UITWERKING CONTINUE WISKUNDE 2 29/3/2016

(1) a) Bepaal eerst de primitieven van $\frac{e^x}{(e^x+3)^{3/2}}$. Er geldt:

$$\int \frac{e^x dx}{(e^x+3)^{3/2}} \underset{\substack{u=e^x+3 \\ du=e^x dx}}{=} \int \frac{du}{u^{3/2}} = -2u^{-\frac{1}{2}}+C = -2(e^x+3)^{-\frac{1}{2}}+C$$

Dus $\int_0^\infty \frac{e^x dx}{(e^x+3)^{3/2}} = \lim_{B\to\infty} \int_0^B \frac{e^x dx}{(e^x+3)^{3/2}} = \lim_{B\to\infty} \left[-2(e^x+3)^{-\frac{1}{2}}\right]_0^B$

$$= \lim_{B\to\infty} \underbrace{-2(e^B+3)^{-\frac{1}{2}}}_{\to 0} + 2.(e^0+3)^{-\frac{1}{2}} = 2(1+3)^{-\frac{1}{2}} = \boxed{1}$$

b) $$\int x \sin 2x\, dx \underset{\substack{f(x)=x \\ g'(x)=\sin 2x \\ g(x)=-\frac{1}{2}\cos 2x}}{=} x.\left(-\frac{1}{2}\cos 2x\right) - \int -\left(\frac{1}{2}\cos 2x\right) 1.dx$$

$$= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x\, dx$$

$$= \boxed{-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C}$$

c) Het omwentelingslichaam van het gebied begrensd door de x-as, de grafiek van $f(x)$ en $x=a$, $x=b$ is $\int_a^b \pi f(x)^2 dx$. In ons geval geeft dit

$$\int_1^2 \pi (x\sqrt{x})^2 dx = \int_1^2 \pi x^3 dx = \left[\frac{1}{4}\pi x^4\right]_1^2 = \frac{1}{4}\pi(16-1) = \boxed{\frac{15}{4}\pi}$$

(2) a) $\lim_{\substack{x\to\infty \\ y=0}} f(x,y) = \lim_{x\to\infty} x^5-5x = \infty$, $\lim_{\substack{x\to-\infty \\ y=0}} f(x,y) = \lim_{x\to-\infty} x^5-5x = -\infty$

Dus f kan geen absoluut maximum of absoluut minimum aannemen

b) $\frac{\partial f}{\partial x} = 5x^4+y^2-5$, $\frac{\partial f}{\partial y} = 2xy$. We moeten oplossen: $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$

Uit $\frac{\partial f}{\partial y}=0$ volgt: $2xy=0$ dus $x=0$ of $y=0$

$x=0$ invullen in $\frac{\partial f}{\partial x}=0$ geeft $y^2-5=0$ dus $y=\pm\sqrt{5}$
Dit geeft de stationaire punten $(0,\sqrt{5})$, $(0,-\sqrt{5})$
$y=0$ invullen in $\frac{\partial f}{\partial x}=0$ geeft $5x^4-5=0$, $x^4=1$, dus $x=\pm 1$
Dit geeft de stationaire punten $(1,0)$, $(-1,0)$

c) $\frac{\partial^2 f}{\partial x^2}=20x^3=A$, $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=2y=B$, $\frac{\partial^2 f}{\partial y^2}=2x=C$ $\quad H=AC-B^2$

	A	B	C	H	
$(0,\sqrt{5})$	0	$2\sqrt{5}$	0	$-20<0$	zadelpunt
$(0,-\sqrt{5})$	0	$-2\sqrt{5}$	0	$-20<0$	zadelpunt
$(1,0)$	20	0	2	$40>0$	minimum, relatief wegens a)
$(-1,0)$	-20	0	-2	$40>0$	maximum, relatief wegens a)

d) Algemeen: de vergelijking van het raakvlak aan de grafiek van f in $(x_0,y_0,f(x_0,y_0))$ is $z=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

In ons geval: $(x_0,y_0)=(1,1)$, $f(1,1)=1+1-5=-3$,
$\frac{\partial f}{\partial x}(1,1)=5+1-5=1$, $\frac{\partial f}{\partial y}(1,1)=2$. Dus de vergelijking wordt:

$$z=\boxed{-3+1.(x-1)+2(y-1)}=x+2y-6$$

3 a) $\frac{1}{3+i}+\frac{1}{7-i}=\frac{3-i}{3^2+1^2}+\frac{7+i}{7^2+(-1)^2}=\frac{3-i}{10}+\frac{7+i}{50}$

$$=\frac{15-5i}{50}+\frac{7+i}{50}=\frac{22-4i}{50}=\boxed{\frac{11}{25}-\frac{2}{25}i}$$

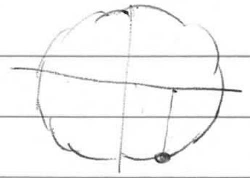
b) Zoek x_1,x_2 zodat $z^2+(5-i)z-5i=(z+x_1)(z+x_2)$ ofwel
$x_1+x_2=5-i$, $x_1x_2=-5i$. $x_1=5$, $x_2=-i$ voldoen. Dus
$z^2+(5-i)z-5i=(z+5)(z-i)$

Hieruit volgt: $z^2+(5-i)z-5i=0 \Leftrightarrow (z+5)(z-i)=0 \Leftrightarrow \boxed{z=-5 \text{ of } z=i}$

c) Schrijf eerst $5-5\sqrt{3}\,i$ in de vorm $r(\cos\varphi + i\sin\varphi)$ met $r\geq 0$

$r = |5-5\sqrt{3}\,i| = \sqrt{5^2+(-5\sqrt{3})^2} = \sqrt{5^2+5^2\cdot 3} = \sqrt{100} = 10$

$\cos\varphi = \frac{5}{10} = \frac{1}{2}, \quad \sin\varphi = \frac{-5\sqrt{3}}{10} = -\frac{1}{2}\sqrt{3}.$



Dus $5-5\sqrt{3}\,i = 10\left(\cos\left(-\tfrac{1}{3}\pi\right) + i\sin\left(-\tfrac{1}{3}\pi\right)\right)$

Hieruit volgt: $(5-5\sqrt{3}\,i)^5 = 10^5\left(\cos\left(-\tfrac{5}{3}\pi\right) + i\sin\left(-\tfrac{5}{3}\pi\right)\right)$

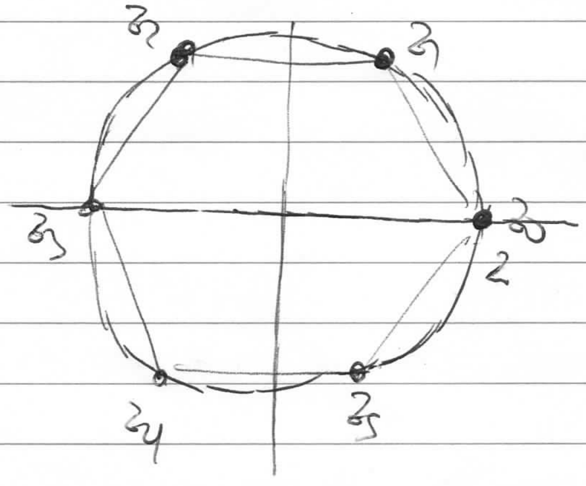
$= 10^5\left(\cos\tfrac{1}{3}\pi + i\sin\tfrac{1}{3}\pi\right) = \boxed{10^5\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i\right)}$

d) Schrijf eerst $64 = \rho(\cos\psi + i\sin\psi)$. Duidelijk: $64 = \rho(\cos 0 + i\sin 0)$

Oplossingen van $z^6 = 64$:

$z_k = \sqrt[6]{64}\left(\cos\frac{0+2k\pi}{6} + i\sin\frac{0+2k\pi}{6}\right) \qquad k=0,1,2,3,4,5$

$= 2\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right)$



4 a) Gebruik het quotiëntkenmerk: als $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$ dan is $\sum_{k=0}^{\infty} a_k$ convergent, als $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$ dan is $\sum_{k=0}^{\infty} a_k$ divergent. Pas dit toe met $a_k = \frac{1}{\sqrt{k!}}$

$a_k = \frac{1}{\sqrt{k!}}$, $a_{k+1} = \frac{1}{\sqrt{(k+1)!}}$, $\frac{a_{k+1}}{a_k} = \frac{1}{\sqrt{(k+1)!}} \Big/ \frac{1}{\sqrt{k!}} = \frac{\sqrt{k!}}{\sqrt{(k+1)!}} = \sqrt{\frac{k!}{(k+1)!}}$

$$= \sqrt{\frac{1\cdot 2 \cdots k}{1 \cdot 2 \cdots k(k+1)}} = \frac{1}{\sqrt{k+1}}.$$

Dus $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{1}{\sqrt{k+1}} = 0$

Dus $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}}$ is $\boxed{\text{convergent.}}$

b) Gebruik het vergelijkingskenmerk: neem aan dat $a_k \geq 0$, $b_k \geq 0$ voor alle k, en $\lim_{k\to\infty} \frac{a_k}{b_k} = l$ met $0 < l < \infty$.

Dan $\sum_{k=0}^{\infty} b_k$ convergent $\Rightarrow \sum_{k=0}^{\infty} a_k$ convergent,

$\sum_{k=0}^{\infty} b_k$ divergent $\Rightarrow \sum_{k=0}^{\infty} b_k$ divergent

Pas dit toe met $a_k = \frac{k^3+1}{2k^7-1}$, $b_k = k^{-\alpha}$ voor geschikte α

Merk op: $a_k \approx \frac{k^3}{2k^7} \approx \frac{1}{2k^4}$ voor k heel groot. Neem $b_k = k^{-4}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^3+1}{2k^7-1} \Big/ k^{-4} = \lim_{k\to\infty} \frac{k^4(k^3+1)}{2k^7-1} = \lim_{k\to\infty} \frac{k^7+k^4}{2k^7-1}$$

$$= \lim_{k\to\infty} \frac{1+k^{-3}}{2-k^{-7}} = \frac{1}{2}.$$

De reeks $\sum_{k=1}^{\infty} k^{-4}$ is convergent. Dus $\sum_{k=1}^{\infty} \frac{k^3+1}{2k^7-1}$ is $\boxed{\text{convergent}}$