SYMMETRIC IMPROVEMENTS OF LIOUVILLE'S INEQUALITY

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Abstract. Let K_1, K_2 be finite extensions of a number field K. For every place w of the composite K_1K_2 we choose a normalised absolute value $|\cdot|_w$ such that the product formula is satisfied. Define the height $H(\alpha) = \prod_w \max(1, |\alpha|_w)$ for $\alpha \in K_1K_2$. Let T be a finite set of places of K_1K_2. Liouville's inequality states that $\prod_{w\in T} |\alpha - \beta|_w \gg \big(H(\alpha)H(\beta)\big)^{-1}$ for α, $\beta \in K_1K_2$ with $\alpha \neq \beta$. We consider inequalities (*) $\prod_{w\in T} |\alpha-\beta|_w \leq \big(H(\alpha)H(\beta)\big)^{-1+\kappa}$ in two unknowns α, β with $K(\alpha) = K_1$, $K(\beta) = K_2$ where $\kappa > 0$. Under certain conditions imposed on K_1, K_2 (i.e., $[K_1 : K] \geq 3$, $[K_2 : K] \geq 3$, $[K_1K_2 : K] = [K_1 : K][K_2 : K]$) we shall describe the collection of sets of places T for which there is a $\kappa > 0$ such that (*) has only finitely many solutions. Our proof goes back to the p-adic Subspace theorem.

1. Introduction.

We have to start with introducing normalised absolute values and heights. Let L be any algebraic number field and M_L its set of places. Denote by L_w the completion of L at a place $w \in M_L$. The set of normalised absolute values $|\cdot|_w$ ($w \in M_L$) on L is defined by requiring

$|x|_w = |x|^{[L_w:\mathbf{R}]/[L:\mathbf{Q}]}$ for $x \in \mathbf{Q}$ if w is archimedean;
$|x|_w = |x|_p^{[L_w:\mathbf{Q}_p]/[L:\mathbf{Q}]}$ for $x \in \mathbf{Q}$ if w lies above the prime number p.

Here $|\cdot|_p$ is the p-adic absolute value with $|p|_p = p^{-1}$. The normalised absolute values satisfy the product formula

$$\prod_{w\in M_L} |x|_w = 1 \quad \text{for } x \in L\backslash\{0\}.$$

Given any other number field K, the set of normalised absolute values $|\cdot|_v$ ($v \in M_K$) on K is defined precisely as for L. Thus, we get for every finite extension of number fields

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L/K and every pair $v \in M_K$, $w \in M_L$ with w lying above v the extension formula

$$|x|_w = |x|_v^{[L_w:K_v]/[L:K]} \quad \text{for } x \in K, \tag{1.1}$$

where K_v denotes the completion of K at v. We define the absolute height of an algebraic number x by taking any number field L with $x \in L$ and putting

$$H(x) := \prod_{w \in M_L} \max(1, |x|_w)\,.$$

By our choice of the normalised absolute values with $[L : \mathbf{Q}]$ in the denominators of the exponents, this quantity is independent of the choice of L.

In what follows, K is an algebraic number field, K_1, K_2 are finite extensions of K and K_1K_2 is their composite. Let T be a finite set of places of K_1K_2. We deal with numbers α, β with $K(\alpha) = K_1$, $K(\beta) = K_2$ and $\alpha \neq \beta$. An immediate consequence of the product formula is the following generalisation of Liouville's inequality:

$$\begin{aligned}\prod_{w \in T} |\alpha - \beta|_w &\geq \prod_{w \in T} \frac{|\alpha - \beta|_w}{\max(1, |\alpha|_w)\max(1, |\beta|_w)} \\ &= H(\alpha)^{-1}H(\beta)^{-1} \cdot \prod_{w \notin T} \frac{\max(1, |\alpha|_w)\max(1, |\beta|_w)}{|\alpha - \beta|_w} \\ &\geq \frac{1}{2}H(\alpha)^{-1}H(\beta)^{-1}. \end{aligned} \tag{1.2}$$

In certain situations it is possible to improve upon the exponents of either $H(\alpha)$ or $H(\beta)$ or both if the degrees $[K_1K_2 : K_1]$ or $[K_1K_2 : K_2]$ are sufficiently large. For instance, if $r := [K_1K_2 : K_2] \geq 3$, then from S. Lang's version of Roth's theorem (cf. [9], Chap. 7) it follows that for every fixed α with $K(\alpha) = K_1$ and for every $\delta > 0$, there are only finitely many β with

$$\prod_{w \in T} |\alpha - \beta|_w \leq H(\beta)^{-(2/r)-\delta}\,, \quad K(\beta) = K_2\,. \tag{1.3}$$

(In Lang's statement there is an exponent -2 since he uses absolute values normalised with respect to K_2 whereas our absolute values are normalised with respect to K_1K_2.) This may be viewed as a one-sided improvement of Liouville's inequality since for every fixed α, we have that for all but finitely many β the right-hand side of (1.2) can be replaced by a power of $H(\beta)$ with exponent larger than -1.

We are interested in so-called symmetric improvements of Liouville's inequality, in which we allow α to vary through K_1 and β through K_2 and in which both the exponents on $H(\alpha)$ and $H(\beta)$ are larger than -1. More precisely, we consider inequalities

$$\prod_{w\in T} |\alpha-\beta|_w \le \big(H(\alpha)H(\beta)\big)^{-1+\kappa} \qquad \text{in } \alpha,\beta \text{ with } K(\alpha)=K_1,\, K(\beta)=K_2, \tag{1.4}$$

with $\kappa > 0$. Any result stating that such an inequality has only finitely many solutions is called a symmetric improvement of Liouville's inequality. We should mention here that from results of Bombieri and van der Poorten [1], Corvaja [3] (Thm. 2) and Vojta [13] it follows that there is a real function f with $f(x) = o(x)$ for $x \to \infty$ such that (1.3) has only finitely many solutions (α, β) with $K(\alpha) = K_1$, $K(\beta) = K_2$ and $H(\alpha) \le f(H(\beta))$. We are interested in the truly symmetric situation in which we do not require the height of one of the numbers $H(\alpha)$, $H(\beta)$ to be much larger than the other.

We recall a symmetric improvement of Liouville's inequality from [6]. Assume

$$[K_1K_2 : K_1] \ge 3\,, \quad [K_1K_2 : K_2] \ge 3\,, \tag{1.5}$$

$$[K_1K_2 : K] = [K_1 : K]\cdot[K_2 : K]\,. \tag{1.6}$$

For instance, for fixed α, Roth's theorem stated above yields a one-sided improvement of Liouville's inequality in terms of $H(\beta)$ only if $[K_1K_2 : K_2] \ge 3$. So in our symmetric situation it is natural to assume (1.5). Condition (1.6) does not seem to be natural but it is essential for the proof.

Denote by S the set of places of K lying below the places in T and write

$$T = \bigcup_{v\in S} T_v\,,$$

where T_v is the set of places in T lying above v. Define

$$W_T := \max_{v\in S} \sum_{w\in T_v} \frac{[(K_1K_2)_w : K_v]}{[K_1K_2 : K]}$$

where $(K_1K_2)_w$ denotes the completion of K_1K_2 at w. Note that always $W_T \le 1$ and that $W_T = 1$ precisely if there is a $v \in S$ such that T_v contains all places of K_1K_2 lying above v. In [6] (Thm. 4) we showed that if

$$W_T < \frac{1}{3}, \qquad \kappa \le \frac{1}{718}\cdot\frac{1-3W_T}{1+3W_T}$$

then (1.4) has only finitely many solutions. On the other hand we showed that if W_T assumes the maximal value 1 then for all $\kappa > 0$ (1.4) has infinitely many solutions.

The result just mentioned does not deal with sets of places T with $\frac{1}{3} \leq W_T < 1$. The purpose of this paper is to fill this gap, i.e., to give a precise description of those sets of places T of K_1K_2 for which there exists a $\kappa > 0$ such that (1.4) has only finitely many solutions.

We continue with the notation introduced above. We will always denote by v a place of K, by w a place of K_1K_2, and by q_i a place of K_i, for $i = 1, 2$. The completion of K_i at q_i is denoted by $(K_i)_{q_i}$. Thus, if w lies above v, then w lies above places q_1 of K_1 and q_2 of K_2 which in turn lie above v.

For the fields K_1, K_2 we assume again (1.5), (1.6) or, equivalently,

$$r := [K_1 : K] \geq 3,\ s := [K_2 : K] \geq 3, \quad [K_1K_2 : K] = [K_1 : K][K_2 : K] = rs\,. \qquad (1.7)$$

Again, condition (1.6) is unnatural but necessary for the proof.

As before, T is a finite set of places of K_1K_2 and we write $T = \cup_{v\in S} T_v$, where S consists of places of K and for $v \in S$, T_v consists of the places in T lying above v. Theorem 1.1 below states in a precise way that there exists a $\kappa > 0$ for which (1.4) has only finitely many solutions if and only if none of the sets T_v ($v \in S$) is "too large." For $v \in S$, let T_v^c denote the set of places of K_1K_2 which lie above v and do not belong to T_v. Then T_v is "too large" or, which is the same, T_v^c is "too small" if

$$\left.\begin{array}{l} \text{either } T_v^c = \emptyset; \\ \text{or there is a place } q_1 \text{ of } K_1 \text{ with } (K_1)_{q_1} = K_v \text{ such that all places in } T_v^c \\ \text{lie above } q_1; \\ \text{or there is a place } q_2 \text{ of } K_2 \text{ with } (K_2)_{q_2} = K_v \text{ such that all places in } T_v^c \\ \text{lie above } q_2. \end{array}\right\} \qquad (1.8)$$

Theorem 1.1. *Assume (1.7). Consider the inequality*

$$\prod_{w\in T} |\alpha-\beta|_w \le \big(H(\alpha)H(\beta)\big)^{-1+\kappa} \qquad \text{in } \alpha,\beta \text{ with } K(\alpha)=K_1,\ K(\beta)=K_2. \tag{1.4}$$

(i). Suppose there is some $v\in S$ for which (1.8) holds. Then for every $\kappa>0$, inequality (1.4) has infinitely many solutions.
(ii). Suppose there is no $v\in S$ for which (1.8) holds. Then for every

$$\kappa \le \frac{1}{718(r+s)^2}$$

inequality (1.4) has only finitely many solutions.

From Theorem 1.1 we derive the following corollary:

Corollary 1.2. *Assume (1.7). For a finite set T of places of K_1K_2, put*

$$W_T := \max_{v\in S} \sum_{w\in T_v} \frac{[(K_1K_2)_w : K_v]}{[K_1K_2 : K]},$$

where S is the set of places of K lying below those in T and T_v is the set of places in T lying above v for $v\in S$.
(i). If $W_T < 1-\max(\frac{1}{r},\frac{1}{s})$ then for every $\kappa \le \frac{1}{718(r+s)^2}$, inequality (1.4) has only finitely many solutions.
(ii). There are finite sets T of places of K_1K_2 with $W_T = 1-\max(\frac{1}{r},\frac{1}{s})$ such that for every $\kappa>0$, inequality (1.4) has infinitely many solutions.

The constant $\frac{1}{718(r+s)^2}$ in part (ii) of Theorem 1.1 just arises from the proof and has no special meaning. Very likely, its dependence on r and s is not best possible. We considered only the problem to prove the existence of some $\kappa>0$ for which (1.4) has only finitely many solutions. We have not done any attempt to obtain the best possible value for κ. It would be very interesting to determine, for a given set of places T, the infimum of the functions Ψ such that the inequality

$$\prod_{w\in T} |\alpha-\beta|_w \le \Psi(H(\alpha),H(\beta))^{-1} \quad \text{in } \alpha,\, \beta \text{ with } K(\alpha)=K_1,\ K(\beta)=K_2$$

has only finitely many solutions. It is plausible that this infimum is the smallest if all sets T_v are small and that it grows larger if one of the sets T_v is made larger. As yet, we are not able to pose a precise conjecture.

We deduce Theorem 1.1 from a slightly more general result. Let K be an algebraic number field and $|\cdot|_v$ ($v \in M_K$) its set of normalised absolute values. Fix an algebraic closure $\overline{K}$ of K and assume that all algebraic extensions of K occurring henceforth are contained in $\overline{K}$. For every $v \in M_K$, we fix an algebraic closure $\overline{K}_v$ of K_v. To be formally correct, we have to choose an isomorphic embedding $\rho : K \hookrightarrow \overline{K}$, and for $v \in M_K$ we have to choose isomorphic embeddings $\sigma_v : K \hookrightarrow K_v$, $\phi_v : K_v \hookrightarrow \overline{K}_v$, $\psi_v : \overline{K} \hookrightarrow \overline{K}_v$ such that $\psi_v \rho = \phi_v \sigma_v$. By identifying elements of $K, \overline{K}, K_v$ with their isomorphic images we can dispose of the isomorphic embeddings and we get for every $v \in M_K$ inclusions $K \subset \overline{K} \subset \overline{K}_v$, $K \subset K_v \subset \overline{K}_v$. For every $v \in M_K$ there is a unique extension of $|\cdot|_v$ to $\overline{K}_v$ which we denote also by $|\cdot|_v$. Note that $|\cdot|_v$ is defined on $\overline{K}$.

Let again K_1, K_2 be extensions of K of degrees r, s, respectively. We denote by $\alpha \mapsto \alpha^{(i)}$ ($i = 1, \dots, r$) the K-isomorphic embeddings of K_1 into $\overline{K}$ and by $\beta \mapsto \beta^{(j)}$ ($j = 1, \dots, s$) the K-isomorphic embeddings of K_2 into $\overline{K}$. Further, let S be a finite set of places of K. Take subsets

$$\mathcal{E}_v \subset \{(i,j) : i = 1, \dots, r,\ j = 1, \dots, s\} \quad (v \in S).$$

Liouville's inequality can be rephrased as

$$\prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v \geq 2^{-rs} \big(H(\alpha)H(\beta)\big)^{-rs}$$

for algebraic numbers α, β with $K(\alpha) = K_1$, $K(\beta) = K_2$ and α, β non-conjugate over K. We consider inequalities

$$\prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v \leq \big(H(\alpha)H(\beta)\big)^{-rs(1-\kappa)}$$
$$\text{in } \alpha,\ \beta \text{ with } K(\alpha) = K_1,\ K(\beta) = K_2\,, \tag{1.9}$$

with $\kappa > 0$.

We view $\{(i,j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ as an $r \times s$-matrix of which the rows are indexed by i and the columns by j. By a K_v-row we mean a subset $\{(i,1), \dots, (i,s)\}$ such that the map $\alpha \mapsto \alpha^{(i)}$ maps K_1 into K_v. By a K_v-column we mean a subset $\{(1,j), \dots, (r,j)\}$ such that $\beta \mapsto \beta^{(j)}$ maps K_2 into K_v. For $v \in S$, let $\mathcal{E}_v^c$ denote the set of pairs from $\{(i,j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ not belonging to $\mathcal{E}_v$. We prove the following:

Theorem 1.3. *Assume*

$$[K_1 : K] = r \geq 3,\ [K_2 : K] = s \geq 3, \quad K_1,\ K_2 \text{ are non-conjugate over } K. \tag{1.10}$$

(i). Suppose there is a $v \in S$ for which either $\mathcal{E}_v^c = \emptyset$ or $\mathcal{E}_v^c$ is contained in a K_v-row or $\mathcal{E}_v^c$ is contained in a K_v-column. Then for every $\kappa > 0$, (1.9) has infinitely many solutions.
(ii). Suppose for each $v \in S$ we have that $\mathcal{E}_v^c \neq \emptyset$, that $\mathcal{E}_v^c$ is not contained in a K_v-row and that $\mathcal{E}_v^c$ is not contained in a K_v-column. Then for every

$$\kappa \leq \frac{1}{718(r+s)^2}$$

inequality (1.9) has only finitely many solutions.

We consider the special case that $K = \mathbf{Q}$ and $S = \{\infty\}$ consists of the infinite place of $\mathbf{Q}$. To agree with the classical notation, we define the Mahler measure $M(\alpha) = H(\alpha)^{\deg(\alpha)}$ for an algebraic number α. Thus, writing $\mathcal{E}$ for $\mathcal{E}_\infty$, (1.9) becomes

$$\prod_{(i,j)\in\mathcal{E}} |\alpha^{(i)} - \beta^{(j)}| \leq \big(M(\alpha)^s M(\beta)^r\big)^{-1+\kappa}$$
$$\text{in } \alpha,\, \beta \text{ with } \mathbf{Q}(\alpha) = K_1,\ \mathbf{Q}(\beta) = K_2\,. \tag{1.11}$$

Note that in this situation, $K_v = \mathbf{R}$ and that for instance an $\mathbf{R}$-row is a set $\{(i,1),\ldots,(i,s)\}$ such that $\alpha \mapsto \alpha^{(i)}$ maps K_1 into $\mathbf{R}$. From Theorem 1.3 we obtain at once the following result which has been stated without proof already in [7]:

Corollary 1.4. *Assume that K_1, K_2 have degrees $r \geq 3$, $s \geq 3$, respectively, over $\mathbf{Q}$ and that K_1, K_2 are non-conjugate over $\mathbf{Q}$.*
If either $\mathcal{E}^c = \emptyset$, or $\mathcal{E}^c$ is contained in an $\mathbf{R}$-row or $\mathcal{E}^c$ is contained in an $\mathbf{R}$-column, then for every $\kappa > 0$, inequality (1.11) has infinitely many solutions.
If on the other hand, $\mathcal{E}^c \neq \emptyset$, $\mathcal{E}^c$ is not contained in an $\mathbf{R}$-row and $\mathcal{E}^c$ is not contained in an $\mathbf{R}$-column, then for every $\kappa \leq \frac{1}{718(r+s)^2}$, inequality (1.11) has only finitely many solutions.

In Section 2 we deduce Theorem 1.1 from Theorem 1.3 and Corollary 1.2 from Theorem 1.1. In the proof of part (i) of Theorem 1.3 we show more precisely, using the p-adic Subspace theorem, that for every pair α_0, β_0 with $K(\alpha_0) = K_1$, $K(\beta_0) = K_2$, there exist infinitely many elements α, β of the form $\alpha = \frac{a\alpha_0+c}{b\alpha_0+d}$, $\beta = \frac{a\beta_0+c}{b\beta_0+d}$ with $a,b,c,d \in K$, $ad-bc \neq 0$, such that (α,β) is a solution of (1.9). The proof of part (ii) uses an (ineffective) lower bound for resultants obtained in [6] which in turn was a consequence of the p-adic Subspace theorem. In Section 3 we introduce some notation. Part (i) is proved in Sections 4 and 5 and part (ii) in Sections 6 and 7.

2. Deduction of Theorem 1.1 and Corollary 1.2.

We deduce Theorem 1.1 from Theorem 1.3 and then Corollary 1.2 from Theorem 1.1. We start with some generalities.

As before, K is a number field. Recall that for every place (equivalence class of absolute values) $v \in M_K$ we have inclusions $K \subset \overline{K} \subset \overline{K}_v$, $K \subset K_v \subset \overline{K}_v$. Further, $|\cdot|_v$ has been extended to $\overline{K}_v$, hence is defined also on $\overline{K}$. We need that numbers γ, $\delta \in \overline{K}_v$ which are conjugate over K_v (i.e., $\delta = \sigma(\gamma)$ for some K_v-invariant isomorphism σ) have $|\gamma|_v = |\delta|_v$.

Let L be a finite extension of K. Denote by $\gamma \mapsto \gamma^{(k)}$ $(k = 1, \dots, t)$ the K-isomorphic embeddings of L into $\overline{K}$. For a place q of L, denote by L_q the completion of L at q. Fix $v \in M_K$ and partition $\{1, \dots, t\}$ into subsets such that k_1, k_2 belong to the same subset if and only if for every $\gamma \in L$, $\gamma^{(k_1)}, \gamma^{(k_2)}$ are conjugate over K_v. For the indices k in a given subset, the absolute values given by $|\gamma^{(k)}|_v$ for $\gamma \in L$ are equal and are all extensions of the absolute value $|\cdot|_v$ on K and therefore represent a place q of L lying above v. In this way, we obtain all places of L lying above v. Thus, $\{1, \dots, t\} = \bigcup_{q|v} \mathcal{F}(q|v)$, where $\mathcal{F}(q|v)$ consists of the indices k such that the absolute value given by $|\gamma^{(k)}|_v$ for $\gamma \in L$ represents q and where the union is taken over all places q of L lying above v.

For γ with $K(\gamma) = L$, the fields $K_v(\gamma^{(k)})$ $(k \in \mathcal{F}(q|v))$ are the isomorphic images of L_q in $\overline{K}_v$. Hence $\mathcal{F}(q|v)$ has cardinality $[L_q : K_v]$. In particular, $L_q = K_v$ if and only if $\mathcal{F}(q|v) = \{k\}$ for some k such that $\gamma \mapsto \gamma^{(k)}$ maps L into K_v. By (1.1) we have for the normalised absolute value on L corresponding to q, $|\gamma|_q = |\gamma^{(k)}|_v^{[L_q:K_v]/[L:K]}$ for $\gamma \in L$, $k \in \mathcal{F}(q|v)$.

Proof of Theorem 1.1. Let K_1, K_2 be finite extensions of K satisfying (1.7). Then certainly they satisfy condition (1.10) of Theorem 1.3. As before, by $\alpha \mapsto \alpha^{(i)}$ $(i = 1, \dots, r)$ we denote the K-isomorphic embeddings of K_1 into $\overline{K}$ and by $\beta \mapsto \beta^{(j)}$ $(j = 1, \dots, s)$ those of K_2 into $\overline{K}$.

Take $v \in S$. As we explained above, the set $\{1, \dots, r\}$ can be partitioned into sets $\mathcal{F}(q_1|v)$, one for each place q_1 of K_1 lying above v, such that for $i \in \mathcal{F}(q_1|v)$ the absolute values given by $|\alpha^{(i)}|_v$ for $\alpha \in K_1$ represent q_1. There is a similar partition of $\{1, \dots, s\}$ into sets $\mathcal{F}(q_2|v)$, one for each place q_2 on K_2 lying above v.

Because of (1.7), there are precisely rs K-isomorphic embeddings of K_1K_2 into $\overline{K}$ and these are given by σ_{ij}: $\alpha \mapsto \alpha^{(i)}$, $\beta \mapsto \beta^{(j)}$ for $\alpha \in K_1$, $\beta \in K_2$ ($i = 1, \ldots, r$, $j = 1, \ldots, s$). Similarly as above, the set $\{(i,j) : i = 1, \ldots, r,\ j = 1, \ldots, s\}$ can be partitioned into sets $\mathcal{F}(w|v)$, one for each place w of K_1K_2 lying above v, such that the absolute values given by $|\sigma_{ij}(\gamma)|_v$ for $\gamma \in K_1K_2$ ($(i,j) \in \mathcal{F}(w|v)$) represent w. We observed above that $\mathcal{F}(w|v)$ has cardinality $[(K_1K_2)_w : K_v]$. Further, by (1.1), (1.7) we have $|\gamma|_w = |\sigma_{ij}(\gamma)|_w^{[(K_1K_2)_w:K_v]/rs}$ for $\gamma \in K_1K_2$, $(i,j) \in \mathcal{F}(w|v)$. Hence $|\gamma|_w = \big(\prod_{(i,j)\in\mathcal{F}(w|v)} |\sigma_{ij}(\gamma)|_v\big)^{1/rs}$ for $\gamma \in K_1K_2$. In particular, we have

$$|\alpha - \beta|_w = \Big(\prod_{(i,j)\in\mathcal{F}(w|v)} |\alpha^{(i)} - \beta^{(j)}|_v\Big)^{1/rs} \quad \text{for } \alpha \in K_1,\ \beta \in K_2. \tag{2.1}$$

We keep the notation of Theorem 1.1. Put

$$\mathcal{E}_v := \bigcup_{w\in T_v} \mathcal{F}(w|v) \quad \text{for } v \in S. \tag{2.2}$$

From (2.1) it follows that for α, β with $K(\alpha) = K_1$, $K(\beta) = K_2$ we have

$$\prod_{w\in T} |\alpha - \beta|_w = \prod_{v\in S} \prod_{w\in T_v} |\alpha - \beta|_w = \prod_{v\in S} \prod_{(i,j)\in\mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v^{1/rs},$$

hence (α, β) is a solution of (1.4) if and only if it satisfies (1.9) with the sets $\mathcal{E}_v$ defined by (2.2).

We claim that (1.8) is equivalent to the condition on the sets $\mathcal{E}_v^c$ in part (i) of Theorem 1.3. Clearly, $\mathcal{E}_v^c = \cup_{w\in T_v^c}\mathcal{F}(w|v)$. So $\mathcal{E}_v^c = \emptyset$ if and only if $T_v^c = \emptyset$. In general, w lies above q_1 if and only if for each pair $(i,j) \in \mathcal{F}(w|v)$ we have $i \in \mathcal{F}(q_1|v)$. Hence $\cup_{w|q_1}\mathcal{F}(w|v) = \mathcal{F}(q_1|v) \times \{1, \ldots, s\}$, where the union is taken over all places w of K_1K_2 lying above q_1. We have $(K_1)_{q_1} = K_v$ if and only if $\mathcal{F}(q_1|v) = \{i\}$ for some i such that $\alpha \mapsto \alpha^{(i)}$ maps K_1 into K_v. Hence $(K_1)_{q_1} = K_v$ if and only if $\cup_{w|q_1}\mathcal{F}(w|v)$ is equal to a set $\{(i,1), \ldots, (i,s)\}$ such that $\alpha \mapsto \alpha^{(i)}$ maps K_1 into K_v, i.e., a K_v-row. Therefore, there is a place q_1 of K_1 with $(K_1)_{q_1} = K_v$ such that all places in T_v^c lie above q_1 if and only if $\mathcal{E}_v^c$ is contained in a K_v-row. Similarly, there is a place q_2 of K_2 with $(K_2)_{q_2} = K_v$ such that all places in T_v^c lie above q_2 if and only if $\mathcal{E}_v^c$ is contained in a K_v-column. This proves our claim. Hence for number fields K_1, K_2 with (1.7) and for sets $\mathcal{E}_v$ with (2.2), Theorem 1.1 is equivalent to Theorem 1.3. □

Proof of Corollary 1.2. Assume again (1.7). We first prove part (i). Suppose (1.8) holds for some $v \in S$. If $T_v^c = \emptyset$ then $W_T = 1$. If T_v^c is contained in the set of places w of K_1K_2 lying above some place q_1 of K_1 with $(K_1)_{q_1} = K_v$ then

$$\sum_{w \in T_v^c} \frac{[(K_1K_2)_w : K_v]}{[K_1K_2 : K]} \le \sum_{w:\, w|q_1} \frac{[(K_1K_2)_w : (K_1)_{q_1}]}{r[K_1K_2 : K_1]} = \frac{1}{r}, \tag{2.3}$$

where the second sum is taken over the places w of K_1K_2 lying above q_1. Hence $W_T \ge 1 - \frac{1}{r}$. Similarly, if T_v^c is contained in the set of places w of K_1K_2 lying above some place q_2 of K_2 with $(K_2)_{q_2} = K_v$, then $W_T \ge 1 - \frac{1}{s}$. Hence $W_T \ge 1 - \max(\frac{1}{r}, \frac{1}{s})$, against our assumption. Therefore, there is no $v \in S$ with (1.8). Now part (ii) of Theorem 1.1 can be applied and part (i) of Corollary 1.2 follows immediately.

We prove part (ii). Suppose for instance $r \le s$. Choose $v \in M_K$ for which there is a place w of K_1K_2 lying above v with $(K_1K_2)_w = K_v$. Let q_1 be the place of K_1 lying below w; then $(K_1)_{q_1} = K_v$. Now let $T = T_v$ consist of all places of K_1K_2 lying above v but not lying above q_1. Then from (2.3) it follows that $W_T = 1 - \frac{1}{r} = 1 - \max(\frac{1}{r}, \frac{1}{s})$. Further, T_v satisfies (1.8). Hence by part (i) of Theorem 1.1, inequality (1.4) has infinitely many solutions for every $\kappa > 0$. □

3. Notation and simple facts.

We introduce some notation to be used throughout the paper and mention some elementary facts.

Let K be an algebraic number field and S a finite set of places of K which from now on contains all infinite places. We define the ring of S-integers and the group of S-units by

$$O_S = \{x \in K : |x|_v \le 1 \quad \text{for } v \notin S\}, \qquad O_S^* = \{x \in K : |x|_v = 1 \quad \text{for } v \notin S\}$$

respectively, where by $v \notin S$ we mean $v \in M_K \backslash S$. For $x \in O_S$ we define

$$|x|_S := \prod_{v \in S} |x|_v \,.$$

Then by the product formula we have

$$|x|_S \ge 1 \quad \text{for } x \in O_S \backslash \{0\}, \qquad |x|_S = 1 \quad \text{for } x \in O_S^*. \tag{3.1}$$

Let $v \in M_K$. There is an extension of $|\cdot|_v$ to $\overline{K}_v$. For $a_1, \dots, a_n \in \overline{K}_v$ we put

$$|a_1, \dots, a_n|_v := \max(|a_1|_v, \dots, |a_n|_v)\,.$$

Further, for a binary form $F(X,Y) = a_0X^r + a_1X^{r-1}Y + \dots + a_rY^r$ with $a_1, \dots, a_r \in \overline{K}_v$ we put

$$|F|_v := |a_0, \dots, a_r|_v\,.$$

For vectors $\mathbf{a} = (a_1, \dots, a_n) \in O_S^n$ we define the truncated height

$$H_S(\mathbf{a}) = H_S(a_1, \dots, a_n) := \prod_{v \in S} |a_1, \dots, a_n|_v$$

and for binary forms F with coefficients in O_S we define

$$H_S(F) := \prod_{v \in S} |F|_v\,.$$

By (3.1) we have for non-zero vectors $\mathbf{a} \in O_S^n$ and for non-zero binary forms $F \in O_S[X,Y]$,

$$H_S(\mathbf{a}) \geq 1\,, \qquad H_S(F) \geq 1. \tag{3.2}$$

We mention some other facts:

Lemma 3.1. *Let $v \in M_K$ and let $F = A\prod_{i=1}^r(\alpha_iX + \gamma_iY)$ be a non-zero binary form with $A \in \overline{K}_v$, $\alpha_i, \gamma_i \in \overline{K}_v$ for $i = 1, \dots, r$. Then*

$$c_v^{-1}|F|_v \leq |A|_v \prod_{i=1}^r |\alpha_i, \gamma_i|_v \leq c_v|F|_v\,, \tag{3.3}$$

where c_v is a constant ≥ 1 depending only on v and r, with $c_v = 1$ if v is finite.

Proof. [9], Chap. 3, Section 2. □

Lemma 3.2. *let α be algebraic over K of degree r. Then there is a binary form $F \in O_S[X,Y]$ of degree r, irreducible over K, such that*

$$F(\alpha, 1) = 0\,, \qquad c^{-1}H(\alpha)^r \leq H_S(F) \leq cH(\alpha)^r\,, \tag{3.4}$$

where c is a constant ≥ 1 depending only on S and $K(\alpha)$.

Proof. [6], Lemma 6. □

We briefly go into discriminants and resultants. Let Ω be an arbitrary integral domain with quotient field of characteristic 0. Let F be a binary form with coefficients in Ω. In an algebraic extension of the quotient field of Ω we can factor F as $F = A\prod_{i=1}^{r}(\alpha_i X + \gamma_i Y)$. The discriminant of F is defined by

$$D(F) := A^{2r-2} \prod_{1\leq i<j\leq r} (\alpha_i\gamma_j - \alpha_j\gamma_i)^2 . \tag{3.5}$$

This is independent of the choice of A and the α_i, γ_i. Moreover, $D(F) \in \Omega$ and $D(F) = 0$ precisely when F has a multiple factor. For binary forms $F \in \Omega[X, Y]$ and non-singular matrices $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in Ω we define

$$F_U := F(aX + bY, cX + dY) . \tag{3.6}$$

Then we have $D(F_U) = (\det U)^{r(r-1)}D(F)$ so in particular

$$D(F_U) = D(F) \quad \text{if } \det U = 1. \tag{3.7}$$

Let F, G be binary forms with coefficients in Ω. In some algebraic closure of the quotient field of Ω, the forms F and G factor as $F = A\prod_{i=1}^{r}(\alpha_i X+\gamma_i Y)$, $G = B\prod_{j=1}^{s}(\beta_j X+\delta_j Y)$. Then the resultant of F and G is given by

$$R(F, G) := A^s B^r \prod_{i=1}^{r}\prod_{j=1}^{s}(\alpha_i\delta_j - \gamma_i\beta_j) . \tag{3.8}$$

This does not depend on the choice of A, B, the α_i, γ_i and the β_j, δ_j. Further, $R(F, G) \in \Omega$ and $R(F, G) = 0$ precisely when F, G have a common factor. It is also clear that for non-singular matrices U with entries in Ω we have $R(F_U, G_U) = (\det U)^{rs}R(F, G)$ and so

$$R(F_U, G_U) = R(F, G) \quad \text{if } \det U = 1. \tag{3.9}$$

Lastly, we have

Lemma 3.3. *Let $v \in M_K$. Let $F = A\prod_{i=1}^{r}(\alpha_i X + \gamma_i Y)$ and $G = B\prod_{j=1}^{s}(\beta_j X + \delta_j Y)$ be non-zero binary forms with A, B, α_i, γ_i $(i = 1, \ldots, r)$, β_j, δ_j $(j = 1, \ldots, s)$ all belonging to $\overline{K}_v$. Then*

$$\frac{|D(F)|_v^{1/2}}{|F|_v^{r-1}} \gg\ll \prod_{1\leq i<j\leq r} \frac{|\alpha_i\gamma_j - \alpha_j\gamma_i|_v}{|\alpha_i, \gamma_i|_v \cdot |\alpha_j, \gamma_j|_v} , \tag{3.10}$$

$$\frac{|R(F, G)|_v}{|F|_v^s|G|_v^r} \gg\ll \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{|\alpha_i\delta_j - \gamma_i\beta_j|_v}{|\alpha_i, \gamma_i|_v \cdot |\beta_j, \delta_j|_v} \tag{3.11}$$

where the constants implied by $\ll$, $\gg$ depend on r, s and v only.

Proof. By (3.5) we have $|D(F)|_v^{1/2} = |A|_v^{r-1} \prod_{1\le i<j\le r} |\alpha_i\gamma_j - \alpha_j\gamma_i|_v$ and by (3.3) we have $|F|_v^{r-1} \gg\ll |A|_v^{r-1} \prod_{1\le i<j\le r} |\alpha_i, \gamma_i|_v \cdot |\alpha_j, \gamma_j|_v$. By taking the quotient, the term $|A|_v^{r-1}$ cancels and we get (3.10). Inequality (3.11) is proved in precisely the same way. □

4. Preparations for the proof of part (i) of Theorem 1.3.

Let K be an algebraic number field. As before, we write $|x, y|_v$ for $\max(|x|_v, |y|_v)$. In this section, S is a finite set of places of K, containing all infinite places.

Our first basic tool is the Subspace theorem, first proved by Schmidt [12] for S consisting of only the archimedean places, and later by Schlickewei [11] in full generality.

Proposition 4.1 (Subspace Theorem). *Let $n \ge 2$, $\delta > 0$. For $v \in S$, let $L_1^{(v)}, \dots, L_n^{(v)}$ be linearly independent linear forms in $\overline{K}[X_1, \dots, X_n]$. Then there are finitely many proper linear subspaces $V_1, \dots, V_t$ of K^n such that the set of solutions of*

$$\prod_{v\in S}\prod_{i=1}^{n} |L_i^{(v)}(\mathbf{x})|_v \le H_S(\mathbf{x})^{-\delta} \quad \text{in } \mathbf{x} \in O_S^n$$

is contained in $V_1 \cup \dots \cup V_t$.

Our second tool is the adèlic generalisation of Minkowski's theorem on successive minima of convex bodies proved by McFeat [10] (see also [2]). We state the special case, needed for our purposes. Let K, S be as above. For $v \in S$, let $A_{1v}, \dots, A_{nv}$ be positive real numbers and $L_1^{(v)}, \dots, L_n^{(v)}$ linear forms with

$$L_1^{(v)}, \dots, L_n^{(v)} \in K_v[X_1, \dots, X_n], \quad L_1^{(v)}, \dots, L_n^{(v)} \text{ linearly independent.} \tag{4.1}$$

Define the set

$$\Pi := \{\mathbf{x} \in O_S^n : |L_i^{(v)}(\mathbf{x})|_v \le A_{iv} \quad \text{for } v \in S,\ i = 1, \dots, n\}.$$

Put

$$s(v) := \frac{[K_v : \mathbf{R}]}{[K : \mathbf{Q}]} \quad \text{if } v \text{ is archimedean,} \qquad s(v) := 0 \quad \text{if } v \text{ is non-archimedean}$$

and define for $\lambda > 0$ the dilatation of Π:

$$\lambda * \Pi := \{\mathbf{x} \in O_S^n : |L_i^{(v)}(\mathbf{x})|_v \leq \lambda^{s(v)} A_{iv} \quad \text{for } v \in S,\, i = 1, \ldots, n\}$$

(note that we have only a dilatation factor at the archimedean places). Then the successive minima $\lambda_1, \ldots, \lambda_n$ of Π are given by

$$\lambda_i := \min\{\lambda > 0 : \lambda * \Pi \text{ contains } i \text{ linearly independent vectors}\}.$$

Proposition 4.2 (Minkowski's Theorem). *Assume (4.1). Then* $0 < \lambda_1 \leq \cdots \leq \lambda_n < \infty$ *and*

$$\lambda_1 \cdots \lambda_n \gg\ll \Big(\prod_{v \in S} \prod_{i=1}^{n} A_{iv} \Big)^{-1}, \tag{4.2}$$

where the constants implied by $\ll$, $\gg$ *depend on* K, S, n *and the linear forms* $L_i^{(v)}$ $(v \in S,\, i = 1, \ldots, n)$ *only.*

We now deduce some specific results needed in the proof of part (i) of Theorem 1.3. Let K, S be as above and let r_v $(v \in S)$ be integers ≥ 2. In what follows we deal with linear forms in two variables with algebraic coefficients but not necessarily in K_v. Thus, let

$$L_i^{(v)} = \alpha_{iv} X + \beta_{iv} Y \in \overline{K}[X, Y] \qquad (v \in S,\, i = 1, \ldots, r_v)$$

be linear forms with

$$\operatorname{rank}\{L_i^{(v)}, L_j^{(v)}\} = 2 \qquad \text{for } v \in S,\, 1 \leq i < j \leq r_v. \tag{4.3}$$

Further, suppose there is a $v_0 \in S$ with

$$\alpha_{1,v_0}, \beta_{1,v_0} \in K_v \backslash \{0\}, \tag{4.4}$$

$$\alpha_{1,v_0}/\beta_{1,v_0} \notin K\,. \tag{4.5}$$

In the remainder of this section, constants implied by $\ll$, $\gg$ will depend on K, S, the linear forms $L_i^{(v)}$ $(v \in S,\, i = 1, \ldots, r_v)$, and a parameter $\delta > 0$.

Lemma 4.3. *Let* u *denote the cardinality of* S *and let* δ *be a real with* $0 < \delta < 1/2u$. *For every* $Q \gg 1$ *and every non-zero vector* $(x, y) \in O_S^2$ *with*

$$\left.\begin{array}{l} |L_1^{(v_0)}(x,y)|_{v_0} \ll Q^{-1+\delta}\,, \\ |L_i^{(v_0)}(x,y)|_{v_0} \ll Q^{1+\delta} \quad \textit{for } i = 2, \ldots, r_{v_0}, \\ |L_i^{(v)}(x,y)|_v \ll Q^{\delta} \quad \textit{for } v \in S \backslash \{v_0\},\, i = 1, \ldots, r_v \end{array}\right\} \tag{4.6}$$

we have in fact

$$\left.\begin{aligned} Q^{-1-3u\delta} \ll |L_1^{(v_0)}(x,y)|_{v_0} &\ll Q^{-1+\delta}\,, \\ Q^{1-3u\delta} \ll |L_i^{(v_0)}(x,y)|_{v_0} &\ll Q^{1+\delta} \quad \textit{for } i=2,\dots,r_{v_0}, \\ Q^{-3u\delta} \ll |L_i^{(v)}(x,y)|_v &\ll Q^{\delta} \quad \textit{for } v\in S\backslash\{v_0\},\ i=1,\dots,r_v. \end{aligned}\right\} \tag{4.7}$$

Proof. Assume there are a positive real Q and a non-zero vector $(x,y)\in O_S^2$ which satisfies (4.6) but does not satisfy (4.7). We have to show that $Q \ll 1$.

Our assumptions on Q and (x,y) imply

$$\left.\begin{aligned} |L_1^{(v_0)}(x,y)|_{v_0} &\ll Q^{-1+\delta-\varepsilon_{1,v_0}}\,, \\ |L_i^{(v_0)}(x,y)|_{v_0} &\ll Q^{1+\delta-\varepsilon_{i,v_0}} \quad \text{for } i=2,\dots,r_{v_0}, \\ |L_i^{(v)}(x,y)|_v &\ll Q^{\delta-\varepsilon_{iv}} \quad \text{for } v\in S\backslash\{v_0\},\ i=1,\dots,r_v, \end{aligned}\right\} \tag{4.8}$$

where $\varepsilon_{iv}=(3u+1)\delta$ for exactly one pair in the set $\{(i,v):\, v\in S,\, i=1,\dots,r_v\}$, and $\varepsilon_{iv}=0$ for all other pairs in this set. In fact, we may assume

$$\left.\begin{aligned} &\varepsilon_{iv}=(3u+1)\delta \text{ for exactly one pair from } \{(i,v):\, v\in S,\, i=1,2\}, \\ &\varepsilon_{iv}=0 \text{ for all other pairs in this set.} \end{aligned}\right\} \tag{4.9}$$

Indeed, if $\varepsilon_{iv}=(3u+1)\delta$ for some $v\in S$, $i>2$ then we can achieve (4.9) by interchanging $L_2^{(v)}$ and $L_i^{(v)}$. This does not affect (4.3), (4.4), (4.5).

We go towards an application of the Subspace Theorem. Assume (4.9). Noting that by (4.3) we can express X,Y as linear combinations of $L_1^{(v)}, L_2^{(v)}$ and using (4.8), (4.9) we obtain

$$|x,y|_{v_0} \ll \max(|L_1^{(v_0)}(x,y)|_{v_0}, |L_2^{(v_0)}(x,y)|_{v_0}) \ll Q^{1+\delta}\,, \tag{4.10}$$

$$|x,y|_{v} \ll \max(|L_1^{(v)}(x,y)|_{v}, |L_2^{(v)}(x,y)|_{v}) \ll Q^{\delta} \quad \text{for } v\in S\backslash\{v_0\}. \tag{4.11}$$

Hence

$$H_S(x,y)=\prod_{v\in S}|x,y|_v \ll Q^{1+u\delta}\,.$$

From (4.8), (4.9) and this last inequality we infer

$$\begin{aligned} \prod_{v\in S}\prod_{i=1}^{2}|L_i^{(v)}(x,y)|_v &\ll Q^{(-1+\delta)+(1+\delta)+2(u-1)\delta-\left(\sum_{v\in S}\sum_{i=1}^{2}\varepsilon_{iv}\right)} = Q^{-(u+1)\delta} \\ &\ll H_S(x,y)^{-\frac{(u+1)\delta}{1+u\delta}}\,. \end{aligned}$$

We can apply Proposition 4.1 because of (4.3). It follows that there are finitely many one-dimensional linear subspaces $V_1, \dots, V_t$ of K^2, independent of Q and (x, y), such that

$$(x, y) \in V_1 \cup \cdots \cup V_t \,.$$

For $i = 1, \dots, t$, fix $(\xi_i, \eta_i) \in V_i \backslash \{\mathbf{0}\}$. By (4.5) we have $L_1^{(v_0)}(\xi_i, \eta_i) \neq 0$. Suppose $(x, y) \in V_j$. Then $(x, y) = \lambda(\xi_j, \eta_j)$ for some $\lambda \in K^*$ and so

$$\frac{|L_1^{(v_0)}(x, y)|_{v_0}}{|x, y|_{v_0}} = \frac{|L_1^{(v_0)}(\xi_j, \eta_j)|_{v_0}}{|\xi_j, \eta_j|_{v_0}} \geq \min_{i=1,\dots,t} \frac{|L_1^{(v_0)}(\xi_i, \eta_i)|_{v_0}}{|\xi_i, \eta_i|_{v_0}} > 0$$

where the right-hand side is independent of Q, (x, y). By combining this with the first inequality of (4.8) we can improve (4.10) to

$$|x, y|_{v_0} \ll Q^{-1+\delta}$$

and together with (4.11) and the assumption $d < 1/2u$ this gives

$$H_S(x, y) = \prod_{v \in S} |x, y|_v \ll Q^{-1+u\delta} \ll Q^{-1/2}.$$

Recalling that $H_S(x, y) \gg 1$ by (3.2), we arrive at $Q \ll 1$. This completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let $\delta > 0$. For every $Q \gg 1$, there are linearly independent vectors (x_1, y_1), $(x_2, y_2) \in O_S$ such that for $k = 1, 2$,*

$$\left.\begin{aligned} Q^{-1-\delta} &\ll |L_1^{(v_0)}(x_k, y_k)|_{v_0} \ll Q^{-1+\delta} \\ Q^{1-\delta} &\ll |L_i^{(v_0)}(x_k, y_k)|_{v_0} \ll Q^{1+\delta} \quad \text{for } i = 2, \dots, r_{v_0}, \\ Q^{-\delta} &\ll |L_i^{(v)}(x_k, y_k)|_v \ll Q^{\delta} \quad \text{for } v \in S \backslash \{v_0\},\ i = 1, \dots, r_v \end{aligned}\right\} \tag{4.12}$$

and such that

$$|x_1 y_2 - x_2 y_1|_v \gg\ll 1 \quad \text{for } v \in S. \tag{4.13}$$

Proof. Without loss of generality we assume $0 < \delta < 1$. Let u denote the cardinality of S.

We are going to apply Minkowski's theorem to the set

$$\Pi := \left\{ (x, y) \in O_S^2 : \begin{array}{c} |L_1^{(v_0)}(x, y)|_{v_0} \leq Q^{-1},\ |y|_{v_0} \leq Q\,, \\ \\ |x|_v \leq 1,\ |y|_v \leq 1 \text{ for } v \in S \backslash \{v_0\} \end{array} \right\}.$$

Condition (4.1) is satisfied because of (4.4). Let λ_1, λ_2 denote the successive minima of Π. By Proposition 4.2 we have

$$\lambda_1\lambda_2 \gg\ll 1\,. \tag{4.14}$$

Choose linearly independent vectors (x_1,y_1), (x_2,y_2) from O_S^2 such that for $k=1,2$ we have $(x_k,y_k)\in\lambda_k*\Pi$, that is,

$$\left.\begin{array}{l} |L_1^{(v_0)}(x_k,y_k)|_{v_0}\le Q^{-1}\lambda_k^{s(v_0)},\ |y_k|_{v_0}\le Q\lambda_k^{s(v_0)},\\ |x_k|_v\le\lambda_k^{s(v)},\ |y_k|_v\le\lambda_k^{s(v)}\quad\text{for } v\in S\backslash\{v_0\}. \end{array}\right\} \tag{4.15}$$

We first show that these vectors satisfy (4.13). By (4.14), (4.15) we have

$$|x_1y_2-x_2y_1|_{v_0}\ll|L_1^{(v_0)}(x_1,y_1)y_2-L_1^{(v_0)}(x_2,y_2)y_1|_{v_0}\ll Q^{-1}Q(\lambda_1\lambda_2)^{s(v_0)}\ll 1\,,$$
$$|x_1y_2-x_2y_1|_v\ll(\lambda_1\lambda_2)^{s(v)}\ll 1\quad\text{for } v\in S\backslash\{v_0\}.$$

Further, since $x_1y_2-x_2y_1$ is a non-zero S-integer, we have by (3.1) that for $v\in S$, $|x_1y_2-x_2y_1|_v\ge\prod_{v'\in S\backslash\{v\}}|x_1y_2-x_2y_1|_{v'}^{-1}\gg 1$. This proves (4.13).

We now prove (4.12). For $k=1,2$ we have

$$\left.\begin{array}{l} |L_1^{(v_0)}(x_k,y_k)|_{v_0}\ll Q^{-1}\lambda_k^{s(v_0)},\\ |L_i^{(v_0)}(x_k,y_k)|_{v_0}\ll Q\lambda_k^{s(v_0)}\quad\text{for } i=2,\ldots,r_{v_0},\\ |L_i^{(v)}(x_k,y_k)|_v\ll\lambda_k^{s(v)}\quad\text{for } v\in S\backslash\{v_0\},\ i=1,\ldots,r_v. \end{array}\right\} \tag{4.16}$$

Indeed, by (4.4), the linear forms $L_1^{(v_0)}$ and Y are linearly independent, hence for $i=2,\ldots,r_{v_0}$, the linear form $L_i^{(v_0)}$ is a linear combination of $L_1^{(v_0)}$, Y. Now the inequalities on the second row follow from (4.15). The other inequalities are obvious consequences of (4.15).

By (4.14) we have $\lambda_1\ll 1$. By inserting this into (4.16) for $k=1$ and being generous we obtain

$$|L_1^{(v_0)}(x_1,y_1)|_{v_0}\ll Q^{-1+\delta/9u^2}\,,$$
$$|L_i^{(v_0)}(x_1,y_1)|_{v_0}\ll Q^{1+\delta/9u^2}\quad\text{for } i=2,\ldots,r_{v_0},$$
$$|L_i^{(v)}(x_1,y_1)|_v\ll Q^{\delta/9u^2}\quad\text{for } v\in S\backslash\{v_0\},\ i=1,\ldots,r_v.$$

Lemma 4.3 yields that for $Q\gg 1$ we have in fact,

$$\left.\begin{array}{r} Q^{-1-\delta/3u}\ll|L_1^{(v_0)}(x_1,y_1)|_{v_0}\ll Q^{-1+\delta/9u^2}\,,\\ Q^{1-\delta/3u}\ll|L_i^{(v_0)}(x_1,y_1)|_{v_0}\ll Q^{1+\delta/9u^2}\quad\text{for } i=2,\ldots,r_{v_0},\\ Q^{-\delta/3u}\ll|L_i^{(v)}(x_1,y_1)|_v\ll Q^{\delta/9u^2}\quad\text{for } v\in S\backslash\{v_0\},\ i=1,\ldots,r_v. \end{array}\right\} \tag{4.17}$$

From (4.17), (4.16) we infer $\lambda_1^{s(v)} \gg Q^{-\delta/3u}$ for $v \in S$ if $Q \gg 1$ and then from (4.14) it follows $\lambda_2^{s(v)} \ll Q^{\delta/3u}$ for $v \in S$. On substituting this into (4.16) for $k = 2$, assuming $Q \gg 1$, we get

$$\begin{aligned} |L_1^{(v_0)}(x_2, y_2)|_{v_0} &\ll Q^{-1+\delta/3u}, \\ |L_i^{(v_0)}(x_2, y_2)|_{v_0} &\ll Q^{1+\delta/3u} \quad \text{for } i = 2, \dots, r_{v_0}, \\ |L_i^{(v)}(x_2, y_2)|_v &\ll Q^{\delta/3u} \quad \text{for } v \in S \backslash \{v_0\},\ i = 1, \dots, r_v. \end{aligned}$$

By applying Lemma 4.3 once more, we obtain for $Q \gg 1$,

$$\left.\begin{aligned} Q^{-1-\delta} \ll |L_1^{(v_0)}(x_2, y_2)|_{v_0} &\ll Q^{-1+\delta/3u}, \\ Q^{1-\delta} \ll |L_i^{(v_0)}(x_2, y_2)|_{v_0} &\ll Q^{1+\delta/3u} \quad \text{for } i = 2, \dots, r_{v_0}, \\ Q^{-\delta} \ll |L_i^{(v)}(x_2, y_2)|_v &\ll Q^{\delta/3u} \quad \text{for } v \in S \backslash \{v_0\},\ i = 1, \dots, r_v. \end{aligned}\right\} \tag{4.18}$$

Now (4.17) and (4.18) together imply (4.12) for $k = 1, 2$. This proves Lemma 4.4. □

5. Proof of part (i) of Theorem 1.3.

We keep the notation and assumptions of the previous sections. Thus, K is an algebraic number field, K_1, K_2 are two extensions of K with (1.10) and S is a finite set of places of K. We assume that S contains all infinite places. This is no loss of generality since if we add a finite number of new places v to S and choose $\mathcal{E}_v = \emptyset$ for these, then this affects neither inequality (1.9) nor the condition on the sets $\mathcal{E}_v^c$ in part (i) of Theorem 1.3. Let $\mathcal{E}_v$ ($v \in S$) be subsets of $\{(i, j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ and suppose that for some $v_0 \in S$, either $\mathcal{E}_{v_0}^c = \emptyset$, or $\mathcal{E}_{v_0}^c$ is contained in a K_v-row, or $\mathcal{E}_{v_0}^c$ is contained in a K_v-column. We pick any α_0, β_0 with $K(\alpha_0) = K_1$, $K(\beta_0) = K_2$. We show that for every $\kappa > 0$, inequality (1.9) has infinitely many solutions (α, β) of the type

$$\alpha = \frac{a\alpha_0 + c}{b\alpha_0 + d}, \quad \beta = \frac{a\beta_0 + c}{b\beta_0 + d} \quad \text{with } a, b, c, d \in O_S,\ ad - bc \neq 0. \tag{5.1}$$

We choose a parameter $\delta > 0$. Below, all constants implied by $\ll$, $\gg$ will depend on K, S, α_0, β_0 and δ.

The following observation is useful:

Lemma 5.1. *Let $a, b, c, d \in O_S$ with $|ad - bc|_v \gg\ll 1$ for $v \in S$ and let α, β be given by (5.1). Then*

$$\begin{aligned} H(\alpha)^r &\gg\ll \prod_{v\in S}\prod_{i=1}^{r} |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_v\,, \\ H(\beta)^s &\gg\ll \prod_{v\in S}\prod_{j=1}^{s} |a\beta_0^{(j)} + c, b\beta_0^{(j)} + d|_v\,. \end{aligned} \tag{5.2}$$

Proof. We prove only the inequality for $H(\alpha)$. Let $v \in M_K$. From the observations in the beginning of Section 2, it follows that $\{1, \ldots, r\}$ can be partitioned into sets $\mathcal{F}(q_1|v)$, one for each place q_1 on K_1 lying above v, such that for $i \in \mathcal{F}(q_1|v)$ the absolute values given by $|\alpha^{(i)}|_v$ for $\alpha \in K_1$ represent q_1. Further, the set $\mathcal{F}(q_1|v)$ has cardinality $[(K_1)_{q_1} : K_v]$ and by (1.1) we have $|\alpha|_{q_1} = |\alpha^{(i)}|_v^{[(K_1)_{q_1}:K_v]/r}$ for $\alpha \in K_1$, $i \in \mathcal{F}(q_1|v)$. A consequence of this is, that $\prod_{q_1|v} |a\alpha_0 + c, b\alpha_0 + d|_{q_1}^r = \prod_{i=1}^r |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_v$ for $v \in M_K$ (with $|x, y|_{q_1} = \max(|x|_{q_1}, |y|_{q_1})$). By taking the product over $v \in M_K$ and applying the product formula we get

$$H(\alpha)^r = \prod_{q_1\in M_{K_1}} |a\alpha_0 + c, b\alpha_0 + d|_{q_1}^r = \prod_{v\in M_K}\prod_{i=1}^{r} |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_v\,.$$

Since $a, b, c, d \in O_S$, the product of the terms with $v \notin S$ is $\ll 1$. On the other hand, using $a(b\alpha_0^{(i)} + d) - b(a\alpha_0^{(i)} + c) = ad - bc$, we get that the product of the terms with $v \notin S$ is $\gg \prod_{v\notin S} |ad-bc|_v^r = \prod_{v\in S} |ad-bc|_v^{-r} \gg 1$. This implies the inequality for $H(\alpha)$ in (5.2).□

In what follows, let u denote the cardinality of S. In the proof of part (ii) of Theorem 1.3 we distinguish two cases.

Case 1. $\mathcal{E}_{v_0}^c = \emptyset$.
Let $Q > 1$. By Proposition 4.2 (the one-dimensional case) or the strong approximation theorem for absolute values, there is a d with

$$d \in O_S\backslash\{0\}\,, \quad |d|_v \le Q^{-1} \quad \text{for } v \in S\backslash\{v_0\}. \tag{5.3}$$

Then by the product formula we have

$$|d|_{v_0} \ge Q^u\,. \tag{5.4}$$

Take

$$\alpha = \frac{1}{\alpha_0 + d}\,, \quad \beta = \frac{1}{\beta_0 + d}\,.$$

By Lemma 5.1, (5.3), (5.4) we have for $Q \gg 1$,

$$H(\alpha) \gg\ll \prod_{v\in S}\prod_{i=1}^{r} |1, \alpha_0^{(i)} + d|_v^{1/r} \gg\ll |d|_{v_0}\,, \quad H(\beta) \gg\ll |d|_{v_0}\,. \tag{5.5}$$

Assuming $Q \gg 1$ we have by (5.3),

$$\prod_{(i,j)\in\mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v = \prod_{(i,j)\in\mathcal{E}_v} \frac{|\alpha_0^{(i)} - \beta_0^{(j)}|_v}{|\alpha_0^{(i)} + d|_v \cdot |\beta_0^{(j)} + d|_v} \ll 1 \quad \text{for } v \in S\backslash\{v_0\}$$

and by (5.4), (5.5),

$$\begin{aligned}\prod_{(i,j)\in\mathcal{E}_{v_0}} |\alpha^{(i)} - \beta^{(j)}|_{v_0} &= \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{|\alpha_0^{(i)} - \beta_0^{(j)}|_{v_0}}{|\alpha_0^{(i)} + d|_{v_0} \cdot |\beta_0^{(j)} + d|_{v_0}} \ll |d|_{v_0}^{-2rs} \\ &\ll H(\alpha)^{-rs} H(\beta)^{-rs}\end{aligned}$$

so altogether,

$$\prod_{v\in S}\prod_{(i,j)\in\mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v \ll H(\alpha)^{-rs} H(\beta)^{-rs}\,. \tag{5.6}$$

From (5.4), (5.5) we infer $H(\alpha) \gg Q^u$, $H(\beta) \gg Q^u$. Thus, letting $Q \to \infty$, we infer that (5.6) has infinitely many solutions and so, for every $\kappa > 0$, (1.9) has infinitely many solutions.

Case 2. $\mathcal{E}_{v_0}^c \neq \emptyset$ and $\mathcal{E}_{v_0}^c$ is contained in a K_{v_0}-row or a K_{v_0}-column.
We deal only with the case that $\mathcal{E}_{v_0}^c$ is contained in a K_{v_0}-row since the argument for K_{v_0}-columns is similar. Without loss of generality we assume

$$\alpha^{(1)} \in K_{v_0} \quad \text{for } \alpha \in K_1, \tag{5.7}$$

$$\mathcal{E}_{v_0} \subseteq \{(1,1), \ldots, (1,s)\}\,. \tag{5.8}$$

Fix $\kappa > 0$ and let $\delta > 0$ be a parameter sufficiently small in terms of κ. By (5.7) and $K(\alpha_0) = K_1 \neq K$ we have $\alpha_0^{(1)} \in K_{v_0}$, $\alpha_0^{(1)} \notin K$. Further, by $K(\alpha_0) = K_1$, $K(\beta_0) = K_2$ and (1.10), the numbers $\alpha_0^{(1)}, \ldots, \alpha_0^{(r)}, \beta_0^{(1)}, \ldots, \beta_0^{(s)}$ are distinct and non-zero. Hence the linear forms $L_1^{(v)} = \alpha_0^{(1)}X + Y, \ldots, L_r^{(v)} = \alpha_0^{(r)}X + Y$, $L_{r+1}^{(v)} = \beta_0^{(1)}X + Y, \ldots, L_{r+s}^{(v)} = \beta_0^{(s)}X + Y$ $(v \in S)$ satisfy the conditions (4.3), (4.4), (4.5) with $r_v = r + s$ for $v \in S$ and so we can apply Lemma 4.4 to these forms. According to this lemma, for every $Q \gg 1$, there are linearly independent vectors (a,c), $(b,d) \in O_S^2$ such that the inequalities

$$Q^{-1-\delta} \ll |x\alpha_0^{(1)} + y|_{v_0} \ll Q^{-1+\delta} \tag{5.9}$$

$$Q^{1-\delta} \ll |x\alpha_0^{(i)} + y|_{v_0} \ll Q^{1+\delta} \quad \text{for } i = 2, \dots, r, \tag{5.10}$$

$$Q^{1-\delta} \ll |x\beta_0^{(j)} + y|_{v_0} \ll Q^{1+\delta} \quad \text{for } j = 1, \dots, s, \tag{5.11}$$

$$Q^{-\delta} \ll |x\alpha_0^{(i)} + y|_v \ll Q^{\delta} \quad \text{for } v \in S\backslash\{v_0\},\ i = 1, \dots, r, \tag{5.12}$$

$$Q^{-\delta} \ll |x\beta_0^{(j)} + y|_v \ll Q^{\delta} \quad \text{for } v \in S\backslash\{v_0\},\ j = 1, \dots, s \tag{5.13}$$

are simultaneously satisfied for $(x,y) = (a,c)$ and for $(x,y) = (b,d)$ and moreover,

$$|ad - bc|_v \gg\ll 1 \quad \text{for } v \in S. \tag{5.14}$$

Let α, β be given by (5.1). We estimate the heights $H(\alpha)$, $H(\beta)$ from above and below in terms of Q. Since $(a,c),(b,d)$ satisfy (5.9), (5.10), (5.12) we have

$$Q^{r-2-r\delta} \ll \prod_{i=1}^{r} |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_{v_0} \ll Q^{r-2+r\delta},$$

$$Q^{-r\delta} \ll \prod_{i=1}^{r} |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_{v} \ll Q^{r\delta} \quad \text{for } v \in S\backslash\{v_0\},$$

so

$$Q^{r-2-ur\delta} \ll \prod_{v\in S}\prod_{i=1}^{r} |a\alpha_0^{(i)} + c, b\alpha_0^{(i)} + d|_{v} \ll Q^{r-2+ur\delta}.$$

Together with (5.14) and Lemma 5.1 this implies

$$Q^{r-2-ur\delta} \ll H(\alpha)^r \ll Q^{r-2+ur\delta}. \tag{5.15}$$

In precisely the same way, using (5.11), (5.13), (5.14) and Lemma 5.1 one shows

$$Q^{s-us\delta} \ll H(\beta)^s \ll Q^{s+us\delta}. \tag{5.16}$$

We now estimate from above $\prod_{v\in S}\prod_{(i,j)\in\mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v$. By (5.1) we have

$$|\alpha^{(i)} - \beta^{(j)}|_v = \frac{|ad-bc|_v \cdot |\alpha_0^{(i)} - \beta_0^{(j)}|_v}{|b\alpha_0^{(i)} + d|_v \cdot |b\beta_0^{(j)} + d|_v}$$

and together with (5.14) and the fact that (b,d) satisfies (5.9)–(5.13) this implies

$$\begin{aligned} |\alpha^{(1)}-\beta^{(j)}|_{v_0} &\ll Q^{2\delta} \quad \text{for } j=1,\ldots,s, \\ |\alpha^{(i)}-\beta^{(j)}|_{v_0} &\ll Q^{-2+2\delta} \quad \text{for } i=2,\ldots,r,\ j=1,\ldots,s, \\ |\alpha^{(i)}-\beta^{(j)}|_{v} &\ll Q^{2\delta} \quad \text{for } v\in S\backslash\{v_0\},\ i=1,\ldots,r,\ j=1,\ldots,s. \end{aligned}$$

By (5.8), the set $\mathcal{E}_{v_0}$ contains all pairs (i,j) with $i=2,\ldots,r$, $j=1,\ldots,s$. By combining the inequalities just mentioned and inserting (5.15), (5.16) we get

$$\prod_{v\in S}\prod_{(i,j)\in\mathcal{E}_v}|\alpha^{(i)}-\beta^{(j)}|_v \ll Q^{-(r-2)s-rs+2urs\delta} \ll \big(H(\alpha)H(\beta)\big)^{-rs(1-\kappa/2)},$$

provided we choose δ sufficiently small. By (5.15), (5.16), the heights $H(\alpha)$, $H(\beta)$ go to infinity with Q. It follows that the last inequality, and consequently (1.9), has infinitely many solutions. This completes the proof of part (i) of Theorem 1.3. □

6. Proof of part (ii) of Theorem 1.3 (modulo a proposition).

We prove part (ii) of Theorem 1.3. In the proof we use a proposition whose proof is postponed to Section 7.

Let K be an algebraic number field and S a finite set of places of K. We assume that S contains all infinite places which, by the observations in the first paragraph of Section 5, is no loss of generality. Further, K_1, K_2 satisfy (1.10). In what follows, constants implied by $\ll$, $\gg$ depend only on K, K_1, K_2 and S. We use the notation introduced in the previous sections.

Pick α, β with $K(\alpha)=K_1$, $K(\beta)=K_2$. By Lemma 3.2 there are binary forms F, $G\in O_S[X,Y]$, irreducible over K, such that

$$\left.\begin{aligned} F(\alpha,1)=0\,,\quad H(\alpha)^r \gg\ll H_S(F)\,,\quad \deg F=r\,, \\ G(\beta,1)=0\,,\quad H(\beta)^s \gg\ll H_S(G)\,,\quad \deg G=s\,. \end{aligned}\right\} \tag{6.1}$$

We can express F, G as $F=A\prod_{i=1}^r(X-\alpha^{(i)}Y)$, $G=B\prod_{j=1}^s(X-\beta^{(j)}Y)$ with A, $B\in O_S$. By applying (3.11) and taking the product over $v\in S$ we get

$$\prod_{v\in S}\prod_{i=1}^r\prod_{j=1}^s\frac{|\alpha^{(i)}-\beta^{(j)}|_v}{|1,\alpha^{(i)}|_v\cdot|1,\beta^{(j)}|_v} \gg\ll \frac{|R(F,G)|_S}{H_S(F)^sH_S(G)^r}\,. \tag{6.2}$$

Recall definition (3.6). The next result is our main tool:

Proposition 6.1. *There is a matrix $U \in SL(2, O_S)$ (i.e., with entries in O_S and determinant 1) such that*

$$|R(F,G)|_S \gg \big(H_S(F_U)^s H_S(G_U)^r\big)^{1/718}\,, \tag{6.3}$$

where the constant implied by $\gg$ is ineffective.

Remark. The matrix U in the right-hand side is necessary because of (3.9).

Proof. We apply Theorem 2 of [6] to F and G. From (1.10), (6.1) it follows that $\deg F = r \geq 3$, $\deg G = s \geq 3$, and that FG has no multiple factor. Hence the conditions of Theorem 2 of [6] are satisfied. It follows from that result that there is a matrix $U_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, O_S)$, i.e., with determinant $ad - bc =: \varepsilon \in O_S^*$, such that

$$|R(F,G)|_S \gg \Big(H_S(F_{U_1})^s H_S(G_{U_1})^r\Big)^{1/718}\,, \tag{6.4}$$

where the implied constant is determined by r, s, S and the splitting field of FG over K, so by K, S, K_1, K_2. The proof of (6.4) uses results from other papers, i.e., [5], [6]. A sketchy overview of the proof is given in [4]. The proof goes back to Schlickewei's p-adic generalisation of Schmidt's Subspace Theorem. Therefore, the constant implied by $\gg$ in (6.4) is ineffective.

From the S-unit theorem it follows that there are $\varepsilon_1, \varepsilon_2 \in O_S^*$ with

$$\varepsilon^{-1} = \varepsilon_1 \varepsilon_2^2\,, \qquad |\varepsilon_1|_v \ll 1 \quad \text{for } v \in S, \tag{6.5}$$

where $\varepsilon = \det U_1$. Now take $U := \left(\begin{smallmatrix} \varepsilon_2 a & \varepsilon_1\varepsilon_2 b \\ \varepsilon_2 c & \varepsilon_1\varepsilon_2 d \end{smallmatrix}\right)$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = U_1$ as above. Thus, $\det U = 1$, i.e., $U \in SL(2, O_S)$. By (6.5) we have

$$F_U(X,Y) = \varepsilon_2 F_{U_1}(X, \varepsilon_1 Y)\,, \qquad |F_U|_v \ll |\varepsilon_2|_v \cdot |F_{U_1}|_v \quad \text{for } v \in S$$

and by taking the product over $v \in S$ and applying (3.1) we obtain

$$H_S(F_U) \ll \big(\prod_{v\in S} |\varepsilon_2|_v\big) H_S(F_{U_1}) = H_S(F_{U_1})\,.$$

Similarly, we get $H_S(G_U) \ll H_S(G_{U_1})$. Together with (6.4) this implies (6.3). $\square$

Proposition 6.2. *Let $\mathcal{E}_v \subseteq \{(i,j) : i = 1, \dots, r, \ j = 1, \dots, s\}$ $(v \in S)$ be sets such that for each $v \in S$ we have that $\mathcal{E}_v^c \neq \emptyset$, $\mathcal{E}_v^c$ is not contained in a K_v-row and $\mathcal{E}_v^c$ is not contained in a K_v-column. Then for every $U \in SL(2, O_S)$ there are pairs $(i_v, j_v) \in \mathcal{E}_v^c$ $(v \in S)$ such that*

$$\prod_{v \in S} \frac{|\alpha^{(i_v)} - \beta^{(j_v)}|_v}{|1, \alpha^{(i_v)}|_v \cdot |1, \beta^{(j_v)}|_v} \ll \frac{H_S(F_U)^{r+s} H_S(G_U)^{r+s}}{H_S(F)^{1/r} H_S(G)^{1/s}} . \tag{6.6}$$

The proof is postponed to Section 7.

Proof of part (ii) of Theorem 1.3. The conditions on the sets $\mathcal{E}_v^c$ in part (ii) of Theorem 1.3 are precisely those of Proposition 6.2 so we can apply the latter. Let U be the matrix from Proposition 6.1 and choose pairs $(i_v, j_v) \in \mathcal{E}_v^c$ $(v \in S)$ according to Proposition 6.2. Let θ be a real with $0 < \theta < 1$ which will be specified later. Put

$$f_{ij}^{(v)} := \frac{|\alpha^{(i)} - \beta^{(j)}|_v}{|1, \alpha^{(i)}|_v \cdot |1, \beta^{(j)}|_v} \quad \text{for } v \in S, \ i = 1, \dots, r, \ j = 1, \dots, s.$$

Now we have

$$\begin{aligned} &\prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v \\ &\quad \gg \prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v} f_{ij}^{(v)} = \Big(\prod_{v \in S} \prod_{i=1}^{r} \prod_{j=1}^{s} f_{ij}^{(v)} \Big) \cdot \Big(\prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v^c} f_{ij}^{(v)} \Big)^{-1} \\ &\quad \gg \frac{|R(F,G)|_S}{H_S(F)^s H_S(G)^r} \cdot \prod_{v \in S} \big(f_{i_v, j_v}^{(v)} \big)^{-\theta} \quad \text{by (6.2) and } f_{ij} \ll 1 \\ &\quad \gg \frac{\big(H_S(F_U)^s H_S(G_U)^r \big)^{1/718}}{H_S(F)^s H_S(G)^r} \cdot \left(\frac{H_S(F)^{1/r} H_S(G)^{1/s}}{H_S(F_U)^{r+s} H_S(G_U)^{r+s}} \right)^{\theta} \quad \text{by (6.3), (6.6).} \end{aligned}$$

By choosing $\theta = \frac{\min(r,s)}{718(r+s)}$ so that the exponents on $H_S(F_U)$ and $H_S(G_U)$ become non-negative and then using (6.1) we get

$$\begin{aligned} \prod_{v \in S} \prod_{(i,j) \in \mathcal{E}_v} |\alpha^{(i)} - \beta^{(j)}|_v &\gg \big(H_S(F)^s H_S(G)^r \big)^{-1 + \frac{1}{718(r+s)\max(r,s)}} \\ &\gg \big(H(\alpha) H(\beta) \big)^{-rs(1 - \frac{1}{718(r+s)\max(r,s)})} . \end{aligned}$$

This implies that for every $\kappa \leq \frac{1}{718(r+s)^2}$ (1.9) has only finitely many solutions. This completes the proof of part (ii) of Theorem 1.3. □

7. Proof of Proposition 6.2.

We keep the notation and assumptions from the previous sections. In particular, K is a number field, K_1, K_2 are finite extensions of K with (1.10), S is a finite set of places of K containing all infinite places, and $\mathcal{E}_v$ ($v \in S$) are subsets of $\{(i,j) : i = 1,\dots,r,\ j = 1,\dots,s\}$ satisfying the conditions of Proposition 6.2. Let α, β be numbers with $K(\alpha) = K_1$, $K(\beta) = K_2$ and F, G corresponding binary forms in $O_S[X,Y]$ with (6.1). Thus,

$$F = A\prod_{i=1}^{r}(X - \alpha^{(i)}Y)\,, \quad G = B\prod_{j=1}^{s}(X - \beta^{(j)}Y) \quad \text{with } A, B \in O_S. \tag{7.1}$$

Let $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, O_S)$. We have

$$F_U = A\prod_{i=1}^{r}(\gamma^{(i)}X + \delta^{(i)}Y)\,, \quad G_U = B\prod_{j=1}^{s}(\xi^{(j)}X + \eta^{(j)}Y) \tag{7.2}$$

with

$$\begin{pmatrix} \gamma^{(i)} \\ \delta^{(i)} \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \alpha^{(i)} \end{pmatrix}, \quad \begin{pmatrix} \xi^{(j)} \\ \eta^{(j)} \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \beta^{(j)} \end{pmatrix} \tag{7.3}$$

for $i = 1,\dots,r$, $j = 1,\dots,s$.

For the moment we fix a place $v \in S$. We define:

$$\left.\begin{aligned}
f_i &:= \frac{|1, \alpha^{(i)}|_v}{|\gamma^{(i)}, \delta^{(i)}|_v} \quad (i = 1,\dots,r),\\
g_j &:= \frac{|1, \beta^{(j)}|_v}{|\xi^{(j)}, \eta^{(j)}|_v} \quad (j = 1,\dots,s),\\
\Delta_{pq} &:= \frac{|\alpha^{(p)} - \alpha^{(q)}|_v}{|\gamma^{(p)}, \delta^{(p)}|_v \cdot |\gamma^{(q)}, \delta^{(q)}|_v} \quad (1 \le p, q \le r,\ p \ne q),\\
\Theta_{pq} &:= \frac{|\beta^{(p)} - \beta^{(q)}|_v}{|\xi^{(p)}, \eta^{(p)}|_v \cdot |\xi^{(q)}, \eta^{(q)}|_v} \quad (1 \le p, q \le s,\ p \ne q),\\
E_{ij} &:= \frac{|\alpha^{(i)} - \beta^{(j)}|_v}{|\gamma^{(i)}, \delta^{(i)}|_v \cdot |\xi^{(j)}, \eta^{(j)}|_v} \quad (i = 1,\dots,r,\ j = 1,\dots,s).
\end{aligned}\right\} \tag{7.4}$$

Below, we have collected some properties of these quantities. Constants implied by $\ll$, $\gg$ depend only on K, K_1, K_2, S, v.

Lemma 7.1. *We have*

$$f_1 \cdots f_r \gg\ll \frac{|F|_v}{|F_U|_v}\,, \tag{7.5}$$

$$g_1 \cdots g_s \gg\ll \frac{|G|_v}{|G_U|_v}, \tag{7.6}$$

$$\frac{|D(F)|_v^{1/2}}{|F_U|_v^{r-1}} \ll \Delta_{pq} \ll 1 \quad \textit{for } 1 \le p, q \le r,\ p \ne q, \tag{7.7}$$

$$\frac{|D(G)|_v^{1/2}}{|G_U|_v^{s-1}} \ll \Theta_{pq} \ll 1 \quad \textit{for } 1 \le p, q \le s,\ p \ne q, \tag{7.8}$$

$$\frac{|R(F,G)|_v}{|F_U|_v^s |G_U|_v^r} \ll E_{ij} \ll 1 \quad \textit{for } i = 1, \ldots, r,\ j = 1, \ldots, s. \tag{7.9}$$

Proof. (7.5) and (7.6) are immediate consequences of (7.1), (7.2) and Lemma 3.1. By (7.3) and $ad - bc = 1$ we have $\alpha^{(p)} - \alpha^{(q)} = \gamma^{(p)}\delta^{(q)} - \gamma^{(q)}\delta^{(p)}$, hence

$$\Delta_{pq} = \frac{|\gamma^{(p)}\delta^{(q)} - \gamma^{(q)}\delta^{(p)}|_v}{|\gamma^{(p)}, \delta^{(p)}|_v \cdot |\gamma^{(q)}, \delta^{(q)}|_v}.$$

This implies $\Delta_{pq} \ll 1$ for $1 \le p, q \le r$, $p \ne q$. From (3.10) and (3.7) it follows that $\prod_{1 \le p < q \le r} \Delta_{pq} \gg |D(F_U)|_v^{1/2}/|F_U|_v^{r-1} = |D(F)|_v^{1/2}/|F_U|_v^{r-1}$. This implies for each Δ_{pq} the lower bound in (7.7). Using that by (7.3), $ad - bc = 1$ we have $\beta^{(p)} - \beta^{(q)} = \xi^{(p)}\eta^{(q)} - \xi^{(q)}\eta^{(p)}$, we can prove (7.8) in precisely the same way. By (7.3) and $ad - bc = 1$ we have $\alpha^{(i)} - \beta^{(j)} = \gamma^{(i)}\eta^{(j)} - \delta^{(i)}\xi^{(j)}$ and so

$$E_{ij} = \frac{|\gamma^{(i)}\eta^{(j)} - \delta^{(j)}\xi^{(i)}|_v}{|\gamma^{(i)}, \delta^{(i)}|_v \cdot |\xi^{(j)}, \eta^{(j)}|_v}.$$

This implies $E_{ij} \ll 1$ for all i, j. Further, by (3.11), (3.9) we have $\prod_{i=1}^r \prod_{j=1}^s E_{ij} \gg |R(F_U, G_U)|_v/|F_U|_v^s |G_U|_v^r = |R(F,G)|_v/|F_U|_v^s |G_U|_v^r$. This implies for each E_{ij} the lower bound in (7.9). □

We assume for the moment

$$f_1 = \min(f_1, \ldots, f_r, g_1, \ldots, g_s). \tag{7.10}$$

Lemma 7.2. *Assume (7.10). Then*

$$f_q \gg |D(F)|_v^{1/2} \cdot |F|_v^{1/r} \cdot |F_U|_v^{-r+1-\frac{1}{r}} \quad \textit{for } q = 2, \ldots, r. \tag{7.11}$$

Proof. By the vector identity

$$(\alpha^{(1)} - \alpha^{(q)}) \begin{pmatrix} 1 \\ \alpha^{(p)} \end{pmatrix} = (\alpha^{(p)} - \alpha^{(q)}) \begin{pmatrix} 1 \\ \alpha^{(1)} \end{pmatrix} + (\alpha^{(1)} - \alpha^{(p)}) \begin{pmatrix} 1 \\ \alpha^{(q)} \end{pmatrix}$$

we get for all p, q with $1 \leq p,\, q \leq r$,

$$|\alpha^{(1)} - \alpha^{(q)}|_v \cdot |1, \alpha^{(p)}|_v \ll \max\left(|\alpha^{(p)} - \alpha^{(q)}|_v \cdot |1, \alpha^{(1)}|_v,\, |\alpha^{(1)} - \alpha^{(p)}|_v \cdot |1, \alpha^{(q)}|_v\right).$$

By dividing this by $|\gamma^{(1)}, \delta^{(1)}|_v \cdot |\gamma^{(p)}, \delta^{(p)}|_v \cdot |\gamma^{(q)}, \delta^{(q)}|_v$ and then using (7.10) and the upper bound from (7.7) we obtain

$$\Delta_{q1} f_p \ll \max(\Delta_{pq} f_1, \Delta_{1p} f_q) \ll f_q \quad \text{for } p = 1, \ldots, r,\, q = 2, \ldots, r.$$

Together with (7.5) and the lower bound in (7.7) this implies

$$f_q \gg \Delta_{q1}(f_1 \cdots f_r)^{1/r} \gg \frac{|D(F)|_v^{1/2}}{|F_U|_v^{r-1}} \cdot \left(\frac{|F|_v}{|F_U|_v}\right)^{1/r}$$

for $q = 2, \ldots, r$, which is (7.11). $\square$

Lemma 7.3. *Assume (7.10). Then*

$$g_q \gg |R(F,G)|_v \cdot |G|_v^{1/s} \cdot |F_U|_v^{-s} \cdot |G_U|_v^{-r-\frac{1}{s}} \quad \textit{for } q = 1, \ldots, s. \tag{7.12}$$

Proof. We have again a vector identity

$$(\alpha^{(1)} - \beta^{(q)}) \begin{pmatrix} 1 \\ \beta^{(p)} \end{pmatrix} = (\beta^{(p)} - \beta^{(q)}) \begin{pmatrix} 1 \\ \alpha^{(1)} \end{pmatrix} + (\alpha^{(1)} - \beta^{(p)}) \begin{pmatrix} 1 \\ \beta^{(q)} \end{pmatrix}$$

from which we deduce

$$|\alpha^{(1)} - \beta^{(q)}|_v \cdot |1, \beta^{(p)}|_v \ll \max\left(|\beta^{(p)} - \beta^{(q)}|_v \cdot |1, \alpha^{(1)}|_v,\, |\alpha^{(1)} - \beta^{(p)}|_v \cdot |1, \beta^{(q)}|_v\right)$$

for all p, q with $1 \leq p,\, q \leq s$. By dividing this by $|\gamma^{(1)}, \delta^{(1)}|_v \cdot |\xi^{(p)}, \eta^{(p)}|_v \cdot |\xi^{(q)}, \eta^{(q)}|_v$ and then using (7.10) and the upper bounds from (7.8), (7.9) we get

$$E_{1q} g_p \ll \max(\Theta_{pq} f_1, E_{1p} g_q) \ll g_q \quad \text{for } p = 1, \ldots, s,\, q = 1, \ldots, s.$$

Using this together with (7.5) and the lower bound in (7.9) we obtain

$$g_q \gg E_{1q}(g_1 \cdots g_s)^{1/s} \gg \frac{|R(F,G)|_v}{|F_U|_v^s |G_U|_v^r} \cdot \left(\frac{|G|_v}{|G_U|_v}\right)^{1/s}$$

for $q = 1, \ldots, s$, which is (7.12). $\square$

Lemma 7.4. *Assume (7.10). Then there is a pair* $(i_v, j_v) \in \mathcal{E}_v^c$ *such that*

$$f_{i_v} g_{j_v} \gg |D(F)|_v^{1/2} \cdot |R(F,G)|_v \cdot |F|_v^{1/r} |G|_v^{1/s} \cdot |F_U|_v^{-(r+s)+1-\frac{1}{r}} |G_U|_v^{-r-\frac{1}{s}}. \tag{7.13}$$

Proof. We distinguish two cases.

Case 1. $\alpha^{(1)} \in K_v$.
Then $x \mapsto x^{(1)}$ maps K_1 into K_v since $K(\alpha) = K_1$. Hence $\{(1,1), \ldots, (1,s)\}$ is a K_v-row. So $\mathcal{E}_v^c \not\subset \{(1,1), \ldots, (1,s)\}$ by our assumption. Therefore, there is a pair $(i_v, j_v) \in \mathcal{E}_v^c$ with $i_v \in \{2, \ldots, r\}$ and $j_v \in \{1, \ldots, s\}$. Now we obtain (7.13) by combining (7.11) with $q = i_v$ and (7.12) with $q = j_v$.

Case 2. $\alpha^{(1)} \notin K_v$.
The set $\mathcal{E}_v^c$ is not empty. Pick any pair $(i_v, j_v) \in \mathcal{E}_v^c$. If $i_v \neq 1$ we derive again (7.13) from (7.11) with $q = i_v$ and from (7.12) with $q = j_v$. Suppose $i_v = 1$. There is a $h \in \{2, \ldots, r\}$ such that $\alpha^{(h)}$ is conjugate to $\alpha^{(1)}$ over K_v. Then $\xi^{(h)}$, $\eta^{(h)}$ are conjugate over K_v to $\xi^{(1)}$, $\eta^{(1)}$, respectively. Since numbers conjugate over K_v have the same $|\cdot|_v$-value, this implies $f_{i_v} = f_1 = f_h$. Now (7.13) follows from (7.11) with $q = h$ and from (7.12) with $q = j_v$. □

We now drop assumption (7.10). Then in general we have:

Lemma 7.5. *There is a pair* $(i_v, j_v) \in \mathcal{E}_v^c$ *such that*

$$\frac{|\alpha^{(i_v)} - \beta^{(j_v)}|_v}{|1, \alpha^{(i_v)}|_v \cdot |1, \beta^{(j_v)}|_v} \ll |D(F)|_v^{-1/2} |D(G)|_v^{-1/2} |R(F,G)|_v^{-1} \cdot$$
$$\cdot |F_U|_v^{(r+s)-1+\frac{1}{r}} |G_U|_v^{(r+s)-1+\frac{1}{s}} |F|_v^{-1/r} |G|_v^{-1/s}. \tag{7.14}$$

Proof. The right-hand side of (7.14) remains unchanged if the pairs (F, r) and (G, s) are interchanged. Further, it remains unchanged if $f_1, \ldots, f_r$ are permuted or if $g_1, \ldots, g_s$ are permuted. Hence there is no loss of generality to assume $f_1 = \min(f_1, \ldots, f_r, g_1, \ldots, g_s)$, i.e., (7.10). Therefore, we may apply Lemma 7.4. Let (i_v, j_v) be the pair from this lemma. Then from (7.4), (7.9) (the upper bound) and (7.13) it follows

$$\frac{|\alpha^{(i_v)} - \beta^{(j_v)}|_v}{|1, \alpha^{(i_v)}|_v \cdot |1, \beta^{(j_v)}|_v} = E_{i_v, j_v} (f_{i_v} g_{j_v})^{-1} \ll (f_{i_v} g_{j_v})^{-1}$$
$$\ll |D(F)|_v^{-1/2} |R(F,G)|_v^{-1} \cdot |F_U|_v^{(r+s)-1+\frac{1}{r}} |G_U|_v^{r+\frac{1}{s}} |F|_v^{-1/r} |G|_v^{-1/s}.$$

By multiplying the right-hand side with $|D(G)|_v^{-1/2} |G_U|_v^{s-1}$ which is $\gg 1$ by (7.8) we arrive at (7.14). □

Proof of Proposition 6.2. Choose for each $v \in S$ a pair (i_v, j_v) with (7.14) and take the product over $v \in S$. Since the binary forms F, G have S-integral coefficients, are irreducible and have no common factor, the numbers $D(F)$, $D(G)$, $R(F,G)$ are non-zero S-integers and so $|D(F)|_S \geq 1$, $|D(G)|_S \geq 1$, $|R(F,G)|_S \geq 1$ by (3.1). Further, we have $H_S(F_U) \geq 1$, $H_S(G_U) \geq 1$ by (3.2). Hence

$$\begin{aligned}
&\prod_{v\in S} \frac{|\alpha^{(i_v)} - \beta^{(j_v)}|_v}{|1, \alpha^{(i_v)}|_v \cdot |1, \beta^{(j_v)}|_v} \\
&\qquad \ll |D(F)|_S^{-1/2} |D(G)|_S^{-1/2} |R(F,G)|_S^{-1} \cdot \\
&\qquad\qquad \cdot H_S(F_U)^{(r+s)-1+\frac{1}{r}} H_S(G_U)^{(r+s)-1+\frac{1}{s}} H_S(F)^{-1/r} H_S(G)^{-1/s} \\
&\qquad \ll H_S(F_U)^{r+s} H_S(G_U)^{r+s} \cdot H_S(F)^{-1/r} H_S(G)^{-1/s},
\end{aligned}$$

which is what we wanted to prove. □

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