_{v\in S\setminus S'} |\mathbf{u_1}\wedge\mathbf{u_2}|_v \cdot \prod_{v\notin S} |\mathbf{u_1}\wedge\mathbf{u_2}|_v \\ &\le \Delta'(\mathbf{i},V)\cdot 2^{c'+c''}(7C/2)\cdot H_1'H_1''\cdot H_2'H_2''\cdot \Big\{H(\mathbf{u_1})^2P\Big\}^{\lambda-1}\cdot (7C)^{-1} \\ &= \Delta'(\mathbf{i},V)\cdot P^{\lambda-1} H(\mathbf{u_1})^{2\lambda-1} H(\mathbf{u_2}) ,\end{aligned}$$

hence

$$1 \le P^{\lambda-1} H(\mathbf{u_1})^{2\lambda}\cdot \frac{H(\mathbf{u_2})}{H(\mathbf{u_1})} .$$

By $H(\mathbf{u_1}) < B$ which is a consequence of (5.8) and the definition of κ we have $H(\mathbf{u_1})^{2\lambda} < B^{2\lambda} = P^{2\lambda\kappa}$ and by (5.10) we have $H(\mathbf{u_2})/H(\mathbf{u_1}) < P^{(k+1)/2}/P^{k/2} = P^{1/2}$. Recalling that $\lambda = 1/\{2(2\kappa+1)\}$, it follows that

$$1 < P^{(\lambda-1)+2\lambda\kappa+1/2} = P^{(2\kappa+1)\lambda-1/2} = 1 .$$

Thus, the negation of (5.11) leads to a contradiction. This completes the proof of Lemma 7. □

We need the following consequence.

Lemma 8. *Let D, A_1, A_2, δ be reals with $\delta > 0$, $D > 0$ and*

$$A_2 \ge A_1 > \max\big(1,\ \big(\tfrac{2}{7}\wedge D\big)^{6/\delta}\big) . \tag{5.18}$$

Then the set of $u \in V \cap \mathcal{O}_{L,S}^*$ with

$$\prod_{v\in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le \Delta(\mathbf{i},V) \cdot \frac{D}{H(\mathbf{u})^{2+\delta}} \ , \qquad A_1 \le H(\mathbf{u}) < A_2 \tag{5.19}$$

is contained in the union of at most

$$\Big(2800(17+12\delta^{-1})\Big)^s \cdot \left(1 + \frac{\log(\log A_2/\log A_1)}{\log(1+\delta)}\right)$$

$\mathcal{O}_S^*$-cosets.

Proof. We assume that $A_2 > A_1$ which is clearly no restriction. Let k be the smallest integer with $A_1^{(1+\delta)^k} \ge A_2$. Then

$$k \le 1 + \frac{\log(\log A_2/\log A_1)}{\log(1+\delta)} \ . \tag{5.20}$$

For every $u \in V \cap \mathcal{O}_{L,S}^*$ satisfying (5.19) there is an integer t with $0 \le t \le k-1$ and

$$A_1^{(1+\delta)^t} \le H(\mathbf{u}) < A_1^{(1+\delta)^{t+1}} \ . \tag{5.21}$$

From the assumption $A_1 > (2D/7)^{6/\delta}$ it follows that each $u \in V \cap \mathcal{O}_{L,S}^*$ with (5.19) and (5.21) satisfies

$$\prod_{v\in S} \frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v} \le \frac{\Delta(\mathbf{i},V)D}{H(\mathbf{u})^2 A_1^{\delta(1+\delta)^t}} \le \Delta(\mathbf{i},V) \cdot \frac{7/2}{H(\mathbf{u})^2 A_1^{(1+\delta)^t(5\delta/6)}} \ .$$

From Lemma 7 with $P = A_1^{(1+\delta)^t(5\delta/6)}$, $B = A_1^{(1+\delta)^{t+1}}$ and $C = 1$, we infer that the set of $u \in V \cap \mathcal{O}_{L,S}^*$ satisfying (5.19) and (5.21) is contained in the union of at most

$$\begin{aligned}\Big(14000\big\{1 + 2\frac{\log B}{\log P}\big\}\Big)^s &= \left(14000\big\{1 + 2\frac{(1+\delta)^{t+1}}{(1+\delta)^t(5\delta/6)}\big\}\right)^s \\ &= \Big(14000\big(1 + \frac{12}{5}\{1+\delta^{-1}\}\big)\Big)^s \\ &= \Big(2800(17+12\delta^{-1})\Big)^s\end{aligned}$$

$\mathcal{O}_S^*$-cosets. By taking into consideration the number of possibilities for t given by the right-hand side of (5.20) this implies Lemma 8. □

§6. The large solutions.

Let as before K be a number field, L a finite extension of K of degree r, and $u \mapsto u^{(1)}, \dots, u \mapsto u^{(r)}$ the K-isomorphic embeddings of L into $\overline{K}$. Further, let S be a finite set of places on K, containing all infinite places. For $x_1, \dots, x_n \in \overline{K}$, $v \in M_K$ we put

$$|x_1, \dots, x_n|_v := \max(|x_1|_v, \dots, |x_n|_v).$$

We define the height of $\beta \in K$ by

$$H(\beta) := \prod_{v \in M_K} |1, \beta|_v \ .$$

More generally, we define the height of $\alpha \in L$ by

$$H(\alpha) := \Big(\prod_{v \in M_K} \prod_{i=1}^{r} |1, \alpha^{(i)}|_v \Big)^{1/r} \ .$$

The following lemma is a slightly modified version of Bombieri's Thue principle [1].

Lemma 9. *(Thue principle). Let $t, \tau, \theta, \delta_1, \delta_2$ be positive real numbers such that*

$$\text{(6.1)} \qquad \sqrt{\frac{2}{r+1}} < t < \sqrt{\frac{2}{r}}, \qquad \tau < t, \qquad t < \theta < t^{-1} \ ,$$

let $\beta_1, \beta_2 \in K$, $\alpha_1, \alpha_2 \in L$, and let $\mathbf{i} = (i_v : v \in S)$ with $i_v \in \{1, \dots, r\}$ for $v \in S$. Then either

$$\text{(6.2)} \quad \prod_{v \in S} \max\left\{ \left(\frac{|\alpha_1^{(i_v)} - \beta_1|_v}{|1, \beta_1|_v} \right)^{\theta\delta_1}, \ \left(\frac{|\alpha_2^{(i_v)} - \beta_2|_v}{|1, \beta_2|_v} \right)^{\theta^{-1}\delta_2} \right\}$$
$$> \left\{ (3H(\alpha_1))^C H(\beta_1) \right\}^{-\frac{\delta_1}{t-\tau}} \cdot \left\{ (3H(\alpha_2))^C H(\beta_2) \right\}^{-\frac{\delta_2}{t-\tau}} \quad \text{with } C = \frac{2}{2 - rt^2},$$

or

$$\text{(6.3)} \qquad \frac{r}{2} \cdot \frac{\delta_2}{\delta_1} > \frac{r}{2} t^2 + \frac{1}{2} \tau^2 - 1 \ .$$

Proof. This is the same result as Theorem 2 of [1], except for the denominators $|1, \beta_i|_v$ in (6.2) and except for the additional assumption $t < \theta < t^{-1}$ which implies that the quantities $\varphi_2(t), \varphi_2(\tau)$ in Bombieri's statement are equal to $\frac{1}{2}t^2$, $\frac{1}{2}\tau^2$, respectively (see the remark at the end of [1],Chap. IV). Further, Bombieri uses another, but equivalent, definition for the height $H(\alpha)$ for $\alpha \in L$. We have to make some minor modifications in the arguments of [1], pp. 288-291 which are indicated below. We mention that our notation $K, L, s(v)$ corresponds to Bombieri's notation $k, K, \varepsilon(v)/[k:\mathbb{Q}]$. Further, by choosing other continuations of $|\cdot|_v$ ($v \in S$) to L if necessary, we may assume that $\alpha_j^{(i_v)} = \alpha_j$ for $j = 1, 2,\ v \in S$. We let S' be the set of those places $v \in S$ for which both quantities $|\alpha_i - \beta_i|_v / |1, \beta_i|_v$ $(i = 1, 2)$ are smaller than 1. Clearly, it suffices to prove Lemma 9 with in the left-hand side of (6.2) the product over $v \in S$ being replaced by the product over $v \in S'$. Our set S' plays the same role as Bombieri's set S.

For pairs $I = (i_1, i_2)$, $J = (j_1, j_2)$ of non-negative integers, we put $I! = i_1! i_2!$ and $\binom{J}{I} = \binom{j_1}{i_1}\binom{j_2}{i_2}$ and we define the differential operator $\Delta_I = (\partial/\partial X_1)^{i_1}(\partial/\partial X_2)^{i_2}$ for polynomials in X_1, X_2. Let $P \in K[X_1, X_2]$ be the polynomial constructed in Section III of [1], with t, τ as in (6.1), and degrees at most d_1, d_2 in X_1, X_2, respectively, such that properties (i)-(v) on p. 288 of [1] are satisfied and such that instead if (vi) we have $|\alpha_i - \beta_i|_v / |1, \beta_i|_v < 1$ for $v \in S'$, $i = 1, 2$. Then $\gamma := (1/I^*!)\Delta^{I^*} P(\beta_1, \beta_2) \neq 0$. We have to estimate $|\gamma|_v$ from above for each $v \in M_K$ and then apply the Product formula. Like in [1], we have to distinguish the four cases:

I. $v \in S'$, v finite; **II.** $v \in S'$, v infinite; **III.** $v \notin S'$, v finite; **IV.** $v \notin S'$, v infinite.

Case I. We indicate the changes on p. 289 of [1]. We have

$$\begin{aligned}\gamma &= \frac{1}{I^*!}\Delta^{I^*} P(\beta_1, \beta_2) \\ &= \sum_I \binom{I^* + I}{I} \frac{1}{(I + I^*)!}\Delta^{I^*+I} P(\alpha_1, \alpha_2)(\beta_1 - \alpha_1)^{i_1}(\beta_2 - \alpha_2)^{i_2} .\end{aligned}$$

By (iii), (iv) on p. 288 we have $\Delta^{I^*+I} P(\alpha_1, \alpha_2) = 0$ for $I = (i_1, i_2)$ with $\theta^{-1} i_1/d_1 + \theta i_2/d_2 < t - \tau$. Let $I = (i_1, i_2)$ be a pair with $\theta^{-1} i_1/d_1 + \theta i_2/d_2 \geq t - \tau$.

Using the notation $\log^+ x = \max(0, \log x)$ we have

$$\log\left|\frac{1}{(I+I^*)!}\Delta^{I^*+I}P(\alpha_1,\alpha_2)\right|_v \leq \log|P|_v + (d_1 - i_1^* - i_1)\log^+|\alpha_1|_v + (d_2 - i_2^* - i_2)\log^+|\alpha_2|_v \ ,$$

where $I^* = (i_1^*, i_2^*)$ and $|P|_v$ is the maximum of the v-adic absolute values of the coefficients of P. From $|\alpha_i - \beta_i|_v < |1, \beta_i|_v$ it follows that $\log^+|\alpha_i|_v \leq \log^+|\beta_i|_v$ for $i = 1, 2$. Hence

$$\log\left|\frac{1}{(I+I^*)!}\Delta^{I^*+I}P(\alpha_1,\alpha_2)\right|_v \leq \log|P|_v + (d_1 - i_1^* - i_1)\log^+|\beta_1|_v + (d_2 - i_2^* - i_2)\log^+|\beta_2|_v \ .$$

Moreover,

$$\begin{aligned}&\log|(\beta_1-\alpha_1)^{i_1}(\beta_2-\alpha_2)^{i_2}|_v \\ &= i_1\log^+|\beta_1|_v + i_2\log^+|\beta_2|_v + i_1\log\left\{\frac{|\beta_1-\alpha_1|_v}{|1,\beta_1|_v}\right\} + i_2\log\left\{\frac{|\beta_2-\alpha_2|_v}{|1,\beta_2|_v}\right\} \\ &\leq i_1\log^+|\beta_1|_v + i_2\log^+|\beta_2|_v \\ &\qquad + (t-\tau)\max\left(\theta d_1\log\left\{\frac{|\beta_1-\alpha_1|_v}{|1,\beta_1|_v}\right\}\ ,\ \theta^{-1}d_2\log\left\{\frac{|\beta_2-\alpha_2|_v}{|1,\beta_2|_v}\right\}\right) \ .\end{aligned}$$

By summing over all I, using that v is finite, we get in case I,

$$\begin{aligned}|\gamma|_v \leq{}& \log|P|_v + d_1\log^+|\beta_1|_v + d_2\log^+|\beta_2|_v \\ &+ (t-\tau)\max\left(\theta d_1\log\left\{\frac{|\beta_1-\alpha_1|_v}{|1,\beta_1|_v}\right\}\ ,\ \theta^{-1}d_2\log\left\{\frac{|\beta_2-\alpha_2|_v}{|1,\beta_2|_v}\right\}\right) \ .\end{aligned}$$

Case II. We modify the arguments in case II on p. 289 of [1] in the same way as above, except that we now have to insert $\log^+|\alpha_i|_v \leq s(v)\log 2 + \log^+|\beta_i|_v$ for $i = 1, 2$. Thus we obtain

$$\begin{aligned}|\gamma|_v \leq{}& \log|P|_v + d_1\log^+|\beta_1|_v + d_2\log^+|\beta_2|_v \\ &+ (t-\tau)\max\left(\theta d_1\log\left\{\frac{|\beta_1-\alpha_1|_v}{|1,\beta_1|_v}\right\}\ ,\ \theta^{-1}d_2\log\left\{\frac{|\beta_2-\alpha_2|_v}{|1,\beta_2|_v}\right\}\right) \ . \\ &+ s(v)(d_1+d_2)\log 6 + o(d_1+d_2) \ .\end{aligned}$$

The arguments of cases III and IV on pp. 289-291 of [1] do not have to be modified, and the proof of our Lemma 9 is then completed in precisely the same way as that of Theorem 2 of [1]. □

Let $K, S, L, r = [L : K]$ be as before, let s denote the cardinality of S, and let V be a K-vector space satisfying (4.3). Then $1 \in V$. We will apply Lemma 9 as follows. Let $u_1, u_2 \in V \cap \mathcal{O}_{L,S}^*$. We will choose an appropriate $b \in V$ such that $\{1, b\}$ is a basis of V and then apply Lemma 9 with $\alpha_1 = \alpha_2 = b$ and with $\beta_i = -x_i/y_i$ for $i = 1, 2$, where $u_i = x_i + y_i b$ with $x_i, y_i \in K$ for $i = 1, 2$. Assume for the moment that there is an element $b \in V$ with

$$b \notin K, \qquad b^{(1)} + \cdots + b^{(r)} = 1 \ . \tag{6.4}$$

It is obvious that $\{1, b\}$ is a basis of V and from (3.2) it follows that

$$|\mathbf{b}|_v = \max(|b^{(1)}|_v, \ldots, |b^{(r)}|_v) \geq r^{-s(v)} \quad \text{for } v \in M_K \ . \tag{6.5}$$

Let $\mathbf{1} := (1, \ldots, 1)$ (r times). We need the following lemma:

Lemma 10. *Let $u \in V$ with $u = x + yb$, where $x, y \in K$ and $y \neq 0$. Then for $v \in M_K$ we have*

$$(i) \qquad |\mathbf{u}|_v \leq (2r)^{s(v)} |\mathbf{b}|_v |x, y|_v \ ;$$

$$(ii) \qquad |x, y|_v \leq (2r)^{s(v)} \frac{|\mathbf{b}|_v}{|\mathbf{1} \wedge \mathbf{b}|_v} \cdot |\mathbf{u}|_v \ .$$

Proof. (i).For $i = 1, \ldots, r$, $v \in M_K$ we have

$$\begin{aligned} |u^{(i)}|_v = |x + yb^{(i)}|_v \leq 2^{s(v)} |1, b^{(i)}|_v |x, y|_v &\leq 2^{s(v)} \max(1, |\mathbf{b}|_v) |x, y|_v \quad \text{by (3.2)} \\ &\leq (2r)^{s(v)} |\mathbf{b}|_v |x, y|_v \quad \text{by (6.5)} \end{aligned}$$

and this implies (i).

(ii). Let $v \in M_K$. We have $x \cdot (\mathbf{1} \wedge \mathbf{b}) = (x\mathbf{1} + y\mathbf{b}) \wedge \mathbf{b} = \mathbf{u} \wedge \mathbf{b}$ and $y \cdot (\mathbf{1} \wedge \mathbf{b}) = \mathbf{1} \wedge \mathbf{u}$. Together with (3.11) this implies that

$$|x|_v = \frac{|\mathbf{u} \wedge \mathbf{b}|_v}{|\mathbf{1} \wedge \mathbf{b}|_v} \leq 2^{s(v)} \frac{|\mathbf{b}|_v}{|\mathbf{1} \wedge \mathbf{b}|_v} \cdot |\mathbf{u}|_v \ ,$$

$$|y|_v = \frac{|\mathbf{1} \wedge \mathbf{u}|_v}{|\mathbf{1} \wedge \mathbf{b}|_v} \leq 2^{s(v)} \frac{1}{|\mathbf{1} \wedge \mathbf{b}|_v} \cdot |\mathbf{u}|_v \ .$$

By taking the maxima of the left- and the right-hand sides and using (6.5) we obtain (ii). □

We recall that by Lemma 4, for every $u \in V \cap \mathcal{O}_{L,S}^*$ there is a tuple $\mathbf{i} = (i_v : v \in S) \in \mathcal{I}$ satisfying (4.4.a)-(4.4.c). Fix $\mathbf{i} \in \mathcal{I}$ and let $\mathcal{S}_{\mathrm{large}}(\mathbf{i})$ be the set of $u \in V \cap \mathcal{O}_{L,S}^*$ satisfying (4.4.a)-(4.4.c) and

$$H(\mathbf{u}) \geq \left\{ \frac{7}{4} H(V) \right\}^{21(1+\theta(\mathbf{i}))} . \tag{6.6}$$

Lemma 11. $\mathcal{S}_{\mathrm{large}}(\mathbf{i})$ *is the union of at most* $\left(4\times 10^6\right)^s$ $\mathcal{O}_S^*$*-cosets.*

Proof. We first choose an appropriate element b of V satisfying (6.4). Clearly, K is a one-dimensional subspace of V and the space $V_0 := \{u \in V : u^{(1)} + \cdots + u^{(r)} = 0\}$ is a proper K-linear subspace of V since $1 \notin V_0$. Hence V_0 has dimension at most 1. Therefore, both K and V_0 contain at most one $\mathcal{O}_S^*$-coset of elements of $V \cap \mathcal{O}_{L,S}^*$. Now let

$$\mathcal{C} := \mathcal{S}_{\mathrm{large}}(\mathbf{i}) \backslash (K \cup V_0).$$

We assume, without loss of generality, that $\mathcal{C}$ is non-empty. Let b' be the element u of $\mathcal{C}$ for which $H(\mathbf{u})$ is minimal. Since $b' \notin V_0$ we have $\lambda := b'^{(1)} + \cdots + b'^{(r)} \neq 0$. Note that $\lambda \in K$. Hence $b := \lambda^{-1} b'$ is an element of V satisfying (6.4). Put

$$H := H(\mathbf{b}).$$

By (3.4) we have $H = H(\lambda^{-1}\mathbf{b}') = H(\mathbf{b}')$. Therefore

$$H \geq \left\{ \frac{7}{4} H(V) \right\}^{21(1+\theta(\mathbf{i}))}, \quad H(\mathbf{u}) \geq H \text{ for } u \in \mathcal{C} . \tag{6.7}$$

We make the following

Claim. *Let* $u_1, \ldots, u_t$ *be a sequence of elements from* $\mathcal{C}$ *with*

$$H(\mathbf{u_1}) \geq H^{10^6 r^2}, \quad H(\mathbf{u}_{i+1}) \geq H(\mathbf{u}_i)^{10^6 r^2} \text{ for } i = 1, \ldots, t-1. \tag{6.8}$$

Then $t \leq (8e)^{s-1}$.

Suppose for the moment that the claim is true. Let $u_1 \in \mathcal{C}$ be such that $H(\mathbf{u_1}) \geq H^{10^6 r^2}$ and subject to this condition $H(\mathbf{u_1})$ is minimal. For $i = 1, 2, \ldots$, let $u_{i+1} \in \mathcal{C}$ be such that $H(\mathbf{u}_{i+1}) \geq H(\mathbf{u}_i)^{10^6 r^2}$ and subject to this condition, $H(\mathbf{u}_{i+1})$ is minimal. Then the sequence $u_1, u_2, u_3, \ldots$ has only a finite number t of elements with $t \leq (8e)^{s-1}$. Now (6.7) and this choice of $u_1, u_2, \ldots, u_t$ imply that for every $u \in \mathcal{C}$ we have either $H \leq H(\mathbf{u}) < H^{10^6 r^2}$ or $H(\mathbf{u}_i) \leq H(\mathbf{u}) < H(\mathbf{u}_i)^{10^6 r^2}$ for some $i \in \{1, \ldots, t\}$. We are going to apply Lemma 8. Note that every $u \in \mathcal{C}$ satisfies (4.4.c), i.e. $\prod_{v \in S} |u^{(i_v)}|_v / |\mathbf{u}|_v \leq \Delta(\mathbf{i}, V) \cdot DH(\mathbf{u})^{-2-\delta}$ with $D = 2^{r-1} H(V)^{r\theta(\mathbf{i})-1}$ and $\delta = r - 2$. Further, by (6.7) and $r \geq 3$ we have $H > \max(1, (2D/7)^{6/\delta})$. Now Lemma 8 with D, δ as defined above and with $A_1 = H$, $A_2 = H^{10^6 r^2}$ implies that the set of elements $u \in \mathcal{C}$ with $H \leq H(\mathbf{u}) < H^{10^6 r^2}$ is contained in the union of at most

$$\left\{2800(17 + \frac{12}{r-2})\right\}^s \left\{1 + \frac{\log(10^6 r^2)}{\log(r-1)}\right\} < 24.2 \times (81200)^s =: T$$

$\mathcal{O}_S^*$-cosets; here we used again that $r \geq 3$. Similarly, for $i = 1, \ldots, t$, the set of $u \in \mathcal{C}$ with $H(\mathbf{u}_i) \leq H(\mathbf{u}) < H(\mathbf{u}_i)^{10^6 r^2}$ is contained in the union of fewer than T $\mathcal{O}_S^*$-cosets. Recalling that $\mathcal{C} = \mathcal{S}_{\text{large}}(\mathbf{i}) \backslash (K \cup V_0)$ and that both K and V_0 contain at most one $\mathcal{O}_S^*$-coset, it follows that $\mathcal{S}_{\text{large}}(\mathbf{i})$ is contained in the union of fewer than

$$2 + (t+1)T \leq 2 + (1 + (8e)^{s-1}) \cdot 24.2 \times (81200)^s < (4 \times 10^6)^s$$

$\mathcal{O}_S^*$-cosets. This proves Lemma 11.

Proof of the claim. We assume the contrary, i.e. that there is a sequence $u_1, \ldots, u_t$ in $\mathcal{C}$ with (6.8) and with

$$t > (8e)^{s-1}. \tag{6.9}$$

Let $u \in \{u_1, \ldots, u_t\}$. From (6.7), (6.8) and $\Delta(\mathbf{i}, V) \leq H(V)$ which is part of Lemma 3 (i), it follows that

$$H(\mathbf{u}) > \left(\Delta(\mathbf{i}, V) \cdot 2^{r-1} H(V)^{r\theta(\mathbf{i})-1}\right)^{10^6}.$$

Further, u satisfies (4.4.c). Hence

$$\prod_{v\in S}\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v}\le \Delta(\mathbf{i},V)\cdot\frac{2^{r-1}H(V)^{r\theta(\mathbf{i})-1}}{H(\mathbf{u})^r}\le H(\mathbf{u})^{-r(1-10^{-6})}\quad\text{for } u\in\{u_1,\dots,u_t\}.$$

By Lemma 6, there is a set Γ of cardinality at most $(8e)^{s-1}$, consisting of tuples $(\gamma_v : v\in S)$ with $\gamma_v\ge 0$ for $v\in S$ and $\sum_{v\in S}\gamma_v = 7/8$, such that for each $u\in\{u_1,\dots,u_t\}$ there is a tuple $(\gamma_v : v\in S)\in\Gamma$ with

$$\frac{|u^{(i_v)}|_v}{|\mathbf{u}|_v}\le\Big(H(\mathbf{u})^{-r(1-10^{-6})}\Big)^{\gamma_v}\quad\text{for } v\in S. \tag{6.10}$$

Since $t>\text{Card}\,\Gamma$, there are distinct elements of $\{u_1,\dots,u_t\}$ satisfying (6.10) with the same tuple $(\gamma_v : v\in S)$. Summarising, it follows that there are $z_1,z_2\in\mathcal{C}$ with

$$H(\mathbf{z_1})\ge H^{10^6r^2}\ , \tag{6.11}$$

$$H(\mathbf{z_2})\ge H(\mathbf{z_1})^{10^6r^2}\ , \tag{6.12}$$

$$\frac{|z_j^{(i_v)}|_v}{|\mathbf{z}_j|_v}\le\Big(H(\mathbf{z}_j)^{-r(1-10^{-6})}\Big)^{\gamma_v}\quad\text{for } j=1,2,\ v\in S\ , \tag{6.13}$$

where $(\gamma_v : v\in S)$ is a tuple of non-negative reals with $\sum_{v\in S}\gamma_v = 7/8$, and where $\mathbf{z}_j=(z_j^{(1)},\dots,z_j^{(r)})$ for $j=1,2$. We apply Lemma 9 to show that such z_1,z_2 can not exist.

Since $\{1,b\}$ is a basis of V, we have

$$z_j = x_j + y_j b\ \text{ with } x_j,y_j\in K \text{ for } j=1,2.$$

Since $\mathcal{C}\cap K=\emptyset$, we have $y_j\ne 0$ for $j=1,2$. Put $\alpha_1=\alpha_2=\alpha:=b$ and $\beta_j:=-x_j/y_j$ for $j=1,2$. We apply Lemma 9 with these $\alpha_j,\ \beta_j$ and with

$$\theta=1,\quad t=\sqrt{\frac{2}{r+0.5\times10^{-4}}}\ ,\quad \tau=\sqrt{2-rt^2+\frac{10^{-4}}{r+0.5\times10^{-4}}}=\frac{t}{100}\ ,$$
$$\delta_1=\frac{1}{\log H(\mathbf{z_1})},\quad \delta_2=\frac{1}{\log H(\mathbf{z_2})}\ . \tag{6.14}$$

Note that the quantity C in Lemma 9 is equal to

$$C=2\times10^4(r+0.5\times10^{-4})=2\times10^4r+1\ . \tag{6.15}$$

Put

$$A_v := \max\left(\left\{\frac{|\alpha^{(i_v)}-\beta_1|_v}{|1,\beta_1|_v}\right\}^{\delta_1}, \left\{\frac{|\alpha^{(i_v)}-\beta_2|_v}{|1,\beta_2|_v}\right\}^{\delta_2}\right) \quad \text{for } v \in S \ ,$$

$$B := \left\{(3H(\alpha))^C H(\beta_1)\right\}^{\frac{\delta_1}{t-\tau}} \cdot \left\{(3H(\alpha))^C H(\beta_2)\right\}^{\frac{\delta_2}{t-\tau}} \ .$$

We estimate each A_v from above. Let $v \in S$ and $j \in \{1,2\}$. By Lemma 10 (i) we have

$$|\mathbf{z}_j|_v \leq (2r)^{s(v)} |\mathbf{b}|_v |x_j, y_j|_v.$$

Hence

$$\frac{|\alpha^{(i_v)}-\beta_j|_v}{|1,\beta_j|_v} = \frac{|x_j + y_j b^{(i_v)}|_v}{|x_j, y_j|_v} \leq C_v \frac{|z_j^{(i_v)}|_v}{|\mathbf{z}_j|_v} \quad \text{with } C_v := (2r)^{s(v)}|\mathbf{b}|_v$$

where the equality is obtained by multiplying numerator and denominator with $|y_j|_v$. Using $\delta_1 \geq \delta_2$ and (6.14), it follows that

$$A_v \leq C_v^{\delta_1} \max\left(\left\{\frac{|z_1^{(i_v)}|_v}{|\mathbf{z}_1|_v}\right\}^{\delta_1}, \ \left\{\frac{|z_2^{(i_v)}|_v}{|\mathbf{z}_2|_v}\right\}^{\delta_2}\right) \leq C_v^{\delta_1} e^{-\gamma_v r(1-10^{-6})} \ . \tag{6.16}$$

By (6.11) we have $\delta_1 \leq (10^6 r^2 \log H)^{-1}$ and by (6.4), (3.1), (3.3) we have

$$\prod_{v\in S} C_v = \prod_{v \in S} (2r)^{s(v)}|\mathbf{b}|_v \leq \prod_{v\in M_K} (2r)^{s(v)}|\mathbf{b}|_v = 2rH(\mathbf{b}) = 2rH \ .$$

By inserting these inequalities into (6.16) and using the lower bound for H from (6.7) we obtain

$$\begin{aligned} \sum_{v\in S} \log A_v &\leq \delta_1 \sum_{v\in S} \log C_v - \big(\sum_{v\in S}\gamma_v\big) r(1-10^{-6}) \\ &\leq \frac{1}{10^6 r^2 \log H} \cdot \log(2rH) - \Big(\frac{7}{8}(1-10^{-6})\Big) r \\ &\leq \frac{3}{10^6 r} - \Big(\frac{7}{8}(1-10^{-6})\Big) r =: a(r) \ . \end{aligned} \tag{6.17}$$

We now estimate B from above. We have

$$\begin{aligned} H(\alpha) &= \Big(\prod_{v\in M_K}\prod_{i=1}^{r} |1, b^{(i)}|_v\Big)^{1/r} \leq \prod_{v\in M_K} \max(1, |\mathbf{b}|_v) \\ &\leq \prod_{v\in M_K} \big(r^{s(v)}|\mathbf{b}|_v\big) = rH \quad \text{by (6.5), (3.1), (3.3).} \end{aligned}$$

Further, the Product formula implies

$$H(\beta_j) = H(x_j/y_j) = \prod_{v\in M_K} |1, x_j/y_j|_v = \prod_{v\in M_K} |x_j, y_j|_v \quad \text{for } j = 1, 2.$$

Therefore,

$$H(\beta_j) \leq \prod_{v\in M_K} (2r)^{s(v)} \frac{|\mathbf{b}|_v}{|\mathbf{1}\wedge \mathbf{b}|_v} |\mathbf{z}_j|_v = 2r\frac{H}{H(V)} H(\mathbf{z}_j) \leq 2rH \cdot H(\mathbf{z}_j) \quad \text{for } j = 1, 2,$$

where the first inequality follows from Lemma 10 (ii), the equality from (3.1), (3.3), (3.8), and the last inequality from Lemma 2 (iii). Using the lower bound for H from (6.7) it follows that

$$(3H(\alpha))^C H(\beta_j) \leq (3rH)^{C+1} H(\mathbf{z}_j) \leq H^{4\times 10^4 r^2} H(\mathbf{z}_j) \quad \text{for } j = 1, 2.$$

Together with (6.11), (6.12) this implies that

$$\begin{aligned}
\log B &\leq \frac{1}{t-\tau}\Big\{2 + \Big(\frac{4\times 10^4 r^2}{\log H(\mathbf{z_1})} + \frac{4\times 10^4 r^2}{\log H(\mathbf{z_2})}\Big)\log H\Big\} \\
&\leq \frac{1}{t-\tau}\Big(2 + \frac{8}{10^2}\Big) \leq \frac{100}{99}\times 2.08\times\sqrt{\frac{r + 0.5\times 10^{-4}}{2}} =: b(r)\ .
\end{aligned}$$

It is easy to check that for $r \geq 3$ we have $a(r) < -b(r)$, where $a(r)$ is the quantity defined in (6.17). Hence

$$\sum_{v\in S} \log A_v < -\log B\ .$$

In other words, (6.2) is not valid and so by Lemma 9, inequality (6.3) holds, that is,

$$\begin{aligned}
\frac{r}{2}\cdot\frac{\log H(\mathbf{z_1})}{\log H(\mathbf{z_2})} = \frac{r}{2}\frac{\delta_2}{\delta_1} &> \frac{r}{2}t^2 + \frac{1}{2}\tau^2 - 1 \\
&= \Big(\frac{r}{2} + 10^{-4}\Big)\frac{2}{r + 0.5\times 10^{-4}} - 1 \\
&= \frac{3}{2\times 10^4 r + 1}\ .
\end{aligned}$$

Hence

$$\frac{\log H(\mathbf{z_2})}{\log H(\mathbf{z_1})} < \frac{2\times 10^4 r^2 + r}{6} < 10^6 r^2$$

which contradicts (6.12). Thus, our assumption that the claim is false leads to a contradiction. This completes our proof of Lemma 11. □

§7. Proof of Theorem 2.

Let $K,\ L,\ r=[L:K],\ S,\ s=$Card S be as before, and let V be a K-vector space satisfying (4.3). We recall that by Lemma 4, for every $u\in V\cap\mathcal{O}^*_{L,S}$ there is an $\mathbf{i}\in\mathcal{I}=\{(i_v:v\in S):i_v\in\{1,\dots,r\}\}$ for which u satisfies (4.4.a)-(4.4.c). Let $\mathcal{S}(\mathbf{i})$ be the set of $u\in V\cap\mathcal{O}^*_{L,S}$ satisfying (4.4.a)-(4.4.c). We divide $\mathcal{S}(\mathbf{i})$ into

$$\begin{aligned}\mathcal{S}_{\text{large}}(\mathbf{i})&=\Big\{u\in\mathcal{S}(\mathbf{i}):\ H(\mathbf{u})\geq\Big(\frac{7}{4}H(V)\Big)^{21(1+\theta(\mathbf{i}))}\Big\},\\ \mathcal{S}_{\text{medium}}(\mathbf{i})&=\Big\{u\in\mathcal{S}(\mathbf{i}):\ \Big(\frac{7}{4}H(V)\Big)^{21}\leq H(\mathbf{u})<\Big(\frac{7}{4}H(V)\Big)^{21(1+\theta(\mathbf{i}))}\Big\},\\ \mathcal{S}_{\text{small}}(\mathbf{i})&=\Big\{u\in\mathcal{S}(\mathbf{i}):\ H(\mathbf{u})<\Big(\frac{7}{4}H(V)\Big)^{21}\Big\}.\end{aligned}$$

Thus,

$$\text{(7.1)}\qquad V\cap\mathcal{O}^*_{L,S}=\bigcup_{\mathbf{i}\in\mathcal{I}}\mathcal{S}(\mathbf{i})=\bigcup_{\mathbf{i}\in\mathcal{I}}\Big(\mathcal{S}_{\text{large}}(\mathbf{i})\cup\mathcal{S}_{\text{medium}}(\mathbf{i})\cup\mathcal{S}_{\text{small}}(\mathbf{i})\Big)\ .$$

Fix $\mathbf{i}\in\mathcal{I}$. By Lemma 11, $\mathcal{S}_{\text{large}}(\mathbf{i})$ is contained in the union of at most $(4\times10^6)^s$ $\mathcal{O}^*_S$-cosets. Every $u\in\mathcal{S}_{\text{medium}}(\mathbf{i})$ satisfies (4.4.b). Hence every $u\in\mathcal{S}_{\text{medium}}(\mathbf{i})$ satisfies (5.19) (cf. Lemma 8) with

$$D=4H(V)^{7/2},\ \delta=1,\quad A_1=\Big(\frac{7}{4}H(V)\Big)^{21},\quad A_2=\Big(\frac{7}{4}H(V)\Big)^{21(1+\theta(\mathbf{i}))}=A_1^{1+\theta(\mathbf{i})}.$$

It is easy to check that these D,δ,A_1,A_2 satisfy (5.18), i.e. $A_2\geq A_1>\max(1,(2D/7)^{6/\delta})$. So Lemma 8 implies that $\mathcal{S}_{\text{medium}}(\mathbf{i})$ is contained in the union of at most

$$(2800\cdot(17+12))^s\Big(1+\frac{\log(1+\theta(\mathbf{i}))}{\log 2}\Big)\leq(81200)^s\Big(1+\frac{3}{2}\theta(\mathbf{i})\Big)$$

$\mathcal{O}^*_S$-cosets. Finally, every $u\in\mathcal{S}_{\text{small}}(\mathbf{i})$ satisfies (4.4.a). Therefore, every $u\in\mathcal{S}_{\text{small}}(\mathbf{i})$ satisfies (5.8) (cf. Lemma 7) with

$$C=1,\quad P=\frac{7}{4}H(V),\quad B=\Big(\frac{7}{4}H(V)\Big)^{21}=P^{21}\ .$$

These C,P,B clearly satisfy (5.7). Hence Lemma 7 implies that $\mathcal{S}_{\text{small}}(\mathbf{i})$ is contained in the union of at most

$$\big(14000(1+2\times21)\big)^s=(602000)^s$$

$\mathcal{O}_S^*$-cosets.

We now apply (7.1). Recalling that $\mathcal{I}$ consists of r^s tuples $\mathbf{i}$ and that $\sum_{\mathbf{i}\in\mathcal{I}} \theta(\mathbf{i}) \leq r^s$ which is part of Lemma 3 (ii), it follows that $V \cap \mathcal{O}_{L,S}^*$ is the union of at most

$$\sum_{\mathbf{i}\in\mathcal{I}} \Big\{(4\times 10^6)^s + (81200)^s\big(1+\frac{3}{2}\theta(\mathbf{i})\big) + (602000)^s\Big\} < (5\times 10^6\, r)^s$$

$\mathcal{O}_S^*$-cosets. This completes the proof of Theorem 2. □

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