THE NUMBER OF ALGEBRAIC NUMBERS OF GIVEN DEGREE APPROXIMATING A GIVEN ALGEBRAIC NUMBER

Jan-Hendrik Evertse

University of Leiden, Mathematical Institute
P.O. Box 9512, 2300 RA Leiden, The Netherlands, email evertse@wi.leidenuniv.nl

§1. Introduction.

In 1955, Roth [15] proved his celebrated theorem, that for every real algebraic number α and every real $\kappa > 2$ the inequality

$$|\alpha - \frac{x}{y}| < \{\max(|x|, |y|)\}^{-\kappa} \quad \text{in } x, y \in \mathbb{Z} \text{ with } \gcd(x, y) = 1 \tag{1.1}$$

has only finitely many solutions. Roth's proof is by contradiction. Assuming that (1.1) has infinitely many solutions, Roth constructed an auxiliary polynomial in a large number of variables, k say, of which all low order partial derivatives vanish in a point $(x_1/y_1, \ldots, x_k/y_k)$ for certain solutions $(x_1, y_1), \ldots, (x_k, y_k)$ of (1.1), and then showed, using a non-vanishing result now known as Roth's lemma, that this is not possible.

Assume that $2 < \kappa < 3$. By making explicit Roth's arguments, Davenport and Roth [3] determined an explicit upper bound for the number of solutions of (1.1) and this was improved later by Mignotte [12]. Bombieri and van der Poorten [1] obtained a much better upper bound by using instead of Roth's lemma a non-vanishing result for polynomials of Esnault and Viehweg [4]. Recently, Corvaja [2] gave an alternative proof of the result of Bombieri and van der Poorten, in which he replaced the construction of an auxiliary polynomial by the use of interpolation determinants as introduced by Laurent in transcendence theory.

We recall the result of Bombieri and van der Poorten. The Mahler measure $M(\alpha)$ of an algebraic number α (always assumed to belong to $\mathbb{C}$) is defined by

$$M(\alpha) := |a_0| \prod_{i=1}^{r} \max(1, |\alpha^{(i)}|) \ ,$$

where $r = \deg \alpha$, $\alpha^{(1)}, \ldots, \alpha^{(r)}$ are the conjugates of α over $\mathbb{Q}$ and a_0 is a rational integer such that the coefficients of the polynomial $f(X) = a_0 \prod_{i=1}^{r}(X - \alpha^{(i)})$ are rational integers with gcd 1. In particular, $M(x/y) = \max(|x|, |y|)$ for $x, y \in \mathbb{Z}$

with $\gcd(x,y)=1$. Now let $\kappa = 2+\delta$ with $0<\delta<1$, and α an algebraic number of degree r. Bombieri and van der Poorten proved that (1.1) has at most

$$c_1 \cdot \delta^{-5} (\log r)^2 \log\big(\frac{\log r}{\delta}\big)$$

solutions with $M(x/y) \geq c_2 M(\alpha)$ and at most

$$c_3 \delta^{-1} \log\big(1+\log M(\alpha)\big)$$

solutions with $M(x/y) < c_2 M(\alpha)$, where c_1, c_2, c_3 are explicitly computable absolute constants. We mention that recently Schmidt [21] gave an explicit upper bound for the number of solutions of (1.1) in the complementary case $\kappa \geq 3$.

We deal with the analogue of (1.1) in which the unknowns are algebraic numbers of given degree, i.e. we consider the inequality

$$|\alpha - \xi| < M(\xi)^{-\kappa} \quad \text{in algebraic numbers } \xi \text{ of degree } t, \tag{1.2}$$

where α is an algebraic number, κ a positive real, and $t \geq 1$. In 1921, Siegel [22], [23] showed that (1.2) has only finitely many solutions if κ exceeds some bound depending on t and the degree of α. In 1966, Ramachandra [14] proved the same with a smaller lower bound for κ, but still depending on the degree of α. In 1971, Wirsing [24] succeeded in proving Roth's conjecture that (1.2) has only finitely many solutions if

$$\kappa > 2t\ . \tag{1.3}$$

Independently, Schmidt [17] (Theorem 3) proved that the number of solutions of (1.2) is finite if

$$\kappa > t+1\ . \tag{1.4}$$

In fact, the latter can be derived from Schmidt's Subspace theorem, cf. [19], p. 278. The lower bound $t+1$ can be shown to be best possible.

It is our purpose to derive an explicit upper bound for the number of solutions of (1.2). For this, one needs, apart from the Diophantine approximation arguments of Wirsing or Schmidt in an explicit form, a "gap principle," which states that solutions of (1.2) are far away from each other. In §2 we derive a simple gap principle for $\kappa > 2t$ which is similar to one which appeared already in Ramachandra's paper [14]. The proof of this gap principle uses a Liouville-type inequality for differences

of algebraic numbers. For obtaining a gap principle for $t+1 < \kappa \le 2t$ one would need an effective improvement of this Liouville-type inequality which, if existing, seems to be very difficult to prove.

We derive an upper bound for the number of solutions of (1.2) with $\kappa > 2t$ by combining the gap principle in §2 with Wirsing's arguments. Another possible approach is to use ideas which are used in the proof of the quantitative Subspace theorem, e.g. in [20] or [6], but this would lead to a larger bound. One of Wirsing's main tools was Leveque's generalisation of Roth's lemma to number fields ([10], Chap. 4). Instead, we use the sharpening of this from [5]. Our result is as follows:

Theorem 1. *Let α be an algebraic number of degree r, t an integer ≥ 1, and $\kappa = 2t + \delta$ with $0 < \delta < 1$.*
(i). (1.2) has at most

$$2 \times 10^7 \cdot t^7 \delta^{-4} \cdot \log 4r \cdot \log\log 4r$$

solutions ξ with $M(\xi) \ge \max\big(4^{t(t+1)/\delta},\ M(\alpha)\big)$.

(ii). (1.2) has at most

$$2^{(t+3)^2} \delta^{-1} \log(2+\delta^{-1}) \ + \ t^2 \delta^{-1} \cdot \log\log 4M(\alpha)$$

solutions ξ with $M(\xi) < \max\big(4^{t(t+1)/\delta},\ M(\alpha)\big)$.

We derive a result more general than Theorem 1. For every algebraic number ξ of degree t we fix an ordering of its conjugates $\xi^{(1)}, \dots, \xi^{(t)}$. Let $\alpha_1, \dots, \alpha_t$ be algebraic numbers. Further, let $\varphi_1, \dots, \varphi_t$ be non-negative reals. We introduce the notation

$$|x, y| := \max(|x|, |y|) \quad \text{for } x, y \in \mathbb{C} \ .$$

Consider the system of inequalities

$$\frac{|\alpha_i - \xi^{(i)}|}{2|1, \alpha_i| \cdot |1, \xi^{(i)}|} \le M(\xi)^{-\varphi_i} \qquad (i = 1, \dots, t) \tag{1.5}$$

in algebraic numbers ξ of degree t .

The denominators have been inserted for technical convenience. Wirsing [24] proved that (1.5) has only finitely many solutions if

$$\max_I \ (\#I)^2 \big(\sum_{i \in I} \varphi_i^{-1} \big)^{-1} > 2t \ , \tag{1.6}$$

where the maximum is taken over all non-empty subsets I of $\{i \in \{1,\dots,t\} : \varphi_i \neq 0\}$ and where $\#$ is used to denote the cardinality of a set. In [24] §3, Wirsing showed that

$$(\sum_{j=1}^{t} \frac{1}{2j-1})^{-1} \leq \frac{\max_I (\#I)^2 \big(\sum_{i\in I} \varphi_i^{-1}\big)^{-1}}{\varphi_1 + \cdots + \varphi_t} \leq 1 \tag{1.7}$$

for all non-negative reals $\varphi_1,\dots,\varphi_t$ with $\varphi_1+\cdots+\varphi_t > 0$, and that the upper and lower bound are best possible. In fact, the upper bound is assumed if and only if all non-zero numbers among $\varphi_1,\dots,\varphi_t$ are equal. So condition (1.6) is in general stronger than

$$\varphi_1 + \cdots + \varphi_t > 2t \ . \tag{1.8}$$

We prove the following quantitative version of Wirsing's result:

Theorem 2. *Let $\alpha_1,\dots,\alpha_t$ be algebraic numbers with*

$$\max_{i=1,\dots,t} M(\alpha_i) \ = M, \quad [\mathbb{Q}(\alpha_1,\dots,\alpha_t) : \mathbb{Q}] = r \tag{1.9}$$

and $\varphi_1,\dots,\varphi_t$ non-negative reals for which

$$\max_I \ (\#I)^2 \big(\sum_{i\in I} \varphi_i^{-1}\big)^{-1} \geq 2t + \delta \quad \textit{with } 0 < \delta < 1 \ . \tag{1.10}$$

Put $\kappa := \varphi_1 + \cdots + \varphi_t$.
(i). (1.5) has at most

$$2\times 10^7 \cdot t^7 \delta^{-4} \cdot \log 4r \cdot \log\log 4r$$

solutions with $M(\xi) \geq \max\big(4^{t(t+1)/(\kappa-2t)},\ M\big)$.

(ii). (1.5) has at most

$$2^{t^2+t+\kappa+4}\Big(1 + \frac{\log(2+\frac{1}{\kappa-2t})}{\log(1+\frac{\kappa-2t}{t})}\Big) \ + \ t \cdot \frac{\log\log 4M}{\log(1+\frac{\kappa-2t}{t})}$$

solutions ξ with $M(\xi) < \max\big(4^{t(t+1)/(\kappa-2t)},\ M\big)$.

It is due to a limitation of Wirsing's method that we have to impose condition (1.6) on $\varphi_1,\dots,\varphi_t$. In §2, we shall derive a gap principle for system (1.5) which is non-trivial if the weaker condition (1.8) holds. It is conceivable that by combining

this gap principle with techniques used in the proof of the quantitative Subspace theorem, one can derive an explicit (but larger) upper bound for the number of solutions of (1.5) with (1.8) replacing (1.6).

Theorem 1 follows at once from Theorem 2 with $\alpha_1 = \alpha$, $\varphi_1 = \kappa$ and $\alpha_i = 0$, $\varphi_i = 0$ for $i = 2, \dots, t$, on observing that in that case we have $\max_I \ (\#I)^2 \big(\sum_{i \in I} \varphi_i^{-1} \big)^{-1}$ $= \kappa$, $\kappa - 2t = \delta < 1$ and $\log(1 + (\kappa - 2t)/t) = \log(1 + \delta/t) \geq \delta/2t$.

Another application of Theorem 2 is to an inequality involving resultants. The resultant $R(f, g)$ of two polynomials $f(X) = a_0 X^r + a_1 X^{r-1} + \cdots + a_r$ and $g(X) = b_0 X^t + b_1 X^{t-1} + \cdots + b_t$ with $a_0 \neq 0$, $b_0 \neq 0$ is defined by the determinant of order $r + t$,

$$
(1.11) \qquad R(f,g) \ = \ \begin{vmatrix}
a_0 & a_1 & \cdots & \cdots & a_r & & & \\
 & a_0 & a_1 & \cdots & \cdots & a_r & & \\
 & & \ddots & & & & \ddots & \\
 & & & a_0 & a_1 & \cdots & \cdots & a_r \\
b_0 & b_1 & \cdots & b_t & & & & \\
 & b_0 & b_1 & \cdots & b_t & & & \\
 & & \ddots & & & \ddots & & \\
 & & & \ddots & & & \ddots & \\
 & & & & b_0 & b_1 & \cdots & b_t
\end{vmatrix} ,
$$

of which the first t rows consist of coefficients of f and the last r rows of coefficients of g. If $f(X) = a_0 \prod_{i=1}^r (X - \alpha_i)$ and $g(X) = b_0 \prod_{j=1}^t (X - \xi_j)$, then

$$
(1.12) \qquad R(f,g) = a_0^t b_0^r \prod_{i=1}^r \prod_{j=1}^t (\alpha_i - \xi_j) \ .
$$

Hence $R(f, g) = 0$ if and only if f and g have a common zero (cf. [9], Chap. V, §10).

We define the Mahler measure of a polynomial $f(X) = a_0 \prod_{i=1}^r (X - \alpha_i) \in \mathbb{C}[X]$ by

$$
M(f) := |a_0| \prod_{i=1}^r \max(1, |\alpha_i|) \ .
$$

We fix a polynomial $f(X) \in \mathbb{Z}[X]$ of degree r and a positive real κ and consider the inequality in unknown polynomials g,

$$
(1.13) \qquad 0 < |R(f,g)| < M(g)^{r-\kappa}
$$

$$
\text{in polynomials } g(X) \in \mathbb{Z}[X] \text{ of degree } t \ .
$$

In [24], Wirsing proved that (1.13) has only finitely many solutions if f has no multiple zeros and if

$$\kappa > 2t\big(1 + \frac{1}{3} + \cdots + \frac{1}{2t-1}\big). \tag{1.14}$$

Schmidt [18] showed that (1.14) can be relaxed to

$$\kappa > 2t \tag{1.15}$$

if f has no multiple zeros and no irreducible factors in $\mathbb{Z}[X]$ of degree $\leq t$. Finally, from a result of Ru and Wong ([16], Thm. 4.1), which is a consequence of the Subspace theorem, it follows that (1.14) can be relaxed to (1.15) for every polynomial f without multiple zeros.

We consider (1.13) only for irreducible polynomials g. Then one can reduce (1.13) to a finite number of systems of inequalities (1.5). Using this, we derive from Theorem 2 an upper bound for the number of irreducible polynomials g satisfying (1.13) with κ satisfying (1.14). We mention that we would be able to derive such an upper bound for all $\kappa > 2t$ if we had an upper bound for the number of solutions of (1.5) for all $\varphi_1, \ldots, \varphi_t$ with $\varphi_1 + \cdots + \varphi_t > 2t$.

A polynomial in $\mathbb{Z}[X]$ is said to be primitive if its coefficients have gcd 1.

Theorem 3. *Let f be a primitive polynomial in $\mathbb{Z}[X]$ of degree r with no multiple zeros. Suppose that*

$$\kappa = (2t+\delta)\big(1 + \frac{1}{3} + \cdots + \frac{1}{2t-1}\big) \quad \textit{with} \;\; 0 < \delta < 1\ . \tag{1.16}$$

Then there are at most

$$10^{15}(\delta^{-1})^{t+3} \cdot (100r)^t \log 4r \cdot \log\log 4r$$

primitive, irreducible polynomials $g(X) \in \mathbb{Z}[X]$ of degree t with

$$0 < |R(f,g)| < M(f)^t \cdot M(g)^{r-\kappa}, \tag{1.17}$$

$$M(g) \geq (2^{8r^2t} M(f)^{4(r-1)t})^{\delta^{-1}(1+\frac{1}{3}+\cdots+\frac{1}{2t-1})^{-1}}\ . \tag{1.18}$$

In (1.17), we have inserted the factor $M(f)^t$ to make the inequality homogeneous in f; without this factor, our bound would not have been better.

§2. A gap principle.

In this section, we derive a gap principle for the system of inequalities

$$\frac{|\alpha_i - \xi^{(i)}|}{2|1,\alpha_i| \cdot |1,\xi^{(i)}|} \le M(\xi)^{-\varphi_i} \quad (i = 1,\dots,t) \tag{1.5}$$
$$\text{in algebraic numbers } \xi \text{ of degree } t \ ,$$

where $\alpha_1,\dots,\alpha_t$ are algebraic numbers, and $\varphi_1,\dots,\varphi_t$ are reals with

$$\varphi_i \ge 0 \ \text{ for } i = 1,\dots,t, \quad \kappa := \varphi_1 + \cdots + \varphi_t > 2t \ . \tag{2.1}$$

After that, we prove part (ii) of Theorem 2. Our gap principle is as follows:

Lemma 1. *(i). Let $\xi_1,\dots,\xi_{t+1}$ be distinct solutions of (1.5) with $M(\xi_{t+1}) \ge M(\xi_t) \ge \cdots \ge M(\xi_1)$. Then*

$$U^{-1}M(\xi_{t+1}) \ge \big(U^{-1}M(\xi_1)\big)^{1+(\kappa-2t)/t} \ \text{ where } U := 2^{t(t+1)/(\kappa-2t)} \ . \tag{2.2}$$

(ii). Put $C := [t \cdot 2^{t^2+\kappa+1}]$. Let $\xi_1,\dots,\xi_{C+1}$ be distinct solutions of (1.5) with $M(\xi_{C+1}) \ge M(\xi_C) \ge \cdots \ge M(\xi_1)$. Then

$$2M(\xi_{C+1}) \ge \big(2M(\xi_1)\big)^{1+(\kappa-2t)/t} \ . \tag{2.3}$$

Proof. Since solutions of (1.5) are assumed to have degree t, at least two numbers among $\xi_1,\dots,\xi_{t+1}$ are not conjugate to each other, $\xi := \xi_i$, $\eta := \xi_j$ with $i < j$, say. Denote the minimal polynomials (in $\mathbb{Z}[X]$ with coefficients having gcd 1) of ξ, η by $f,\ g$, respectively. Then f and g have no common zeros, i.e. their resultant $R(f,g)$ is a non-zero integer. Let $f(X) = a_0 \prod_{k=1}^{t}(X - \xi^{(k)})$, $g(X) = b_0 \prod_{l=1}^{t}(X - \eta^{(l)})$. Then by (1.12) (on noting that a_0, b_0 are cancelled) we have

$$\frac{|R(f,g)|}{M(\xi)^t M(\eta)^t} \ = \ \prod_{k=1}^{t}\prod_{l=1}^{t} \frac{|\xi^{(k)} - \eta^{(l)}|}{|1,\xi^{(k)}| \cdot |1,\eta^{(l)}|} \ , \tag{2.4}$$

and since $R(f,g)$ is a non-zero integer, this implies the Liouville-type inequality,

$$\prod_{k=1}^{t}\prod_{l=1}^{t} \frac{|\xi^{(k)} - \eta^{(l)}|}{|1,\xi^{(k)}| \cdot |1,\eta^{(l)}|} \ge \frac{1}{M(\xi)^t M(\eta)^t} \ . \tag{2.5}$$

We estimate the left-hand side from above. For $k \ne l$ we use the trivial estimate

$$\frac{|\xi^{(k)} - \eta^{(l)}|}{|1,\xi^{(k)}| \cdot |1,\eta^{(l)}|} \le 2 \ . \tag{2.6}$$

Let $k = l \in \{1, \dots, t\}$. We apply the following variation on the triangle inequality:

$$\frac{|x-y|}{|1,x| \cdot |1,y|} \le \frac{|x-z|}{|1,x| \cdot |1,z|} + \frac{|z-y|}{|1,z| \cdot |1,y|} \quad \text{for } x, y, z \in \mathbb{C} \ . \tag{2.7}$$

Thus, using that ξ, η satisfy (1.5),

$$\begin{aligned} \frac{|\xi^{(k)} - \eta^{(k)}|}{|1,\xi^{(k)}| \cdot |1,\eta^{(k)}|} &\le \frac{|\xi^{(k)} - \alpha_k|}{|1,\xi^{(k)}| \cdot |1,\alpha_k|} + \frac{|\eta^{(k)} - \alpha_k|}{|1,\eta^{(k)}| \cdot |1,\alpha_k|} \\ &\le 2M(\xi)^{-\varphi_k} + 2M(\eta)^{-\varphi_k} \le 4M(\xi)^{-\varphi_k} \ . \end{aligned}$$

Together with (2.5), (2.6) this implies

$$\frac{1}{M(\xi)^t M(\eta)^t} \le 2^{t^2+t} M(\xi)^{-(\varphi_1+\cdots+\varphi_t)} = U^{\kappa-2t} M(\xi)^{-\kappa} \ , \tag{2.8}$$

whence

$$U^{-1} M(\eta) \ge \left(U^{-1} M(\xi)\right)^{1+(\kappa-2t)/t} \ .$$

Together with $M(\xi_1) \le M(\xi) \le M(\eta) \le M(\xi_{t+1})$ this implies (2.2).

(ii). Let p be a prime number which will be chosen later. We partition the solutions of (1.5) into equivalence classes as follows. Let ξ and η be solutions of (1.5) with minimal polynomials $f,\ g$, respectively. By definition, both f and g have $t+1$ integer coefficients without a common factor. We call $\xi,\ \eta$ equivalent if there is an integer λ, not divisible by p, such that $\frac{1}{p}(f - \lambda g)$ has its coefficients in $\mathbb{Z}$, in other words, if the reductions modulo p of the vectors of coefficients of $f,\ g$, respectively, represent the same point in the t-dimensional projective space $\mathbb{P}^t(\mathbb{F}_p)$. Clearly, the number of equivalence classes is at most the number of points in $\mathbb{P}^t(\mathbb{F}_p)$, which is

$$\frac{p^{t+1}-1}{p-1} \le 2p^t \ . \tag{2.9}$$

Now for equivalent $\xi,\ \eta$ with minimal polynomials $f,\ g$ and with λ as above we have by (1.11), that

$$R(f,g) = R(f - \lambda g, g) = p^t R(\tfrac{1}{p}(f - \lambda g), g) \equiv 0 \pmod{p^t} \ . \tag{2.10}$$

Choose p such that $2^{t-1+\kappa/t} \le p < 2^{t+\kappa/t}$. Then by (2.9), the number of equivalence classes is at most $2p^t < 2^{t^2+\kappa+1}$. So among the solutions $\xi_1, \dots, \xi_{C+1}$ there must be at least $t+1$ belonging to the same equivalence class. Among these $t+1$ solutions we can choose two, $\xi := \xi_i$ and $\eta := \xi_j$, say, with $i < j$, which are not

conjugate to each other. Now if f, g are the minimal polynomials of $\xi,\ \eta$, then in view of (2.4), (2.10), we can replace (2.5) by

$$\prod_{k=1}^{t}\prod_{l=1}^{t}\frac{|\xi^{(k)}-\eta^{(l)}|}{|1,\xi^{(k)}|\cdot|1,\eta^{(l)}|}\geq\frac{p^t}{M(\xi)^tM(\eta)^t}\geq\frac{2^{t^2-t+\kappa}}{M(\xi)^tM(\eta)^t}\ .$$

By repeating the argument of (i) we obtain instead of (2.8),

$$\frac{2^{t^2-t+\kappa}}{M(\xi)^tM(\eta)^t}\leq 2^{t^2+t}M(\xi)^{-(\varphi_1+\cdots+\varphi_t)}=2^{t^2+t}M(\xi)^{-\kappa}$$

and so

$$2M(\eta)\geq\big(2M(\xi)\big)^{1+(\kappa-2t)/t}\ .$$

Together with $M(\xi_1)\leq M(\xi)\leq M(\eta)\leq M(\xi_{C+1})$ this implies (2.3). □

We need the following simple consequence of Lemma 1:

Lemma 2. *(i). Let A, B be reals with $B\geq A\geq U^2=4^{t(t+1)/(\kappa-2t)}$. Then the number of solutions ξ of (1.5) with $A\leq M(\xi)<B$ is at most*

$$t\cdot\Big(1+\frac{\log(2\log B/\log A)}{\log(1+(\kappa-2t)/t)}\Big)\ .$$

(ii). Let A, B be reals with $B\geq A\geq 1$. Then the number of solutions ξ of (1.5) with $A\leq M(\xi)<B$ is at most

$$C\cdot\Big(1+\frac{\log(\log 2B/\log 2A)}{\log(1+(\kappa-2t)/t)}\Big)\ .$$

Proof. (i). Put $\theta:=1+(\kappa-2t)/t$. Let k be the smallest integer with

$$\big(U^{-1}A\big)^{\theta^k}\geq U^{-1}B\ .$$

Part (i) of Lemma 1 implies that for each $i\in\{0,\dots,k-1\}$, (1.5) has at most t solutions ξ with $(U^{-1}A)^{\theta^i}\leq U^{-1}M(\xi)<(U^{-1}A)^{\theta^{i+1}}$. Hence (1.5) has at most $t\cdot k$ solutions with $A\leq M(\xi)<B$. Now part (i) follows since in view of our assumption $A\geq U^2$ we have

$$k\leq 1+\frac{\log(\log U^{-1}B/\log U^{-1}A)}{\log\theta}\leq 1+\frac{\log(2\log B/\log A)}{\log\theta}\ .$$

(ii). Use part (ii) of Lemma 1 and repeat the argument given above with 2 replacing U^{-1} and $C\cdot k$ replacing $t\cdot k$. □

Proof of part (ii) of Theorem 2.
Put $\theta := 1 + (\kappa - 2t)/t$. We first estimate the number of solutions ξ of (1.5) with $4^{t(t+1)/(\kappa-2t)} \le M(\xi) < \max(4^{t(t+1)/(\kappa-2t)}, M)$. Assuming that $M \ge 4^{t(t+1)/(\kappa-2t)}$, we infer from part (i) of Lemma 2 that this number is at most

$$t \cdot \Big(1 + \frac{\log\big(2\log M / \frac{t(t+1)}{\kappa-2t}\log 4\big)}{\log\theta}\Big) \le t \cdot \Big(1 + \frac{\log(\theta \log M)}{\log\theta}\Big) \le t \cdot \Big(2 + \frac{\log\log 4M}{\log\theta}\Big) . \tag{2.12}$$

This is clearly also true if $M < 4^{t(t+1)/(\kappa-2t)}$.

We now estimate the number of solutions ξ of (1.5) with $M(\xi) < 4^{t(t+1)/(\kappa-2t)}$. From part (ii) of Lemma 2 with $A = 1$, $B = 4^{t(t+1)/(\kappa-2t)}$ it follows that this number is at most

$$t\cdot 2^{t^2+\kappa+1}\cdot\Big(1+\frac{\log(1+\frac{2t(t+1)}{\kappa-2t})}{\log\theta}\Big) \le t\cdot 2^{t^2+\kappa+1}\cdot 3\log\big(2t(t+1)\big)\cdot\Big(1+\frac{\log(2+\frac{1}{\kappa-2t})}{\log\theta}\Big) .$$

Together with (2.12) this implies that the total number of solutions of (1.5) with $M(\xi) < \max(4^{t(t+1)/(\kappa-2t)}, M)$ is at most

$$2^{t^2+t+\kappa+4}\Big(1 + \frac{\log(2+\frac{1}{\kappa-2t})}{\log\theta}\Big) + t \cdot \frac{\log\log 4M}{\log\theta}$$

which is precisely the upper bound in part (ii) of Theorem 2. □

§3. Construction of the auxiliary polynomial.

For an algebraic number ξ we put

$$|| \xi || := \max(|\xi^{(1)}|, \dots, |\xi^{(r)}|),$$

where $\xi^{(1)}, \dots, \xi^{(r)}$ are the conjugates of ξ over $\mathbb{Q}$. More generally, for a vector $\mathbf{x} := (\xi_1, \dots, \xi_R)$ with algebraic coordinates we put

$$|| \mathbf{x} || := \max(|| \xi_1 ||, \dots, || \xi_R ||) .$$

The ring of integers of an algebraic number field K (assumed to be contained in $\mathbb{C}$) is denoted by O_K. We need the following consequence of Siegel's lemma:

Lemma 3. *Let K be an algebraic number field of degree r. Further, let R, S be rational integers with*

$$0 < S \le R, \quad rS > (r-1)R \ , \tag{3.1}$$

let A be a positive real and let $\mathbf{a}_1, \dots, \mathbf{a}_S \in K^R$ be K-linearly independent vectors for which there are rational integers $q_1, \dots, q_S$ with

$$0 < q_i \le A, \quad q_i \mathbf{a}_i \in O_K^R, \quad ||\, q_i \mathbf{a}_i \,|| \le A \ \text{ for } i = 1, \dots, S \ . \tag{3.2}$$

Then there are $\beta_1, \dots, \beta_S \in O_K$ such that

$$\mathbf{x} := \sum_{i=1}^{S} \beta_i \mathbf{a}_i \in \mathbb{Z}^R \backslash \{\mathbf{0}\}, \tag{3.3}$$

$$||\, \mathbf{x} \,|| \le \big\{ C(K) \cdot SA \big\}^{\frac{rS}{rS-(r-1)R}}, \tag{3.4}$$

$$|\beta_i| \le \big\{ C(K) \cdot SA \big\}^{\frac{rS}{rS-(r-1)R}} \ \text{ for } i = 1, \dots, S \ , \tag{3.5}$$

where $C(K)$ is a constant depending only on K.

Proof. Lemma 3 may be proved by applying a sophisticated version of Siegel's lemma of Bombieri-Vaaler type, but then some extra work must be done to get a good upper bound for the numbers $|\beta_i|$. Instead, we give a direct proof of Lemma 3, following Wirsing [24]. $C_1(K)$, $C_2(K), \dots$ denote constants depending only on K.

Put $\mathbf{a}_i' := q_i \mathbf{a}_i$ for $i = 1, \dots, S$. We search for $\beta_1', \dots, \beta_S' \in O_K$ such that

$$\mathbf{x} := \sum_{i=1}^{S} \beta_i' \mathbf{a}_i' \in \mathbb{Z}^R \backslash \{\mathbf{0}\} \ . \tag{3.6}$$

Then (3.3) holds with

$$\beta_i = q_i \beta_i' \ \text{ for } i = 1, \dots, S \ . \tag{3.7}$$

Let $\{\omega_1, \dots, \omega_r\}$ be a $\mathbb{Z}$-basis of O_K with $\omega_1 = 1$. We can express $\alpha \in O_K$ as

$$\alpha = \sum_{i=1}^{r} x_i \omega_i \ \text{ with } x_i \in \mathbb{Z}, \ |x_i| \le C_1(K) ||\, \alpha \,|| \text{ for } i = 1, \dots, r \ ; \tag{3.8}$$

the upper bounds for $|x_i|$ follow by taking conjugates and solving $x_1, \ldots, x_r$ from the system of linear equations $\alpha^{(j)} = \sum_{i=1}^{r} x_i \omega_i^{(j)}$ $(j = 1, \ldots, r)$, using Cramer's rule. Now we have

$$\mathbf{a}'_i = \sum_{j=1}^{r} \omega_j \mathbf{b}_{ij} \quad \text{with } \mathbf{b}_{ij} \in \mathbb{Z}^R \text{ for } i = 1, \ldots, S,\ j = 1, \ldots, r, \tag{3.9}$$

$$\beta'_i = \sum_{k=1}^{r} \omega_k z_{ik} \quad \text{with } z_{ik} \in \mathbb{Z} \text{ for } i = 1, \ldots, S,\ k = 1, \ldots, r. \tag{3.10}$$

Define the integers u_{jkl} by

$$\omega_j \omega_k = \sum_{l=1}^{r} u_{jkl} \omega_l \quad \text{for } j, k \in \{1, \ldots, r\}\ .$$

Then we obtain

$$\begin{aligned} \sum_{i=1}^{S} \beta'_i \mathbf{a}'_i &= \sum_{i=1}^{S} \sum_{j=1}^{r} \sum_{k=1}^{r} \omega_j \omega_k z_{ik} \mathbf{b}_{ij} \\ &= \sum_{l=1}^{r} \omega_l \Big\{ \sum_{i=1}^{S} \sum_{k=1}^{r} z_{ik} \mathbf{c}_{ikl} \Big\} \quad \text{with} \quad \mathbf{c}_{ikl} := \sum_{j=1}^{r} u_{jkl} \mathbf{b}_{ij} \in \mathbb{Z}^R\ . \end{aligned} \tag{3.11}$$

By (3.8), (3.2) we have

$$|| \mathbf{b}_{ij} || \le C_2(K) || \mathbf{a}'_i || \le C_2(K) A \quad \text{for } i = 1, \ldots, S,\ j = 1, \ldots, r, \tag{3.12}$$

so

$$|| \mathbf{c}_{ikl} || \le C_3(K) A \quad \text{for } i = 1, \ldots, S,\ k = 1, \ldots, r,\ l = 1, \ldots, r\ . \tag{3.13}$$

Recalling that $\omega_1 = 1$, we infer that $\sum_{i=1}^{S} \beta'_i \mathbf{a}'_i \in \mathbb{Z}^R$ if and only if the coefficients of $\omega_2, \ldots, \omega_r$ in (3.11) are 0, i.e.

$$\sum_{i=1}^{S} \sum_{k=1}^{r} z_{ik} \mathbf{c}_{ikl} = 0 \quad \text{for } l = 2, \ldots, r\ . \tag{3.14}$$

Since the vectors $\mathbf{c}_{ikl}$ have R coordinates, (3.14) is a system of $R(r-1)$ equations in Sr unknowns. Since $Sr > R(r-1)$, we have by the most basic form of Siegel's lemma (cf. [19], p. 127), that system (3.14) has a non-trivial solution in integers z_{ik} with

$$\begin{aligned} \max_{i,k} |z_{ik}| &\le \Big\{ rS \cdot \max_{i,k,l} || \mathbf{c}_{ikl} || \Big\}^{\frac{R(r-1)}{Sr - R(r-1)}} \\ &\le \Big\{ C_4(K) \cdot SA \Big\}^{\frac{R(r-1)}{Sr - R(r-1)}} \quad \text{by (3.13).} \end{aligned} \tag{3.15}$$

By (3.14) we have that $\mathbf{x} := \sum_{i=1}^{S} \beta_i' \mathbf{a}_i'$ is equal to the coefficient of $\omega_1 = 1$ in (3.11), i.e.

$$\mathbf{x} = \sum_{i=1}^{S} \sum_{k=1}^{r} z_{ik} \mathbf{c}_{1kl} \ .$$

Together with (3.15), (3.13) this implies

$$\begin{aligned} || \, \mathbf{x} \, || &\leq Sr \cdot \Big(\max_{i,k} |z_{ik}| \Big) \Big(\max_{k,l} || \, \mathbf{c}_{1kl} \, || \Big) \\ &\leq \Big(C_5(K) \cdot SA \Big)^{1 + \frac{R(r-1)}{Sr - R(r-1)}} = \Big(C_5(K) \cdot SA \Big)^{\frac{Sr}{Sr - R(r-1)}} \ . \end{aligned}$$

Moreover, (3.10) and (3.15) imply

$$|\beta_i'| \leq C_6(K) \Big\{ C_4(K) \cdot SA \Big\}^{\frac{R(r-1)}{Sr - R(r-1)}}$$

and so, by (3.7),

$$|\beta_i| = |q_i| |\beta_i'| \leq A |\beta_i'| \leq \Big(C_7(K) \cdot SA \Big)^{\frac{Sr}{Sr - R(r-1)}} \quad \text{for } i = 1, \ldots, S \ .$$

This completes the proof of Lemma 3. □

Let $\alpha_1, \ldots, \alpha_t$ be the algebraic numbers from Theorem 2 and put $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$. By assumption we have

$$\max_{i=1,\ldots,t} M(\alpha_i) = M, \quad [K : \mathbb{Q}] = r \ . \tag{1.9}$$

Let $\gamma_1, \ldots, \gamma_t$ be non-negative real numbers with $\gamma_1 + \cdots + \gamma_t = 1$. For $i = 0, 1, 2, \ldots$ we define the polynomial of degree i,

$$\begin{aligned} p_i(X) &:= (X - \alpha_1)^{j_1(i)} \cdots (X - \alpha_t)^{j_t(i)}, \\ &\text{with } j_l(i) = [\gamma_l \cdot i] \ \text{ for } l = 1, \ldots, t-1, \quad j_t(i) - \sum_{l=1}^{t-1} [\gamma_l \cdot i] \ . \end{aligned} \tag{3.16}$$

Let $k, d_1, \ldots, d_k$ be positive integers and put

$$\mathcal{I}_k = \{0, \ldots, d_1\} \times \cdots \times \{0, \ldots, d_k\} \ .$$

By $\mathbf{i}$ we denote a tuple $(i_1, \ldots, i_k) \in \mathcal{I}_k$. For a polynomial P with integer coefficients, we denote by $|| \, P \, ||$ the maximum of the absolute values of its coefficients. The next lemma gives our auxiliary polynomial:

Lemma 4. *Assume that*

$$\frac{2^{d_1+\cdots+d_k}}{(d_1+1)\cdots(d_k+1)} \geq C(K) \tag{3.17}$$

where $C(K)$ is the constant from Lemma 3. Let $\mathcal{I}$ be a subset of $\mathcal{I}_k$ with

$$\#\mathcal{I} \leq \frac{1}{2r}(d_1+1)\cdots(d_k+1)\ . \tag{3.18}$$

Then there are $\beta_{\mathbf{i}} \in O_K$ for $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$ such that

$$P(X_1,\ldots,X_k) := \sum_{\mathbf{i}\in\mathcal{I}_k\backslash\mathcal{I}} \beta_{\mathbf{i}} p_{i_1}(X_1)\cdots p_{i_k}(X_k) \in \mathbb{Z}[X_1,\ldots,X_k]\backslash\{0\}\ , \tag{3.19}$$

$$||\, P \,|| \leq \big(4M\big)^{2r(d_1+\cdots+d_k)}\ , \tag{3.20}$$

$$|\beta_{\mathbf{i}}| \leq \big(4M\big)^{2r(d_1+\cdots+d_k)} \quad \text{for } \mathbf{i} \in \mathcal{I}_k\backslash\mathcal{I}\ . \tag{3.21}$$

Proof. Let $i > 0$. Then

$$p_i(X) = (X-\alpha_{i1})\cdots(X-\alpha_{ii}) \quad \text{with } \alpha_{i1},\ldots,\alpha_{ii} \in \{\alpha_1,\ldots,\alpha_t\}.$$

Let $q_{ij} \in \mathbb{Z}_{>0}$ be the leading coefficient of the minimal polynomial of α_{ij}. Clearly, by (1.9) we have

$$q_{ij} \leq M(\alpha_{ij}) \leq M, \quad q_{ij}||\,\alpha_{ij}\,|| \leq M(\alpha_{ij}) \leq M\ . \tag{3.22}$$

The coefficient of $X_1^{j_1}\cdots X_k^{j_k}$ in $p_{i_1}(X_1)\cdots p_{i_k}(X_k)$ is equal to

$$\alpha(\mathbf{i},\mathbf{j}) = \pm \prod_{h=1}^{k} \sum_{S_h} \prod_{l_h\in S_h} \alpha_{i_h,l_h}\ , \tag{3.23}$$

where for $h = 1,\ldots,k$, the sum is taken over all subsets S_h of $\{1,\ldots,i_h\}$ of cardinality $i_h - j_h$. Define the rational integer

$$q(\mathbf{i},\mathbf{j}) := \prod_{h=1}^{k}\prod_{j=1}^{i_h} q_{i_h,j}\ .$$

By (3.23) we have $q(\mathbf{i},\mathbf{j})\alpha(\mathbf{i},\mathbf{j}) \in O_K$ and by (3.23), (3.22) we have

$$q(\mathbf{i},\mathbf{j}) \leq M^{d_1+\cdots+d_k}, \tag{3.24}$$

$$||\, q(\mathbf{i},\mathbf{j})\alpha(\mathbf{i},\mathbf{j})\,|| \leq \prod_{h=1}^{k} \binom{i_h}{i_h-j_h} M^{i_1+\cdots+i_k} \leq (2M)^{d_1+\cdots+d_k}\ .$$

We apply Lemma 3 with $R = (d_1+1)\cdots(d_k+1)$, $S = R - \#\mathcal{I}$ and $\{\mathbf{a}_1, \ldots, \mathbf{a}_S\} = \{p_{i_1}(X_1)\cdots p_{i_k}(X_k) : \mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}\}$. (3.18) implies condition (3.1) and (3.24) implies (3.2) with $A = (2M)^{d_1+\cdots+d_k}$. Note that by (3.17) we have

$$C(K) \cdot SA \le C(K)(d_1+1)\cdots(d_k+1)A \le (4M)^{d_1+\cdots+d_k}$$

and that by (3.18), we have $Sr \ge (r - \frac{1}{2})R$, whence

$$\frac{Sr}{Sr - R(r-1)} \le \frac{(r-\frac{1}{2})R}{\frac{1}{2}R} \le 2r \ .$$

Together with Lemma 3 this implies at once that there are $\beta_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$) with (3.19)-(3.21). □

§4. Combinatorial lemmas.

We will have to estimate the values of the auxiliary polynomial constructed in §3 in certain points and for this purpose we need some combinatorial lemmas. We use the arguments from elementary probability theory introduced by Wirsing [24], except that we obtain better estimates by using the following lemma instead of Chebyshev's inequality:

Lemma 5. *Let $X_1, \ldots, X_k$ be mutually independent random variables on some probability space with probability measure P, such that for $i = 1, \ldots, k$, X_i has expectation μ_i and $P(X_i \in [0,1]) = 1$. Let $\mu := \mu_1 + \cdots + \mu_k$ and let ϵ be a real with $0 < \epsilon < 2/3$. Then*

$$P(|X_1 + \cdots + X_k - \mu| \ge \epsilon k) \le 2e^{-\epsilon^2 k/3} \quad (e = 2.7182\ldots) \ . \tag{4.1}$$

Proof. Clearly, (4.1) follows from

$$P(X_1 + \cdots + X_k - \mu \ge \epsilon k) \le e^{-\epsilon^2 k/3} \ , \tag{4.2}$$

$$P(X_1 + \cdots + X_k - \mu \le -\epsilon k) \le e^{-\epsilon^2 k/3} \ , \tag{4.3}$$

and (4.3) follows from (4.2) by replacing X_i by $1 - X_i$, μ_i by $1 - \mu_i$ in (4.2) for $i = 1, \ldots, k$. So it suffices to prove (4.2).

For $i = 1, \ldots, k$, denote by σ_i^2 the variance of X_i, i.e. the expectation of $(X_i - \mu_i)^2$; since $P(X_i \in [0,1]) = 1$ this variance exists and is ≤ 1. Put $s^2 := \sum_{i=1}^k \sigma_i^2$. We may assume that $s^2 > 0$ since otherwise $P(X_i = \mu_i) = 1$ for $i = 1, \ldots, t$ and we

are done. By the inequality at the bottom of p. 267, Section 19 of Loève [11] we have

$$P\big(\frac{X_1+\cdots+X_k-\mu}{s}\ge\epsilon'\big)\le\exp\big(-t\epsilon'+\frac{t^2}{2}(1+\frac{tc}{2})\big)\ \text{ for }\epsilon'>0\ , \tag{4.4}$$

where c is such that $P(|(X_i-\mu_i)/s|\le c)=1$ for $i=1,\dots,k$ and t is any real with $0<t\le c^{-1}$. (Loève uses the notation S' for $(X_1+\cdots+X_k-\mu)/s$). We apply (4.4) with $\epsilon'=k\epsilon/s$, $c=1/s$ and $t=\epsilon s$. Then the right-hand side of (4.4) becomes

$$\exp\big(-\epsilon^2k+\frac{\epsilon^2s^2}{2}(1+\frac{\epsilon}{2})\big)\le\exp\left(-\epsilon^2k/3\right)$$

since $s^2\le k$, $0<\epsilon<2/3$. This implies (4.2). □

Let ϵ be a real and let $k,t,d_1,\dots,d_k$ be positive integers with

$$0<\epsilon<\frac{1}{6t}\ , \tag{4.5}$$

$$d_h>\frac{10^4}{\epsilon}\ \text{ for }h=1,\dots,k\ . \tag{4.6}$$

Define the sets

$$\mathcal{I}_k=\{0,\dots,d_1\}\times\cdots\times\{0,\dots,d_k\}\ ,$$
$$\mathcal{C}_k=\{1,\dots,t\}^k\ .$$

We will use $\mathbf{i}$ to denote a tuple $(i_1,\dots,i_k)\in\mathcal{I}_k$ and $\mathbf{c}$ to denote a tuple $(c_1,\dots,c_k)\in\mathcal{C}_k$.

Lemma 6. *There is a subset $\mathcal{I}$ of $\mathcal{I}_k$ with*

$$\#\mathcal{I}\le24\epsilon^{-1}e^{-\epsilon^2k/4}(d_1+1)\cdots(d_k+1)$$

such that for all $\mathbf{i}\in\mathcal{I}_k\backslash\mathcal{I}$ and all $x\in[0,1]$ we have

$$\big|\#\big\{h\in\{1,\dots,k\}:\ \frac{i_h}{d_h}\le x\big\}-kx\big|\le\epsilon k\ .$$

Proof. For $x\in[0,1]$, $\mathbf{i}\in\mathcal{I}_k$ we put $s(\mathbf{i},x):=\#\big\{h\in\{1,\dots,k\}:\ \frac{i_h}{d_h}\le x\big\}$. We endow $\mathcal{I}_k$ with the probability measure P such that each tuple $\mathbf{i}=(i_1,\dots,i_k)\in\mathcal{I}_k$ has probability $1/\#\mathcal{I}_k=1/(d_1+1)\cdots(d_k+1)$. Fix $x\in[0,1]$. For $h=1,\dots,k$, define the random variable $X_h=X_h(\mathbf{i})$ on $\mathcal{I}_k$ by $X_h=1$ if $0\le i_h/d_h\le x$ and $X_h=0$ if $x<i_h/d_h\le1$. Thus, $X_1,\dots,X_k$ are mutually independent and X_h has expectation $\mu_h=P(X_h=1)=([xd_h]+1)/(d_h+1)$ for $h=1,\dots,k$. By Lemma 5 with $(0.9-10^{-4})\epsilon$ replacing ϵ we have

$$P\big(|X_1+\cdots+X_k-(\mu_1+\cdots+\mu_k)|>(0.9-10^{-4})\epsilon k\big)\le2e^{-\epsilon^2k/4}\ .$$

By (4.6) we have

$$|\mu_h - x| = \frac{|[xd_h] + 1 - xd_h - x|}{d_h + 1} \le \frac{1}{d_h + 1} < 10^{-4}\epsilon \ \text{ for } h = 1, \dots, k \ .$$

Hence

$$P(|X_1 + \cdots + X_k - kx| > 0.9\epsilon k) \le 2e^{-\epsilon^2 k/4} \ .$$

This implies that for each *fixed* $x \in [0,1]$ there exists a subset $\mathcal{I}(x)$ of $\mathcal{I}_k$ with

$$|s(\mathbf{i}, x) - kx| \le 0.9\epsilon k \ \text{ for } \mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}(x) \ , \quad \#\mathcal{I}(x) \le 2e^{-\epsilon^2 k/4} \ .$$

Now let $n = [10/\epsilon] + 1$ and take

$$\mathcal{I} := \bigcup_{m=0}^{n} \mathcal{I}(\frac{m}{n}) \ .$$

Then

$$\#\mathcal{I} \le 2(n+1)e^{-\epsilon^2 k/4} \le 24\epsilon^{-1}e^{-\epsilon^2 k/4} \ .$$

Let $x \in [0,1]$ and choose $m \in \{0, \dots, n-1\}$ with $m/n \le x \le (m+1)/n$. Then for $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I} = \cap_{m=0}^{n} (\mathcal{I}_k \backslash \mathcal{I}(\frac{m}{n}))$ we have

$$s(\mathbf{i}, x) \le s(\mathbf{i}, \frac{m+1}{n}) \le k(\frac{m+1}{n} + 0.9\epsilon) \le k(x + \frac{1}{n} + 0.9\epsilon) \le k(x + \epsilon),$$
$$s(\mathbf{i}, x) \ge s(\mathbf{i}, \frac{m}{n}) \ge k(\frac{m}{n} - 0.9\epsilon) \ge k(x - \frac{1}{n} - 0.9\epsilon) \ge k(x - \epsilon),$$

which is what we wanted to prove. □

Lemma 7. *Let $\mathcal{I}$ be the set from Lemma 6. Then for $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$, $h = 1, \dots, k$ we have*

$$|\frac{i_{\pi(h)}}{d_{\pi(h)}} - \frac{h}{k}| \le \epsilon \ ,$$

where π is the permutation of $(1, \dots, k)$ such that

$$\frac{i_{\pi(1)}}{d_{\pi(1)}} \le \cdots \le \frac{i_{\pi(k)}}{d_{\pi(k)}} \ .$$

Proof. Fix $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$, $h \in \{1, \dots, k\}$ and put $x := i_{\pi(h)}/d_{\pi(h)}$. By definition, the number of integers j with $j \in \{1, \dots, k\}$, $i_j/d_j \le x$ is equal to h. Lemma 6 implies that $|h - kx| \le \epsilon k$. This implies Lemma 7. □

Lemma 8. *There is a subset $\mathcal{C}$ of $\mathcal{C}_k$ with*

$$\#\mathcal{C} \le 2te^{-\epsilon^2 k/3} \cdot t^k \ ,$$

such that for each $\mathbf{c} \in \mathcal{C}_k \backslash \mathcal{C},\ c \in \{1, \dots, t\}$ *we have*

$$|\#\big\{h \in \{1, \dots, k\} :\ c_k = c\big\} - \frac{k}{t}| \le \epsilon k \ . \tag{4.7}$$

Proof. Lemma 8 follows once we have proved that for each $c \in \{1, \dots, t\}$ there is a subset $\mathcal{C}^{(c)}$ of $\mathcal{C}_k$ with $\#\mathcal{C}^{(c)} \le 2e^{-\epsilon^2 k/3} t^k$ such that for each $\mathbf{c} \in \mathcal{C}_k \backslash \mathcal{C}^{(c)}$ we have (4.7). We endow $\mathcal{C}_k$ with the probability measure P such that each $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{C}_k$ has probability $1/\#\mathcal{C}_k = 1/t^k$. Fix $c \in \{1, \dots, t\}$. For $h = 1, \dots, k$, define the random variable $X_h = X_h(\mathbf{c})$ on $\mathcal{C}_k$ by $X_h = 1$ if $c_h = c$ and $X_h = 0$ if $c_h \ne c$. Then $X_1, \dots, X_k$ are mutually independent and X_h has expectation $1/t$ for $h = 1, \dots, k$. Now by Lemma 5 we have

$$\frac{\#\Big\{\mathbf{c} \in \mathcal{C}_k :\ |\#\big\{h \in \{1, \dots, k\} :\ c_k = c\big\} - \frac{k}{t}| \ge \epsilon k\Big\}}{t^k}$$
$$= P(|X_1 + \cdots + X_k - \frac{k}{t}| \ge \epsilon k) \le 2e^{-\epsilon^2 k/3}$$

which is what we wanted to prove. □

The next lemma is the main result of this section:

Lemma 9. *Let* $\varphi_1, \dots, \varphi_t$ *be non-negative reals satisfying (1.10), and let* ϵ *be a real and* $k, t, d_1, \dots, d_k$ *integers satisfying (4.5),(4.6). Then there are subsets* $\mathcal{I}$ *of* $\mathcal{I}_k = \{0, \dots, d_1\} \times \cdots \times \{0, \dots, d_k\}$ *and* $\mathcal{C}$ *of* $\mathcal{C}_k = \{1, \dots, t\}^k$ *with*

$$\#\mathcal{I} \le 24\epsilon^{-1} e^{-\epsilon^2 k/4} \cdot (d_1 + 1) \cdots (d_k + 1) \ , \tag{4.8}$$

$$\#\mathcal{C} \le 2te^{-\epsilon^2 k/3} \cdot t^k \ , \tag{4.9}$$

and non-negative reals $\gamma_1, \dots, \gamma_t$ *with* $\gamma_1 + \cdots + \gamma_t = 1$, *such that for all tuples* $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$, $\mathbf{c} \in \mathcal{C}_k \backslash \mathcal{C}$ *we have*

$$\sum_{h=1}^{k} \frac{i_h}{d_h} \gamma_{c_h} \varphi_{c_h} \ge \Big(\frac{k}{2t^2} - \frac{3\epsilon k}{t}\Big) \cdot (2t + \delta) \ . \tag{4.10}$$

Remark. The lower bound of (4.10) cannot be improved by another choice of $\gamma_1, \dots, \gamma_t$.

Proof. We prove Lemma 9 with the sets $\mathcal{I}$ from Lemmas 6 and and $\mathcal{C}$ from Lemma 8. These sets satisfy (4.8), (4.9), respectively. By (1.10), there is a subset I of $\{1, \dots, t\}$ such that $(\#I)^2\big(\sum_{j \in I} \varphi_j^{-1}\big)^{-1} \ge 2t + \delta$. Choose

$$\gamma_i := 0 \ \text{ for } i \in \{1, \dots, t\} \backslash I, \qquad \gamma_i := \varphi_i^{-1} \big/ (\sum_{j \in I} \varphi_j^{-1})^{-1} \ \text{ for } i \in I \ .$$

Then (4.10) follows once we have proved that for every $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}_k \backslash \mathcal{I}$, $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{C}_k \backslash \mathcal{C}$,

$$\sum_{h:\ c_h \in I} \frac{i_h}{d_h} \geq \Big(\frac{k}{2t^2} - \frac{3\epsilon k}{t} \Big) \big(\#I\big)^2. \tag{4.11}$$

Fix $\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}$, $\mathbf{c} \in \mathcal{C}_k \backslash \mathcal{C}$. Let $T := \#\{h \in \{1, \dots, k\} :\ c_h \in I\}$ and let π be a permutation of $(1, \dots, k)$ such that $i_{\pi(1)}/d_{\pi(1)} \leq \cdots \leq i_{\pi(k)}/d_{\pi(k)}$. By Lemma 8, Lemma 7, respectively, we have

$$(\#I)k(\frac{1}{t} - \epsilon) \leq T \leq (\#I)k(\frac{1}{t} + \epsilon)\ , \qquad \frac{i_{\pi(h)}}{d_{\pi(h)}} \geq \frac{h}{k} - \epsilon \ \text{ for } h = 1, \dots, k.$$

Hence

$$\begin{aligned} \sum_{h:\ c_h \in I} \frac{i_h}{d_h} &\geq \sum_{h=1}^{T} \frac{i_{\pi(h)}}{d_{\pi(h)}} \geq \sum_{h=1}^{T} \big(\frac{h}{k} - \epsilon\big) \geq \frac{1}{2k} T^2 - \epsilon T \\ &\geq \frac{1}{2k} k^2 (\#I)^2 (\frac{1}{t} - \epsilon)^2 - \epsilon k (\frac{1}{t} + \epsilon) \#I \geq \Big(\frac{k}{2t^2} - \frac{3\epsilon k}{t} \Big) \big(\#I\big)^2 \end{aligned}$$

where we used that $\epsilon < \frac{1}{t}$ by (4.5) and $\#I \leq (\#I)^2$. This proves (4.11). □

§5. Estimation of certain values of the auxiliary polynomial.

Let $\alpha_1, \dots, \alpha_t$ be the algebraic numbers and $\varphi_1, \dots, \varphi_t$ the reals from Theorem 2. Thus, $\max_I\ (\#I)^2 \big(\sum_{i \in I} \varphi_i^{-1} \big)^{-1} \geq 2 + \delta$ with $0 < \delta < 1$. Define

$$\epsilon = \frac{\delta}{34t^2}\ , \tag{5.1}$$

$$k = \big[3.5 \times 10^4 \cdot t^4 \delta^{-2} \big(1 + \frac{1}{2} \log t\big) \big(1 + \frac{1}{2} \log \delta^{-1}\big) \log 4r \big] \tag{5.2}$$

and let $d_1, \dots, d_k$ be integers satisfying

$$d_1 \geq d_2 \geq \cdots \geq d_k \geq \max(\frac{10^4 t}{\epsilon}, C(K)) \tag{5.3}$$

where $C(K)$ is the constant from Lemma 3. Thus, (4.5) and (4.6) are satisfied and Lemma 9 is applicable. Let $\mathcal{I}$ and $\mathcal{C}$ be the sets, and $\gamma_1, \dots, \gamma_t$ the reals from Lemma 9. Then

$$\#\mathcal{I} \leq \frac{1}{2r} (d_1 + 1) \cdots (d_t + 1)\ , \tag{5.4}$$

$$\#\mathcal{C} \leq t\epsilon \cdot t^k\ . \tag{5.5}$$

Namely, (5.4) and (5.5) follow from (4.8), (4.9) and the inequalities $24\epsilon^{-1}e^{-k\epsilon^2/4} \le \frac{1}{2r}$, $2te^{-k\epsilon^2/3} \le t\epsilon$, and these inequalities hold true since by (5.1), (5.2) we have

$$\begin{aligned}\max\Big(4\epsilon^{-2}\log\frac{48r}{\epsilon}, 3\epsilon^{-2}\log\frac{2}{\epsilon}\Big)&\\ = 4624t^4\delta^{-2}\Big(\log 1632 + 2\log t + \log\delta^{-1} + \log r\Big)&\\ < 3.5\times10^4\cdot t^4\delta^{-2}\big(1+\frac{1}{2}\log t\big)\big(1+\frac{1}{2}\log\delta^{-1}\big)\log 4r - 1 < k\ .&\end{aligned}$$

We apply Lemma 4 with these $k, d_1, \dots, d_k, \mathcal{I}$ and $\gamma_1, \dots, \gamma_t$; this is possible since (5.3) and (5.4) imply the conditions (3.17) and (3.18) of Lemma 4. Let P be the auxiliary polynomial from Lemma 4, i.e.

$$P(X_1, \dots, X_k) = \sum_{\mathbf{i}\in\mathcal{I}_k\setminus\mathcal{I}} \beta_{\mathbf{i}} p_{i_1}(X_1)\cdots p_{i_k}(X_k)\ , \tag{5.6}$$

where for $i = 1, 2, \dots$ $p_i(X)$ is given by (3.16). Further, let $\xi_1, \dots, \xi_k$ be solutions of (1.5) with

$$M(\xi_1) \ge (6M)^{68t^2(2r+t)/\delta}\ , \tag{5.7}$$

$$M(\xi_1)^{d_1} \le M(\xi_h)^{d_h} \le M(\xi_1)^{d_1(1+\epsilon^2)} \quad \text{for } h = 1, \dots, k\ . \tag{5.8}$$

For a polynomial in k variables $X_1, \dots, X_k$ and a tuple of non-negative integers $\mathbf{j} = (j_1, \dots, j_k)$ define the differential operator

$$D^{\mathbf{j}} = \frac{1}{j_1!\cdots j_k!}\frac{\partial^{j_1+\cdots+j_k}}{\partial X_1^{j_1}\cdots\partial X_k^{j_k}}\ ;$$

note that $D^{\mathbf{j}}$ maps polynomials with coefficients in $\mathbb{Z}$ to polynomials with coefficients in $\mathbb{Z}$. We need the following estimate:

Lemma 10. *(i). For each tuple $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{C}_k\setminus\mathcal{C}$ and for each tuple of non-negative integers $\mathbf{j} = (j_1, \dots, j_k)$ with*

$$\sum_{h=1}^{k}\frac{j_h}{d_h} < \frac{\epsilon k}{t} \tag{5.9}$$

we have

$$\begin{aligned}|D^{\mathbf{j}}P(\xi_1^{(c_1)}, \dots, \xi_k^{(c_k)})| \le (6M)^{(2r+t)(d_1+\cdots+d_k)}\Big(\prod_{h=1}^{k}|1, \xi_h^{(c_h)}|^{d_h}\Big)\cdot&\\ \cdot\big(M(\xi_1)^{d_1}\big)^{-(2t+\delta)((k/2t^2)-5k\epsilon/t)}\ .&\end{aligned} \tag{5.10}$$

(ii). For each tuple $\mathbf{c} \in \mathcal{C}_k$ *and each tuple of non-negative integers* $\mathbf{j}$ *we have*

$$|D^{\mathbf{j}} P(\xi_1^{(c_1)}, \dots, \xi_k^{(c_k)})| \le (6M)^{(2r+t)(d_1+\cdots+d_k)} \Big(\prod_{h=1}^{k} |1, \xi_h^{(c_h)}|^{d_h} \Big) \ . \tag{5.11}$$

Proof. In addition to the hypotheses made above we assume that

$$0 \le \varphi_l \le 2t + \delta \ \text{ for } l = 1, \dots, t \ . \tag{5.12}$$

This is no loss of generality. Namely, suppose that for instance $\varphi_1 > 2t + \delta$. Then $\xi_1, \dots, \xi_t$ satisfy (1.5) with $\varphi_1 = 2t + \delta$ and $\varphi_l = 0$ for $l = 2, \dots, t$. Then these new φ_l satisfy (1.10) and we can prove Lemma 10 with these new φ_l.

For every non-negative integer j we define the differential operator for polynomials in one variable X, $D^j = (1/j!)d^j/dX^j$. Then for each $i \ge 0$, $j \ge 0$ we have

$$D^j p_i(X) = D^j \big(\prod_{l=1}^{t} (X - \alpha_l)^{j_l(i)} \big) = \sum_{\substack{0 \le j_l \le j_l(i) \\ j_1 + \cdots + j_t = j}} \prod_{l=1}^{t} \binom{j_l(i)}{j_l} (X - \alpha_l)^{j_l(i) - j_l} \ .$$

For $h \in \{1, \dots, k\}$, $c \in \{1, \dots, t\}$ we have by (1.5),

$$|\alpha_c - \xi_h^{(c)}| \le 2|1, \alpha_c| \cdot |1, \xi_h^{(c)}| \cdot M(\xi_h)^{-\varphi_c} \tag{5.13}$$

and, trivially,

$$|\alpha_l - \xi_h^{(c)}| \le 2|1, \alpha_l| \cdot |1, \xi_h^{(c)}| \ \text{ for } l = 1, \dots, t \ . \tag{5.14}$$

Further, by (3.16) we have that $p_i(X) = \prod_{l=1}^{t} (X - \alpha_l)^{j_l(i)}$ where the $j_l(i)$ are non-negative integers with

$$\sum_{l=1}^{t} j_l(i) = i, \quad j_l(i) \ge \gamma_l i - 1 \ \text{ for } l = 1, \dots, t \ .$$

Together with (5.8), (5.12) these imply

$$\begin{aligned} &|D^j p_i(\xi_h^{(c)})| \\ &\le \sum_{\substack{0 \le j_l \le j_l(i) \\ j_1 + \cdots + j_l = j}} \prod_{l=1}^{t} \binom{j_l(i)}{j_l} \big(2|1, \alpha_l| \cdot |1, \xi_h^{(c)}|\big)^{j_l(i) - j_l} \cdot M(\xi_h)^{-\varphi_c(j_c(i) - j_c)} \\ &\le (4M^t |1, \xi_h^{(c)}|)^i \cdot M(\xi_h)^{-\gamma_c \varphi_c i + \varphi_c(j+1)} \\ &\le (4M^t |1, \xi_h^{(c)}|)^i \cdot \big(M(\xi_1)^{d_1}\big)^{-\gamma_c \varphi_c (i/d_h) + (1+\epsilon^2)(2t+\delta)(j+1)/d_h} \ . \end{aligned} \tag{5.15}$$

We now use Lemma 9. Let $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{C}_k \backslash \mathcal{C}$, $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}_k \backslash \mathcal{I}$, and $\mathbf{j} = (j_1, \dots, j_k)$ a tuple of non-negative integers satisfying (5.9). Then by (5.15) we have

$$|D^{\mathbf{j}} \prod_{h=1}^{k} p_{i_h}(\xi_h^{(c_h)})| \le AB\big(M(\xi_1)^{d_1}\big)^{-C}$$

with $A = (4M^t)^{d_1+\cdots+d_t}$, $B = \prod_{h=1}^{k} |1, \xi_h^{(c_h)}|^{d_h}$ and

$$C = \sum_{h=1}^{k} \frac{i_h}{d_h} \gamma_{c_h} \varphi_{c_h} \;-\; (1+\epsilon^2)(2t+\delta)\big(\sum_{h=1}^{k} \frac{j_h}{d_h} + \sum_{h=1}^{k} \frac{1}{d_h}\big) \; .$$

We have

$$\sum_{h=1}^{k} \frac{i_h}{d_h} \gamma_{c_h} \varphi_{c_h} \ge (2t+\delta)\big(\frac{k}{2t^2} - \frac{3\epsilon k}{t}\big) \quad \text{by Lemma 9,}$$

$$(1+\epsilon^2)\big(\sum_{h=1}^{k} \frac{j_h}{d_h} + \sum_{h=1}^{k} \frac{1}{d_h}\big) < \frac{2\epsilon k}{t} \quad \text{by (5.9), (5.1), (5.3),}$$

hence

$$C \ge (2t+\delta)\big(\frac{k}{2t^2} - \frac{5\epsilon k}{t}\big) \; .$$

Now (5.6) and the estimates for $\beta_{\mathbf{i}}$ in Lemma 4 give

$$|D^{\mathbf{j}} P(\xi_1^{(c_1)}, \dots, \xi_k^{(c_k)})| \le A' B\big(M(\xi_1)^{d_1}\big)^{-C}$$

with

$$\begin{aligned} A' &= A\Big(\sum_{\mathbf{i} \in \mathcal{I}_k \backslash \mathcal{I}} |\beta_{\mathbf{i}}|\Big) \\ &\le (4M^t)^{d_1+\cdots+d_k} \cdot 2^{d_1+\cdots+d_k} (4M)^{2r(d_1+\cdots+d_k)} \\ &\le (6M)^{(2r+t)(d_1+\cdots+d_k)} \; . \end{aligned}$$

This proves part (i) of Lemma 10. We obtain part (ii) by observing that, as a consequence of (5.14), we can replace (5.15) by the trivial estimate $|D^j p_i(\xi_h^{(c)}| \le (4M^t |1, \xi_h^{(c)}|)^i$ and so all estimates made above remain valid if we replace the exponent C on $M(\xi_1)^{d_1}$ by 0. □

Lemma 11. *Suppose that $\epsilon, k, d_1, \dots, d_k$ satisfy (5.1)-(5.3) and that $\xi_1, \dots, \xi_k$ are solutions of (1.5) satisfying (5.7),(5.8). Let P be the polynomial from Lemma 4, with the set $\mathcal{I}$ and the reals $\gamma_1, \dots, \gamma_t$ from Lemma 9. Then there is a tuple $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{C}_k = \{1, \dots, t\}^k$ such that for each tuple $\mathbf{j} = (j_1, \dots, j_k)$ of non-negative integers with $\sum_{h=1}^{k} j_h/d_h < \epsilon k/t$ we have*

$$D^{\mathbf{j}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) = 0 \; . \tag{5.16}$$

Proof. We assume the contrary. Let L be the normal extension of $\mathbb{Q}$ generated by the numbers $\xi_h^{(c)}$ $(h = 1, \dots, k,\ c = 1, \dots, t)$. We call two tuples $\mathbf{c} = (c_1, \dots, c_k)$, $\mathbf{c}' = (c_1', \dots, c_k') \in \mathcal{C}_k$ *conjugate* if there is an $\mathbb{Q}$-automorphism of L mapping the tuple $(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)})$ to $(\xi_1^{(c_1')}, \dots, \xi_t^{(c_t')})$. From our assumption, it follows that for every $\mathbf{c} \in \mathcal{C}_k$ there is a tuple $\mathbf{j_c}$ with (5.9) such that $D^{\mathbf{j_c}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) \neq 0$. Since P has its coefficients in $\mathbb{Z}$, there is no loss of generality in assuming that $\mathbf{j_c} = \mathbf{j_{c'}}$ whenever $\mathbf{c}$ and $\mathbf{c}'$ are conjugate. For $h = 1, \dots, k$, let $q_h \in \mathbb{Z}_{>0}$ denote the leading coefficient of the minimal polynomial of ξ_h. Define the number

$$Z := \big(q_1^{d_1} \cdots q_k^{d_k}\big)^{t^{k-1}} \prod_{\mathbf{c} \in \mathcal{C}_k} D^{\mathbf{j_c}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) \ .$$

Then $Z \neq 0$. We will obtain a contradiction by showing that $Z \in \mathbb{Z}$ and $|Z| < 1$.

We first show that $Z \in \mathbb{Z}$. Since for conjugate tuples $\mathbf{c}$, $\mathbf{c}'$ we have $\mathbf{j_c} = \mathbf{j_{c'}}$, the number Z is invariant under automorphisms of L, i.e. $Z \in \mathbb{Q}$. Denote the fractional ideal with respect to the ring of integers of L generated by $\mu_1, \dots, \mu_m \in L$ by $(\mu_1, \dots, \mu_m)$. For $\mathbf{c} \in \mathcal{C}_k$ we have

$$D^{\mathbf{j_c}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) \in (1, \xi_1^{(c_1)})^{d_1} \cdots (1, \xi_k^{(c_k)})^{d_k} \tag{5.17}$$

since the polynomial $D^{\mathbf{j_c}} P$ has its coefficients in $\mathbb{Z}$ and has degree $\leq d_h$ in X_h. The minimal polynomial of ξ_h is $q_h \prod_{c=1}^{t} (X - \xi_h^{(c)})$. The coefficients of this polynomial are integers with gcd 1. On the other hand, by Gauss' lemma for fractional ideals in number fields, the ideal generated by the coefficients of this polynomial is equal to $q_h \prod_{c=1}^{t} (1, \xi_h^{(c)})$; therefore, $q_h \prod_{c=1}^{t} (1, \xi_h^{(c)}) = (1)$. Together with (5.17) this implies

$$\begin{aligned} Z &\in (q_1^{d_1} \cdots q_k^{d_k})^{t^{k-1}} \Big(\prod_{\mathbf{c} \in \mathcal{C}_k} (1, \xi_1^{(c_1)})^{d_1} \cdots (1, \xi_k^{(c_k)})^{d_k} \Big) \\ &= \Big(\prod_{h=1}^{k} \big\{ q_h (1, \xi_h^{(1)}) \cdots (1, \xi_h^{(t)}) \big\}^{d_h} \Big)^{t^{k-1}} = (1) \ . \end{aligned}$$

Hence $Z \in \mathbb{Z}$.

We now show that $|Z| < 1$. Lemma 10 gives

$$|Z| \leq A_1 B_1 \big(M(\xi_1)^{d_1} \big)^{-C_1} \ ,$$

with

$$\begin{aligned} A_1 &= (6M)^{(2r+t)(d_1 + \cdots + d_k) t^k} \ , \\ B_1 &= (q_1^{d_1} \cdots q_k^{d_k})^{t^{k-1}} \prod_{\mathbf{c} \in \mathcal{C}_k} \Big(\prod_{h=1}^{k} |1, \xi_h^{(c_h)}|^{d_h} \Big) = \Big(\prod_{h=1}^{k} M(\xi_h)^{d_h} \Big)^{t^{k-1}} \ , \\ C_1 &= \big(\# \mathcal{C}_k \backslash \mathcal{C} \big) \cdot (2t + \delta) \{ \frac{k}{2t^2} - \frac{5\epsilon k}{t} \} \ . \end{aligned}$$

Further,

$$A_1 \le (6M)^{(2r+t)kt^k d_1} \text{ by (5.3)},$$
$$B_1 \le M(\xi_1)^{kt^{k-1}d_1(1+\epsilon^2)} \text{ by (5.8)},$$
$$C_1 \ge (1-\epsilon t)t^k(2t+\delta)\{\frac{k}{2t^2} - \frac{5\epsilon k}{t}\} = kt^{k-1}\cdot(1-\epsilon t)\big(\frac{1}{2t} - 5\epsilon\big)(2t+\delta) \text{ by (5.5)}.$$

Therefore,

$$|Z| \le \Big(A_2 M(\xi_1)^{-C_2}\Big)^{kt^{k-1}d_1},$$

with

$$\begin{aligned} A_2 &= (6M)^{t(2r+t)}\ , \\ C_2 &= -(1+\epsilon^2) + (1-\epsilon t)\big(\frac{1}{2t} - 5\epsilon\big)(2t+\delta) \\ &= \frac{\delta}{2t} - 11\epsilon t - \frac{11}{2}\epsilon\delta + 10\epsilon^2 t^2 + 5\epsilon^2 t\delta - \epsilon^2 > \frac{\delta}{2t} - \frac{33}{2}\epsilon t \text{ since } \delta < t \\ &\ge \frac{\delta}{68t} \text{ by (5.1)}\ . \end{aligned}$$

Together with (5.7) this implies that $|Z| < 1$. □

§6. Completion of the proof of part (i) of Theorem 2.

We apply Lemma 12 below, which is the sharpening of Roth's lemma from [5]. We mention that this sharpening was proved by making explicit the arguments in Faltings' proof of his Product theorem [7]. A result slightly weaker than Lemma 12 follows from Ferretti's work [8]. For further information on Faltings' Product theorem we refer to [13]. We recall that for a polynomial P with coefficients in $\mathbb{Z}$, $||\,P\,||$ denotes the maximum of the absolute values of its coefficients.

Lemma 12. *Let σ be a real and $k, d_1, \ldots, d_k$ integers such that $k \ge 2$, $0 < \sigma \le k+1$ and*

$$\frac{d_h}{d_{h+1}} \ge \omega_1 := \frac{2k^3}{\sigma} \quad \text{for } h = 1, \ldots, k-1. \tag{6.1}$$

Further, let P be a non-zero polynomial in $\mathbb{Z}[X_1, \ldots, X_k]$ of degree at most d_h in X_h for $h = 1, \ldots, k$ and $\xi_1, \ldots, \xi_k$ non-zero algebraic numbers such that

$$M(\xi_h)^{d_h/\deg \xi_h} \ge \Big(4^{d_1+\cdots+d_k}||\,P\,||\Big)^{\omega_2} \quad \text{for } h = 1, \ldots, k \tag{6.2}$$
$$\text{with } \omega_2 := \big(\frac{3k^3}{\sigma}\big)^k\ .$$

Then there is a tuple $\mathbf{j} = (j_1, \dots, j_k)$ *of non-negative integers with*

$$\sum_{h=1}^{k} \frac{j_h}{d_h} < \sigma \ , \qquad D^{\mathbf{j}} P(\xi_1, \dots, \xi_k) \neq 0 \ .$$

Proof. This follows from Theorem 3 and the Remark on pp. 221,222 of [5]. We mention that Theorem 3 of [5] has instead of (6.2) the assumption $H(\xi_h)^{d_h} > \{e^{d_1+\cdots+d_k} H(P)\}^{\omega_2}$, with heights $H(\xi_h)$, $H(P)$ defined in [5]. This is implied by (6.2) since $H(\xi_h) \geq M(\xi_h)^{1/\deg \xi_h}$ and since for polynomials $P \in \mathbb{Z}[X_1, \dots, X_k]$, $H(P)$ is equal to the Euclidean norm of the vector of coefficients of P so $H(P) \leq \{(d_1+1)\cdots(d_k+1)\}^{1/2} || \, P \, ||$. □

Let $\varphi_1, \dots, \varphi_t$ be non-negative reals satisfying (1.10). Let ϵ and k be given by (5.1), (5.2), respectively. Put

$$\sigma := \frac{\epsilon k}{t}. \tag{6.3}$$

Thus, the quantities ω_1, ω_2 in Lemma 12 are equal to

$$\omega_1 = \frac{2k^2 t}{\epsilon}, \qquad \omega_2 = \Big(\frac{3k^2 t}{\epsilon}\Big)^k = \Big(\frac{3\omega_1}{2}\Big)^k \ . \tag{6.4}$$

We prove the following:

Lemma 13. *(1.5) has no solutions* $\xi_1, \dots, \xi_k$ *with*

$$M(\xi_1) \geq \big(4M\big)^{3rk\omega_2} \ , \tag{6.5}$$

$$M(\xi_{h+1}) \geq M(\xi_h)^{3\omega_1/2} \ \text{ for } h = 1, \dots, k-1 \ . \tag{6.6}$$

Proof. We assume the contrary and obtain a contradiction by applying Lemmas 11 and 12. We choose integers $d_1, \dots, d_k$ as follows: take

$$d_k \geq \max\Big(\frac{10^4 t}{\epsilon^2}, C(K)\Big) \tag{6.7}$$

and let $d_1, \dots, d_{k-1}$ be the integers defined by

$$d_k \log M(\xi_k) - \log M(\xi_1) < d_1 \log M(\xi_1) \leq d_k \log M(\xi_k),$$
$$d_1 \log M(\xi_1) \leq d_h \log M(\xi_h) < d_1 \log M(\xi_1) + \log M(\xi_h) \ \text{ for } h = 2, \dots, k-1 \ .$$

(6.7) implies that

$$\frac{\log M(\xi_1)}{d_k \log M(\xi_k)} < 10^{-4}\epsilon^2 \ ,$$
$$\frac{\log M(\xi_h)}{d_1 \log M(\xi_1)} \leq \big(1 - 10^{-4}\epsilon^2\big)^{-1} \frac{\log M(\xi_h)}{d_k \log M(\xi_k)} \leq \epsilon^2 \ \text{ for } h = 2, \dots, k-1,$$

so

$$M(\xi_1)^{d_1} \le M(\xi_h)^{d_h} \le M(\xi_1)^{d_1(1+\epsilon^2)} \quad \text{for } h = 1, \dots, k. \tag{6.8}$$

Further, (6.8) and (6.6) imply that

$$\frac{d_h}{d_{h+1}} \ge (1+\epsilon^2)^{-1} \cdot \frac{\log M(\xi_{h+1})}{M(\xi_h)} \ge (1+\epsilon^2)^{-1} \frac{3\omega_1}{2} > \omega_1 \ . \tag{6.9}$$

We apply Lemma 11. Let P be the polynomial from Lemma 4. We assumed (5.1) and (5.2), and (5.3) is a consequence of (6.7) and (6.9). Further, (5.7) follows from (6.5),(5.1),(5.2) and (6.4), while (5.8) follows from (6.8). So by Lemma 11 we have that there is a tuple $\mathbf{c} \in \mathcal{C}_k$ such that for each tuple of non-negative integers $\mathbf{j} = (j_1, \dots, j_k)$ with

$$\sum_{h=1}^{k} \frac{j_h}{d_h} < \frac{\epsilon k}{t} \tag{5.9}$$

we have $D^{\mathbf{j}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) = 0$.

We now apply Lemma 12 with $\sigma = \frac{\epsilon k}{t}$ and with $\xi_h^{(c_h)}$ replacing ξ_h for $h = 1, \dots, k$. From (6.9) we know already that (6.1) holds. Further, we have for $h = 1, \dots, k$,

$$\begin{aligned} M(\xi_h)^{d_h} &\ge M(\xi_1)^{d_1} \ge (4M)^{3rkd_1\omega_2} \quad \text{by (6.8), (6.5)} \\ &\ge (4M)^{3r(d_1+\cdots+d_k)\omega_2} \quad \text{by (6.9)} \\ &\ge \left(4^{d_1+\cdots+d_k} || \, P \, ||\right)^{\omega_2} \quad \text{by (3.20).} \end{aligned}$$

Hence (6.2) is also satisfied. It follows that there is a tuple $\mathbf{j}$ with (5.9) for which $D^{\mathbf{j}} P(\xi_1^{(c_1)}, \dots, \xi_t^{(c_t)}) \neq 0$. This is contrary to what we proved above. Thus, our assumption that Lemma 13 is false leads to a contradiction. □

We now complete the proof of part (i) of Theorem 2. Define a sequence of solutions $\xi_1, \xi_2, \dots$ of (1.5) as follows: ξ_1 is a solution ξ of (1.5) such that $M(\xi) \ge (4M)^{3rk\omega_2}$ and $M(\xi)$ is minimal; and for $h = 1, 2, \dots$, ξ_{h+1} is a solution ξ of (1.5) such that $M(\xi) \ge M(\xi_h)^{3\omega_1/2}$ and $M(\xi)$ is minimal. From Lemma 13 it follows, that this sequence has at most $k-1$ elements.

Let $A := \max(4^{t(t+1)/(\kappa-2t)}, M)$ be the lower bound in part (i) of Theorem 2. Put $\theta := 1 + (\kappa - 2t)/t$. By assumption, the solutions of (1.5) lie in the union of the intervals $I_0 = [A, (4M)^{3rk\omega_2}]$ and $I_h = [M(\xi_h), M(\xi_h)^{3\omega_1/2}]$ $(h = 1, 2, \dots)$. By

part (i) of Lemma 2 and $4M \le A^\theta$ we have that the number of solutions ξ in I_0 is at most

$$\begin{aligned} t\Big(1+\frac{\log(2\log\{(4M)^{3rk\omega_2}\}/\log A)}{\log\theta}\Big) &\le t\Big(1+\frac{\log\{6rk\omega_2\theta\}}{\log\theta}\Big)\\ &\le t\Big(2+\frac{\log 6rk}{\log\theta}+k\frac{\log 3\omega_1/2}{\log\theta}\Big)\\ &\le t\Big(2+k\frac{\log 3\omega_1}{\log\theta}\Big) \ \text{ by (5.1), (5.2), (6.4).} \end{aligned}$$

Moreover, by part (i) of Lemma 2 we have for $h = 1, 2, \ldots$, that the number of solutions in I_h is at most

$$t\Big(1+\frac{\log(2\log\{M(\xi_h)^{3\omega_1/2}\}/\log M(\xi_h))}{\log\theta}\Big) \le t\Big(1+\frac{\log 3\omega_1}{\log\theta}\Big)\ .$$

Since we have at most $k-1$ intervals I_h $(h \ge 1)$, it follows that (1.5) has at most

$$N := t\Big(k+1+(2k-1)\frac{\log 3\omega_1}{\log\theta}\Big)$$

solutions with $M(\xi) \ge A$. We estimate this from above. From (1.7) it follows that $\kappa = \sum_{l=1}^{t}\varphi_l \ge 2t+\delta$ so

$$\log\theta \ge \log(1+\frac{\delta}{t}) \ge \frac{\delta}{2t}.$$

Further,

$$\begin{aligned} &\log 3\omega_1 = \log\frac{6k^2t}{\epsilon} \ \text{ by (6.4)}\\ &\le \log\Big(2.5\times10^{11}\cdot t^{11}\delta^{-5}\big(1+\frac{1}{2}\log t\big)^2\big(1+\frac{1}{2}\log\delta^{-1}\big)^2(\log 4r)^2\Big) \ \text{ by (5.1), (5.2)}\\ &< 27+12\log t+6\log\delta^{-1}+2\log\log 4r \quad \text{using } \big(1+\frac{1}{2}\log x\big)^2 \le x \text{ for } x\ge 1\\ &< 85\big(1+\frac{1}{2}\log t\big)\big(1+\frac{1}{2}\log\delta^{-1}\big)\cdot\log\log 4r \quad \text{using } \log\log 4r \ge \log\log 4. \end{aligned}$$

Together with (5.2) this implies

$$\begin{aligned} N &\le kt\cdot(1+\frac{4t}{\delta}\log 3\omega_1) < 5kt^2\delta^{-1}\log 3\omega_1\\ &< 5\times 3.5\times10^4\times85\cdot t^6\big(1+\frac{1}{2}\log t\big)^2\cdot\delta^{-3}\big(1+\frac{1}{2}\log\delta^{-1}\big)^2\log 4r\log\log 4r\\ &< 2\times10^7\cdot t^7\delta^{-4}\log 4r\cdot\log\log 4r\ . \end{aligned}$$

This completes the proof of part (i) of Theorem 2. □

§7. Proof of Theorem 3.

We need the following combinatorial lemma:

Lemma 14. *Let θ be a real with $0 < \theta < 1$ and t an integer ≥ 1. There exists a set P, consisting of tuples $\underline{\rho} = (\rho_1, \ldots, \rho_t)$ with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_t \geq 0$ and $1 - \theta \leq \sum_{i=1}^{t} \rho_i \leq 1$, such that $\#P \leq 4\{e^2(\frac{1}{2} + \frac{1+\theta^{-1}}{t})\}^{t-1}$ and such that for all reals $F_1, \ldots, F_t, \Lambda$ with*

$$0 < F_1 \leq F_2 \leq \cdots \leq F_t \leq 1, \quad F_1 \cdots F_t \leq \Lambda$$

there is a tuple $\underline{\rho} \in P$ with $F_i \leq \Lambda^{\rho_i}$ for $i = 1, \ldots, t$.

Proof. We assume without loss of generality that $F_1 \cdots F_t = \Lambda$ and that $t \geq 2$ (otherwise we may take $\rho_1 = 1$). Define c_i by $F_i = \Lambda^{c_i}$ for $i = 1, \ldots, t$; thus, $c_1 \geq \cdots \geq c_t \geq 0$ and $c_1 + \cdots + c_t = 1$. Put

$$g := [\theta^{-1}(t-1)] + 1, \quad f_i = [c_i g], \quad \rho_i = f_i/g \ \text{ for } i = 1, \ldots, t\ .$$

Then clearly, $F_i \leq \Lambda^{\rho_i}$ for $i = 1, \ldots, t$. Since $c_i g - 1 < f_i \leq c_i g$, we have $g - t < \sum_{i=1}^{t} f_i \leq g$ and therefore, $g - t + 1 \leq \sum_{i=1}^{t} f_i \leq g$ since the f_i are integers. It follows that $1 - \theta \leq \sum_{i=1}^{t} \rho_i \leq 1$. Further, the tuple $\underline{\rho} = (\rho_1, \ldots, \rho_t)$ belongs to the set

$$P := \Big\{ (\frac{f_1}{g}, \ldots, \frac{f_t}{g}) : \ f_1, \ldots, f_t \in \mathbb{Z},\ f_1 \geq \cdots \geq f_t \geq 0,\ g - t + 1 \leq \sum_{i=1}^{t} f_i \leq g \Big\}\ .$$

The map $(f_1/g, \ldots, f_t/g) \mapsto (f_1 + t - 1, f_2 + t - 2, \ldots, f_t)$ maps P bijectively onto

$$P' := \Big\{ (h_1, \ldots, h_t) \in \mathbb{Z}^t : \ h_1 > h_2 > \cdots > h_t \geq 0,\ \ g' - t + 1 \leq \sum_{i=1}^{t} h_i \leq g' \Big\}\ ,$$

with $g' = g + \frac{1}{2}t(t-1) = [\theta^{-1}(t-1)] + \frac{1}{2}t(t-1) + 1$. Clearly, the cardinality of P' is at most $1/t!$ times the cardinality of the set of all (not necessarily decreasing) tuples of non-negative integers $(h_1, \ldots, h_t)$ with $g' - t + 1 \leq \sum_{i=1}^{t} h_i \leq g'$. Using that

$$\binom{x+y}{y} \leq \frac{(x+y)^{x+y}}{x^x y^y} = (1 + \frac{y}{x})^x (1 + \frac{x}{y})^y \leq \big(e(1 + \frac{x}{y})\big)^y \ \text{ for } x, y \geq 1$$

we infer

$$\begin{aligned}
\#P = \#P' &\le \frac{1}{t!} \sum_{h=g'-t+1}^{g'} \binom{h+t-1}{t-1} \le \frac{1}{t!} \cdot t \cdot \binom{g'+t-1}{t-1} \\
&\le \frac{e^t}{t^{t-1}} \cdot \Big(e\big(1+\theta^{-1}+\frac{t}{2}+\frac{1}{t-1}\big)\Big)^{t-1} \\
&\le \frac{e^t}{t^{t-1}} \cdot \Big(e\big(1+\theta^{-1}+\frac{t}{2}\big)\Big)^{t-1} \cdot \Big(1+\frac{1}{3(t-1)}\Big)^{t-1} \quad \text{since } t \ge 2,\ \theta < 1 \\
&\le 4 \cdot \Big(e^2\big(\frac{1}{2}+\frac{1+\theta^{-1}}{t}\big)\Big)^{t-1} \ . \qquad \square
\end{aligned}$$

Let f be the polynomial from Theorem 3, i.e.

$$f(X) = a_0(X-\alpha_1)\cdots(X-\alpha_r)$$

where the coefficients of f are rational integers, f is primitive, and $\alpha_1, \ldots, \alpha_r$ are distinct. Further, let g be a primitive, irreducible polynomial in $\mathbb{Z}[X]$ of degree t satisfying

$$0 < |R(f,g)| < M(f)^t \cdot M(g)^{r-\kappa}, \tag{1.17}$$

$$M(g) \ge (2^{8r^2t} M(f)^{4(r-1)t})^{\delta^{-1}(1+\frac{1}{3}+\cdots+\frac{1}{2t-1})^{-1}} \tag{1.18}$$

where

$$\kappa = (2t+\delta)\big(1+\frac{1}{3}+\cdots+\frac{1}{2t-1}\big) \quad \text{with} \ \ 0 < \delta < 1 \ . \tag{1.16}$$

Then

$$g(X) = b_0(X-\xi^{(1)})\cdots(X-\xi^{(t)})$$

where $\xi^{(1)}, \ldots, \xi^{(t)}$ are the conjugates of an algebraic number ξ of degree t and $b_0 \in \mathbb{Z}$. We order $\xi^{(1)}, \ldots, \xi^{(t)}$ in such a way that

$$\min_{j=1,\ldots,r} \frac{|\alpha_j - \xi^{(1)}|}{|1,\alpha_j|} \le \cdots \le \min_{j=1,\ldots,r} \frac{|\alpha_j - \xi^{(t)}|}{|1,\alpha_j|} \ . \tag{7.1}$$

We show that ξ satisfies one from a finite collection of systems (1.5) to which Theorem 2 is applicable. From (1.12) it follows that

$$\frac{|R(f,g)|}{M(f)^t M(g)^r} = \prod_{i=1}^{t} \prod_{j=1}^{r} \frac{|\alpha_j - \xi^{(i)}|}{|1,\alpha_j| \cdot |1,\xi^{(i)}|} \ . \tag{7.2}$$

For $i = 1, \dots, t$, let α_{j_i} be the zero of f for which

$$\frac{|\alpha_{j_i} - \xi^{(i)}|}{|1, \alpha_{j_i}|} = \min_{j=1,\dots,r} \frac{|\alpha_j - \xi^{(i)}|}{|1, \alpha_j|} \ .$$

From triangle inequality (2.7) it follows that for $j \neq j_i$,

$$\frac{|\alpha_j - \xi^{(i)}|}{|1, \alpha_j| \cdot |1, \xi^{(i)}|} \geq \frac{1}{2}\Big(\frac{|\alpha_j - \xi^{(i)}|}{|1, \alpha_j| \cdot |1, \xi^{(i)}|} + \frac{|\alpha_{j_i} - \xi^{(i)}|}{|1, \alpha_{j_i}| \cdot |1, \xi^{(i)}|}\Big) \geq \frac{|\alpha_{j_i} - \alpha_j|}{2|1, \alpha_{j_i}| \cdot |1, \alpha_j|} \ .$$

Further, using that the discriminant $D(f) = a_0^{2r-2} \prod_{1 \leq p < q \leq r} (\alpha_p - \alpha_q)^2$ is a non-zero rational integer,

$$\begin{aligned} \prod_{j \neq j_i} \frac{|\alpha_{j_i} - \alpha_j|}{2|1, \alpha_{j_i}| \cdot |1, \alpha_j|} &\geq \prod_{1 \leq p < q \leq r} \frac{|\alpha_p - \alpha_q|}{2|1, \alpha_p| \cdot |1, \alpha_q|} = \frac{|D(f)|^{1/2}}{2^{r(r-1)/2} M(f)^{r-1}} \\ &\geq \frac{1}{2^{r(r-1)/2} M(f)^{r-1}} \ . \end{aligned}$$

Together with (7.2) this implies that

$$\frac{|R(f,g)|}{M(f)^t M(g)^r} \geq C^{-1} \prod_{i=1}^{t} \frac{|\alpha_{j_i} - \xi^{(i)}|}{2|1, \alpha_{j_i}| \cdot |1, \xi^{(i)}|} \ , \quad \text{with } C = \Big(2^{1+r(r-1)/2} M(f)^{r-1}\Big)^t \ . \tag{7.3}$$

Put $\kappa' := (1 + \frac{1}{3} + \cdots + \frac{1}{2t-1})(2t + \frac{3}{4}\delta)$. From (1.18) it follows that $M(g) \geq C^{(\kappa - \kappa')^{-1}}$. By combining this with (7.3), (1.17) and using that $M(g) = M(\xi)$ we get

$$\prod_{i=1}^{t} \frac{|\alpha_{j_i} - \xi^{(i)}|}{2|1, \alpha_{j_i}| \cdot |1, \xi^{(i)}|} \leq C \cdot M(g)^{-\kappa} \leq M(\xi)^{-\kappa'} \ .$$

We now apply Lemma 14 to $F_j := |\alpha_{j_i} - \xi^{(i)}|/(2 \cdot |1, \alpha_{j_i}| \cdot |1, \xi^{(i)}|)$ for $j = 1, \dots, t$ and $\Lambda = M(\xi)^{-\kappa'}$. It is trivial that $F_t \leq 1$ and together with (7.1) this gives $0 < F_1 \leq \cdots \leq F_t \leq 1$. Put

$$\kappa'' := (1 + \frac{1}{3} + \cdots + \frac{1}{2t-1})(2t + \frac{1}{2}\delta), \quad \theta := 1 - \kappa''/\kappa' = \delta/(8t + 3\delta) \ . \tag{7.4}$$

Letting P be the set from Lemma 14, we infer that there is a tuple $\underline{\rho} = (\rho_1, \dots, \rho_t) \in P$ such that

$$\frac{|\alpha_{j_i} - \xi^{(i)}|}{2|1, \alpha_{j_i}| \cdot |1, \xi^{(i)}|} \leq M(\xi)^{-\rho_i \kappa'} = M(\xi)^{-\varphi_i} \ \text{ for } i = 1, \dots, t, \tag{7.5}$$

where $\varphi_i := \rho_i \kappa'$. Note that $\sum_{i=1}^{t} \varphi_i \geq \kappa''$. Together with (1.7) this implies

$$\max_I (\#I)^2 \big(\sum_{i \in I} \varphi_i^{-1} \big)^{-1} \geq \big(\sum_{j=1}^{t} \frac{1}{2j-1} \big)^{-1} \cdot \kappa'' = 2t + \frac{\delta}{2} \ .$$

Further, we have

$$\begin{aligned} M(f) &\geq \max_{i=1,\ldots,r} M(\alpha_i), \quad [\mathbb{Q}(\alpha_{j_1}, \ldots, \alpha_{j_t}) : \mathbb{Q}] \leq r^t, \\ M(\xi) = M(g) &\geq \max(4^{t(t+1)/(\kappa''-2t)}, M(f)) \ \text{ by (1.18).} \end{aligned}$$

Hence from part (i) of Theorem 2 with $\delta/2$ replacing δ it follows that each system (7.5) has at most $3.2 \times 10^8 t^7 \delta^{-4} \log 4r^t \log\log 4r^t$ solutions ξ coming from an irreducible polynomial g satisfying (1.17), (1.18).

By (7.4) we have

$$\begin{aligned} \#P &\leq 4\Big(e^2 \big(\frac{1}{2} + \frac{1+\theta^{-1}}{t} \big) \Big)^{t-1} = 4\Big(e^2 \big(\frac{1}{2} + \frac{8}{\delta} + \frac{4}{t} \big) \Big)^{t-1} \\ &\leq 4\Big(e^2 \big(\frac{1}{2} + \frac{8}{\delta} \big) \Big)^{t-1} \Big(1 + \frac{1}{2t} \Big)^{t-1} \leq 7\big(63\delta^{-1} \big)^{t-1} \ . \end{aligned}$$

Further, for the tuple $(j_1, \ldots, j_t)$ we have at most r^t possiblilities. Therefore, we have at most $7r^t(63\delta^{-1})^{t-1}$ possibilities for the system (7.5). We conclude that the total number of primitive, irreducible polynomials g satisfying (1.17), (1.18) is at most

$$\begin{aligned} &7r^t(63\delta^{-1})^{t-1} \cdot 3.2 \times 10^8 t^7 \delta^{-4} \log 4r^t \log\log 4r^t \\ &\qquad \leq 10^{15} (\delta^{-1})^{t+3} \cdot (100r)^t \log 4r \log\log 4r \ . \end{aligned}$$

This completes the proof of Theorem 3. □

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