(1)

UITWERKING HERKANSING 2^E DEELTENTAMEN
CONTINUE WISKUNDE, 27-3-2012

(1) $f(x) = \dfrac{x^5+x^4+1}{x^4}$

a) $f(-\tfrac{3}{2}) = \dfrac{-(3/2)^5+(3/2)^4+1}{(3/2)^4} = \dfrac{-\frac{1}{2}\cdot\frac{81}{16}+1}{(3/2)^4} = \dfrac{-\frac{65}{32}}{(3/2)^4} < 0,$

$f(-1) = \dfrac{-1+1+1}{1^4} = 1 > 0$

Dus f heeft een nulpunt in $(-\tfrac{3}{2}, -1)$.

b) Het domein van f is $\mathbb{R}\setminus\{0\}$

Verticale asymptoten: alleen in $x=0$.
$\lim_{x\downarrow 0} f(x) = \infty$ (noemer $>0$, teller $>0$ in de buurt van 0)
$\lim_{x\uparrow 0} f(x) = \infty$

$x^4 / x^5+x^4+1 \backslash x+1$
$x^5$
$x^4+1$
$x^4$
$1$

Scheve asymptoten: $f(x) = x+1+\dfrac{1}{x^4}$, $\lim_{x\to\infty} f(x)-x-1=0$, $\lim_{x\to-\infty} f(x)-x-1=0$,

Dus $y=x+1$ is een scheve asymptoot van $f(x)$ zowel voor $x\to\infty$ als $x\to-\infty$, en de grafiek van f ligt boven de lijn $y=x+1$.

Er zijn scheve asymptoten voor $x\to\pm\infty$, dus geen horizontale asymptoten

$f'(x) = \dfrac{x^4\cdot(5x^4+4x^3)-4x^3(x^5+x^4+1)}{x^8} = \dfrac{5x^8+4x^7-4x^8-4x^7-4x^3}{x^8}$

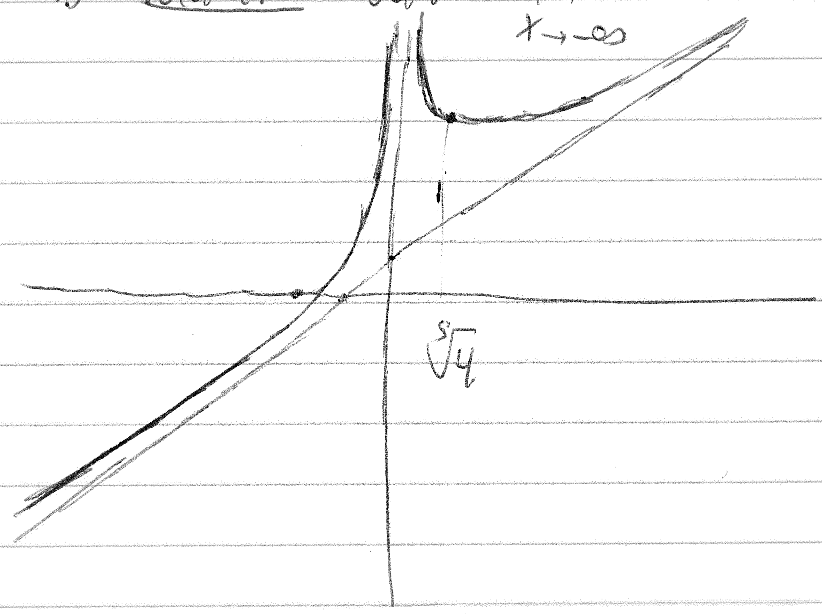
$= \dfrac{x^8-4x^3}{x^8} = \dfrac{x^5-4}{x^5}$

f': + NG − +
0, $\sqrt[5]{4}$
f: ↗ ↘ ↗

f heeft een minimum voor $x = \sqrt[5]{4}$, grootte

$$f(\sqrt[5]{4}) = \frac{(\sqrt[5]{4})^5 + (\sqrt[5]{4})^4 + 1}{(\sqrt[5]{4})^4} = \frac{5 + (\sqrt[5]{4})^4}{(\sqrt[5]{4})^4} = 1 + 5\sqrt[5]{\frac{1}{4}}$$

Dit minimum is relatief want $\lim_{x \to -\infty} f(x) = -\infty$




2 a) $$\int_1^e \frac{dx}{x\sqrt{1+\ln x}} \underset{\uparrow}{=} \int_1^2 \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_1^2 = \boxed{2\sqrt{2} - 2}$$

$u = 1 + \ln x$
$du = \frac{1}{x} dx$
$x = 1 \to u = 1$
$x = e \to u = 2$

b) $$\int \cos x \cdot \ln(\sin x)\, dx \underset{\uparrow}{=} \int \ln u \cdot du \underset{\uparrow}{=} (\ln u) u - \int u \cdot \ln' u\, du$$

$u = \sin x$
$du = \cos x\, dx$

$f(u) = \ln u$
$g'(u) = 1$
$g(u) = u$

$= u \ln u - \int du$

$= u \ln u - u + C$

$= \boxed{\sin x \cdot \ln(\sin x) - \sin x + C}$

3 $f(x,y) = x^4 + x^2y^2 - x^2 - y^2$

a) $\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} x^4 - x^2 = \lim_{x\to\infty} x^4(1-\frac{1}{x^2}) = \infty$

$\lim_{y\to\infty} f(0,y) = \lim_{y\to\infty} -y^2 = -\infty$

b) $\frac{\partial f}{\partial x} = 4x^3 + 2x.y^2 - 2x$, $\quad \frac{\partial f}{\partial y} = 2x^2.y - 2y = 2y(x^2-1) = 2y(x+1)(x-1)$

$(x,y)$ stationair punt van $f \Leftrightarrow \frac{\partial f}{\partial x} = 0$ en $\frac{\partial f}{\partial y} = 0$

Er geldt: $\frac{\partial f}{\partial y} = 0 \Leftrightarrow y=0$ of $x=1$ of $x=-1$.

$y=0$ invullen in $\frac{\partial f}{\partial x} = 0$ geeft: $4x^3 - 2x = 0 \Leftrightarrow 2x(2x^2-1) = 0$
$\Leftrightarrow x=0$ of $x^2 = \frac{1}{2}$
$\Leftrightarrow x=0$ of $x = \frac{1}{2}\sqrt{2}$ of $x = -\frac{1}{2}\sqrt{2}$

Dit geeft 3 stationaire punten $(0,0)$, $(\frac{1}{2}\sqrt{2},0)$, $(-\frac{1}{2}\sqrt{2},0)$

$x=1$ invullen in $\frac{\partial f}{\partial x} = 0$ geeft: $2+2y^2=0$; dit heeft geen opl$^n$.
$x=-1$ invullen in $\frac{\partial f}{\partial x} = 0$ geeft: $-2-2y^2=0$; dit heeft ook geen opl$^n$.

c) $\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2y^2 - 2$, $\quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 4xy$, $\quad \frac{\partial^2 f}{\partial y^2} = 2x^2 - 2$

| stat. pnt | $A = \frac{\partial^2 f}{\partial x^2}$ | $B = \frac{\partial^2 f}{\partial y \partial x}$ | $C = \frac{\partial^2 f}{\partial y^2}$ | $H = AC - B^2$ | conclusie |
|---|---|---|---|---|---|
| $(0,0)$ | $-2<0$ | $0$ | $-2$ | $4>0$ | maximum. |
| $(\frac{1}{2}\sqrt{2},0)$ | $4>0$ | $0$ | $-1$ | $-4<0$ | zadelpunt. |
| $(-\frac{1}{2}\sqrt{2},0)$ | $4>0$ | $0$ | $-1$ | $-4<0$ | zadelpunt. |

Het maximum in $(0,0)$ is <u>relatief</u> want $\lim_{x\to\infty} f(x,0) = \infty$ (zie a)
De <u>grootte</u> van het maximum is $f(0,0) = 0$.

4 a) We gebruiken $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ als $-1<r<1$.

$$\sum_{n=0}^{\infty}\left\{\left(\frac{2}{3}\right)^n + 3\left(\frac{1}{5}\right)^n\right\} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + 3 \cdot \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$$

$$= \frac{1}{1-\frac{2}{3}} + \frac{3}{1-\frac{1}{5}} = \frac{1}{\frac{1}{3}} + \frac{3}{\frac{4}{5}} = 3 + \frac{15}{4} = \boxed{\frac{27}{4}}$$

b) We gebruiken het <u>criterium van d'Alembert</u>

Stel $a_n \geq 0$ voor alle $n$. , ~~Dan is~~ en $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

Dan is $\sum_{n=0}^{\infty} a_n$ convergent

Pas dit toe met $a_n = \frac{n}{2^n}$. Dan is $\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+1}{2^{n+1}}\right)}{\left(\frac{n}{2^n}\right)} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$

Dus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2} < 1.$$

Dus $\sum_{n=0}^{\infty} a_n$ <u>convergeert</u>

5 a) $$\int \frac{dx}{x(\ln x)^{3/2}} \underset{\substack{u=\ln x \\ du=\frac{1}{x}dx}}{=} \int \frac{du}{u^{3/2}} = -2 \cdot u^{-\frac{1}{2}} + C = \boxed{-2(\ln x)^{-\frac{1}{2}} + C}$$

b) We gebruiken het integraalkenmerk:
Zij $f(x)$ continu, $\geq 0$ en ~~monotoon~~ niet-stijgend op $[a,\infty)$ dan geldt:

$$\sum_{n=a}^{\infty} f(n) \text{ convergent} \Leftrightarrow \int_a^{\infty} f(x)dx \text{ convergent}$$

Pas dit toe met $f(x) = \dfrac{1}{x(\ln x)^{3/2}}$

Dan is $$\int_2^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{A\to\infty} \int_2^A \frac{dx}{x(\ln x)^{3/2}} = \lim_{A\to\infty} \left[-2(\ln x)^{-\frac{1}{2}}\right]_2^A$$

$$= \lim_{A\to\infty} \left(2(\ln 2)^{-\frac{1}{2}} - 2(\ln A)^{-\frac{1}{2}}\right) = 2(\ln 2)^{-\frac{1}{2}} < \infty$$

Verder is $f(x) \geq 0$ en niet-stijgend op $[2,\infty)$ omdat $x$ en $\ln x$ stijgend zijn op $[2,\infty)$. Dus
$\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^{3/2}}$ convergeert.

HERKANSING CONTINUE WISKUNDE (HELE STOF)
27-3-2012

① a) $\lim_{x\to\infty} \sqrt{x^4+x} - x^2 = \lim_{x\to\infty} \frac{(\sqrt{x^4+x}-x^2)(\sqrt{x^4+x}+x^2)}{\sqrt{x^4+x}+x^2}$

$= \lim_{x\to\infty} \frac{x^4+x-x^4}{\sqrt{x^4+x}+x^2} = \lim_{x\to\infty} \frac{x}{\sqrt{x^4+x}+x^2} = \lim_{x\to\infty} \frac{\overbrace{\frac{1}{x}}^{\to 0}}{\sqrt{1+\underbrace{\frac{1}{x^3}}_{\to 0}}+1} = \boxed{0}$

b) $\lim_{x\to 0} \frac{\cos x - 1}{e^{x^2}-1} \underset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\sin x}{2x \cdot e^{x^2}} =$

$= \lim_{x\to 0} \underbrace{\frac{1}{e^{x^2}}}_{\to 1} \cdot \lim_{x\to 0} \frac{-\sin x}{2x} \underset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\cos x}{2} = \boxed{-\frac{1}{2}}$

② $f(x) = \ln x + \ln(x+1)$

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ | $f^{(n)}(1)/n!$ |
|---|---|---|---|
| 0 | $\ln x + \ln(x+1)$ | $\ln 2$ | $\ln 2$ |
| 1 | $\frac{1}{x} + \frac{1}{x+1}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| 2 | $-\frac{1}{x^2} - \frac{1}{(x+1)^2}$ | $-\frac{5}{4}$ | $-\frac{5}{4} \cdot \frac{1}{2} = -\frac{5}{8}$ |
| 3 | $\frac{2}{x^3} + \frac{2}{(x+1)^3}$ | $\frac{9}{4}$ | $\frac{9}{4} \cdot \frac{1}{6} = \frac{9}{24}$ |

$\boxed{P_3(x) = \ln 2 + \frac{3}{2}(x-1) - \frac{5}{8}(x-1)^2 + \frac{9}{24}(x-1)^3}$

③ $f_c(x) = \begin{cases} c + cx - x^2 & \text{voor } x > 1 \\ c^2 \sin \frac{\pi x}{2} & \text{voor } x \leq 1 \end{cases}$

~~lim~~ Er geldt: $f_c$ continu in $x=1 \iff$

$\lim_{x\downarrow 1} f_c(x) = \lim_{x\uparrow 1} f_c(x) = f_c(1)$

Er geldt:

$$f_c(1) = c^2 \sin\frac{\pi}{2} = c^2,$$

$$\lim_{x \downarrow 1} f_c(x) = \lim_{x \downarrow 1} c + cx - x^2 = 2c, \quad \lim_{x \uparrow 1} f_c(x) = \lim_{x \uparrow 1} c^2 \sin\frac{\pi x}{2} = c^2$$

Dus $f_c$ is continu in $x=1$

$$\Leftrightarrow \quad 2c = c^2 = c^2 \quad \Leftrightarrow \quad c^2 - 2c = 0 \quad \Leftrightarrow \quad c(c-2) = 0 \Leftrightarrow \boxed{c=0 \text{ of } c=2}$$

opgaven 4, 5, 6, 7:

Zie de uitwerking van de opgaven 1, 2, 3, 4 van de herhaling van het 2e deeltentamen.