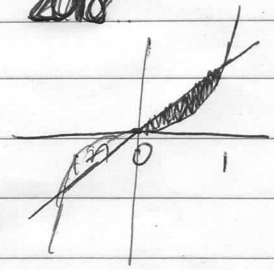
UITWERKING HERKANSING CONTINUE WISKUNDE 2

5 JULI 2018

1 a) 

Snijpunten van de grafieken van f en g

$x^3 = x \Leftrightarrow x^3 - x = 0 \Leftrightarrow x(x^2-1) = 0$

$\Leftrightarrow x(x+1)(x-1) = 0 \Leftrightarrow x = 0, 1, -1.$

We bekijken alleen het stuk rechts van de y-as. Dus de oppervlakte van het gevraagde gebied is

$$\int_0^1 (f(x)-g(x))\,dx = \int_0^1 (x - x^3)\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{4} = \boxed{\tfrac{1}{4}}$$

b) Gebruik de substitutieregel:

$$\int e^{-(2\sin x)^2} \sin x \cos x\,dx = \int e^{-u} \cdot \tfrac{1}{8}\,du = -\tfrac{1}{8}e^{-u} + C = \boxed{-\tfrac{1}{8}e^{-(2\sin x)^2} + C}$$

$u = (2\sin x)^2 = 4\sin^2 x$

$du = 8\sin x \cos x\,dx$

$\sin x \cos x\,dx = \tfrac{1}{8}\,du$

c) $$\int \ln x^{-1}\,dx = (\ln x^{-1})x - \int x \cdot \left(-\frac{1}{x^2}\right)\frac{1}{x^{-1}}\,dx = (-\ln x)x + \int dx$$

$f(x) = \ln x^{-1}$
$g'(x) = 1$
$g(x) = x$

$$= -x\ln x + x + C$$

Dus $$\int_0^1 (\ln x^{-1})\,dx = \lim_{\delta \downarrow 0} \int_\delta^1 (\ln x^{-1})\,dx = \lim_{\delta \downarrow 0} \left[-x\ln x + x\right]_\delta^1$$

$$= \lim_{\delta \downarrow 0} \left(-1 \cdot \ln 1 + 1 - (-\delta \ln \delta + \delta)\right) = 1$$

wl. $\lim_{\delta \downarrow 0} \delta \ln \delta = 0$ (standaardlimiet $\lim_{\delta \downarrow 0} \delta^a \ln \delta = 0$ voor $a > 0$)

2 a) $f(x,y) = x^6 + xy^2 - x$. Als we $y = x^3$ substitueren krijgen we $x^6 + x^7 - x$. Nu geldt

$$\lim_{\substack{x\to\infty \\ y=x^3}} f(x,y) = \lim_{x\to\infty} x^6 + x^7 - x = \infty, \quad \lim_{\substack{x\to-\infty \\ y=x^3}} f(x,y) = \lim_{x\to-\infty} x^6 + x^7 - x = -\infty$$

Dus f neemt geen absolute maxima of minima aan.

b) $\frac{\partial f}{\partial x} = 6x^5 + y^2 - 1$, $\frac{\partial f}{\partial y} = 2xy$

stationaire punten: $6x^5 + y^2 - 1 = 0$ en $2xy = 0$

uit $2xy = 0$ volgt dat $x = 0$ of $y = 0$

$x = 0$ geeft: $y^2 - 1 = 0$ dus $y = 1$ of $y = -1$. Dit geeft de stationaire punten $(0,1), (0,-1)$

$y = 0$ geeft $6x^5 - 1 = 0$, dus $x^5 = \frac{1}{6}$, $x = \left(\frac{1}{6}\right)^{1/5}$. Dit geeft het stationaire punt $\left(\left(\frac{1}{6}\right)^{1/5}, 0\right) = (6^{-1/5}, 0)$

c) $\frac{\partial^2 f}{\partial x^2} = 30x^4 = A$, $\frac{\partial^2 f}{\partial y \partial x} = 2y = B$, $\frac{\partial^2 f}{\partial y^2} = 2x = C$, $H = AC - B^2$

	A	B	C	H	
$(6^{-1/5}, 0)$	$30 \times 6^{-4/5} > 0$	0	$2 \times 6^{-1/5}$	$60 \times 6^{-1} = 10 > 0$	relatief minimum *)
$(0, 1)$	0	2	0	$-4 < 0$	zadelpunt
$(0, -1)$	0	-2	0	$-4 < 0$	zadelpunt

*) Het minimum kan niet absoluut zijn wegens a)

d) vergelijking raakvlak

$$z = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$$

$$= 0 + (-1)x + 0 \cdot y, \quad \boxed{z = -x}$$

③

③ a) $|z| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$,

$\bar{w} = 5-12i$, $|\bar{w}| = \sqrt{5^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$,

$$\left|\frac{z^2}{\bar{w}}\right| = \frac{|z^2|}{|\bar{w}|} = \frac{|z|\cdot|z|}{|\bar{w}|} = \frac{5\cdot 5}{13} = \boxed{\frac{25}{13}}$$

b) $e^{\frac{1}{6}\pi i} = \cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$e^{\frac{1}{4}\pi i} = \cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,i$

$e^{\frac{5}{12}\pi i} = e^{\frac{1}{6}\pi i}\cdot e^{\frac{1}{4}\pi i} = \left(\frac{1}{2}\sqrt{3}+\frac{1}{2}i\right)\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\,i\right)$

$= \frac{1}{2}\sqrt{3}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3}\cdot\frac{1}{2}\sqrt{2}\,i + \frac{1}{2}i\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\cdot\frac{1}{2}\sqrt{2}\cdot i^2$

$= \frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{6}\,i + \frac{1}{4}\sqrt{2}\,i - \frac{1}{4}\sqrt{2} = \boxed{\frac{1}{4}\sqrt{6} - \frac{1}{4}\sqrt{2} + \left(\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2}\right)i}$

c) Schrijf $5-5\sqrt{3}i = r(\cos\varphi + i\sin\varphi)$

$r = |5-5\sqrt{3}i| = \sqrt{5^2+(-5\sqrt{3})^2} = \sqrt{25+75} = \sqrt{100} = 10$

$\cos\varphi = \frac{5}{10} = \frac{1}{2}$, $\sin\varphi = \frac{-5\sqrt{3}}{10} = -\frac{1}{2}\sqrt{3}$

$5-5\sqrt{3}i = 10\left(\cos(-\frac{1}{3}\pi) + i\sin(-\frac{1}{3}\pi)\right)$,

$(5-5\sqrt{3}i)^{10} = 10^{10}\left(\cos(-\frac{10}{3}\pi) + i\sin(-\frac{10}{3}\pi)\right) = 10^{10}\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$

$= \boxed{10^{10}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\,i\right)}$

d) $z^8 = -6561$

Schrijf $-6561 = \rho(\cos\psi + i\sin\psi)$, $\rho = |-6561| = 6561$,

$\cos\psi = -1$, $\sin\psi = 0 \Rightarrow -6561 = 6561(\cos\pi + i\sin\pi)$

$$\Rightarrow z = \sqrt[8]{6561}\left(\cos\left(\frac{\pi}{8}+\frac{2k\pi}{8}\right) + i\sin\left(\frac{\pi}{8}+\frac{2k\pi}{8}\right)\right) = \boxed{3\left(\cos\left(\frac{\pi}{8}+\frac{k\pi}{4}\right) + i\sin\left(\frac{\pi}{8}+\frac{k\pi}{4}\right)\right) \quad (k=0,1,\ldots,7)}$$

4 a) $$\sum_{k=0}^{\infty} \frac{10^k + (-11)^k}{12^k} = \sum_{k=0}^{\infty} \frac{10^k}{12^k} + \sum_{k=0}^{\infty} \frac{(-11)^k}{12^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{10}{12}\right)^k + \sum_{k=0}^{\infty} \left(\frac{-11}{12}\right)^k = \frac{1}{1-\frac{10}{12}} + \frac{1}{1-\frac{-11}{12}} = \frac{1}{2/12} + \frac{1}{23/12}$$

$$= \frac{12}{2} + \frac{12}{23} = 6 + \frac{12}{23} = \boxed{6\,\frac{12}{23}}$$

b) Voor grote waarden van k geldt:

$$\frac{k^2+\ln k}{k^6+k} \approx \frac{k^2}{k^6} \approx \frac{1}{k^4}$$

Pas het vergelijkingskenmerk toe met $a_k = \frac{k^2+\ln k}{k^6+k}$, $b_k = \frac{1}{k^4}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{k^2+\ln k}{k^6+k} \Big/ \frac{1}{k^4} = \lim_{k\to\infty} \frac{k^4(k^2+\ln k)}{k^6+k} = \lim_{k\to\infty} \frac{k^6+k^4\ln k}{k^6+k}$$

$$= \lim_{k\to\infty} \frac{1+\overbrace{\frac{\ln k}{k^2}}^{\to 0}}{1+\underbrace{\frac{1}{k^5}}_{\to 0}} = 1$$

De reeks $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$ convergeert. Dus de reeks

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2+\ln k}{k^6+k}$ $\boxed{\text{convergeert}}$

c) Pas het quotiëntkenmerk toe met $a_k = \frac{3^k}{\sqrt[4]{k!}}$

$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}}{\sqrt[4]{(k+1)!}} \Big/ \frac{3^k}{\sqrt[4]{k!}} = \frac{3^{k+1}}{3^k} \cdot \frac{\sqrt[4]{k!}}{\sqrt[4]{(k+1)!}} = \frac{3^{k+1}}{3^k} \cdot \sqrt[4]{\frac{1\cdot 2 \ldots k}{1\cdot 2 \ldots k(k+1)}}$$

$$= 3 \cdot \sqrt[4]{\frac{1}{k+1}} = \frac{3}{\sqrt[4]{k+1}}.$$ Dus $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 0.$

Volgens het quotiëntkenmerk is $\sum_{k=0}^{\infty} \frac{3^k}{\sqrt[4]{k!}}$ $\boxed{\text{convergent}}$