UITWERKING CONTINUE WISKUNDE 2, 2/7/2020
VERSIE 1

1 a) De inhoud van het omwentelingslichaam van het gebied begrensd door de lijnen $x=a$, $x=b$ en de grafiek van $f(x)$ is $\int_a^b \pi f(x)^2 dx$. In ons geval geeft dit

$$\int_0^1 \pi (x^2+x)^2 dx = \int_0^1 \pi (x^4+2x^3+x^2) dx = \left[\pi\left(\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3\right)\right]_0^1$$

$$= \pi\left(\tfrac{1}{5}+\tfrac{1}{2}+\tfrac{1}{3}\right) = \pi\left(\frac{6+15+10}{30}\right) = \boxed{\pi \cdot \frac{31}{30}}$$

b) Partiële integratie met $f(x)=\ln x$, $g'(x)=x^4+2x$, $g(x)=\frac{1}{5}x^5+x^2$

$$\int (x^4+2x)\ln x\, dx = \left(\tfrac{1}{5}x^5+x^2\right)\ln x - \int\left(\tfrac{1}{5}x^5+x^2\right)\cdot \ln' x\, dx$$
$$= \left(\tfrac{1}{5}x^5+x^2\right)\ln x - \int\left(\tfrac{1}{5}x^5+x^2\right)\cdot \tfrac{1}{x}dx = \left(\tfrac{1}{5}x^5+x^2\right)\ln x - \int\left(\tfrac{1}{5}x^4+x\right)dx$$
$$= \boxed{\left(\tfrac{1}{5}x^5+x^2\right)\ln x - \left(\tfrac{1}{25}x^5+\tfrac{1}{2}x^2\right)+C}$$

c) Substitueer $u=-2x^2$. Dan is $du=-4x\,dx$, $x\,dx=-\frac{1}{4}du$

$$\int xe^{-2x^2}dx = \int e^u \cdot -\tfrac{1}{4}du = -\tfrac{1}{4}e^u+C = -\tfrac{1}{4}e^{-2x^2}+C$$

Dus $\int_0^\infty xe^{-2x^2}dx = \lim_{B\to\infty}\int_0^B xe^{-2x^2}dx = \lim_{B\to\infty}\left[-\frac{1}{4}e^{-2x^2}\right]_0^B$

$$= \lim_{B\to\infty} -\tfrac{1}{4}e^{-2B^2} - \left(-\tfrac{1}{4}\right) = \boxed{\tfrac{1}{4}}$$

2 $f(x,y) = 3x^5+5x^3y^3-5y^3$.

a) $\frac{\partial f}{\partial x} = 15x^4+15x^2y^3 = 15x^2(x^2+y^3)$, $\frac{\partial f}{\partial y} = 15x^3y^2-15y^2 = 15y^2(x^3-1)$

We moeten de stationaire punten bepalen uit $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$
Geval 1: $x=0$. Uit $\frac{\partial f}{\partial y}=0$ volgt dan $y^2=0$ dus $y=0$
Geval 2: $y=0$ Uit $\frac{\partial f}{\partial x}=0$ volgt dan $x^4=0$ dus $x=0$

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Geval 3: $x \neq 0, y \neq 0$. Dan moeten we oplossen: $x^2+y^3=0$, $x^3-1=0$

Gevallen 1 en 2 geven het stationaire punt $(0,0)$
Geval 3 geeft het stationaire punt $(1,-1)$

b) $\frac{\partial^2 f}{\partial x^2} = 60x^3+30xy^3 = A$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 45x^2y^2 = B$,

$\frac{\partial^2 f}{\partial y^2} = 30x^3y - 30y = C$ $\qquad H = AC - B^2$

	A	B	C	H	
$(0,0)$	0	0	0	0	geen uitsluitsel
$(1,-1)$	30	45	0	<0	zadelpunt.

f neemt in $(1,-1)$ een zadelpunt aan. Voor $(0,0)$ is $H=0$ en hebben we nog geen uitsluitsel

c) $f(x,0) = 3x^5$. Er geldt $f(0,0)=0$, $f(x,0)<0$ voor $x<0$, $f(x,0)>0$ voor $x>0$, dus f neemt in de buurt van $(0,0)$ zowel waarden >0 als <0 aan. Bijgevolg neemt f in $(0,0)$ geen maximum of minimum aan, m.a.w. $(0,0)$ is een $\boxed{\text{zadelpunt}}$ van f

d) Vergelijking raakvlak: $z = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$
Er geldt $f(1,1)=3$, $\frac{\partial f}{\partial x}(1,1)=30$, $\frac{\partial f}{\partial y}(1,1)=0$
De vergelijking wordt dus:
$\boxed{z = 3+30(x-1)}$

3) a) $(2+i)^2 = (2+i)(2+i) = 4+2i+i\cdot 2+i^2 = 3+4i$,

$\frac{(2+i)^2}{3+i} = \frac{(2+i)^2\cdot(3-i)}{3^2+1^2} = \frac{(3+4i)(3-i)}{10} = \frac{3\cdot 3 - 3\cdot i + 4i\cdot 3 + 4i\cdot(-i)}{10}$

$= \frac{9-3i+12i+4}{10} = \frac{13+9i}{10} = \boxed{\frac{13}{10} + \frac{9}{10}i}$

b) Schrijf eerst $\sqrt{3}+i = r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$, $\cos\varphi = \frac{\sqrt{3}}{2}$, $\sin\varphi = \frac{1}{2}$, neem $\varphi = \frac{1}{6}\pi$.

Dus

$$(\sqrt{3}+i)^{50} = 2^{50}\left(\cos\frac{50}{6}\pi + i\sin\frac{50}{6}\pi\right) = 2^{50}\left(\cos 8\tfrac{1}{3}\pi + i\sin 8\tfrac{1}{3}\pi\right)$$
$$= 2^{50}\left(\cos\tfrac{1}{3}\pi + i\sin\tfrac{1}{3}\pi\right) = 2^{50}\cdot\left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,i\right)$$

Schrijf $\sqrt{3}-i = r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$, $\cos\varphi = \frac{\sqrt{3}}{2}$, $\sin\varphi = -\frac{1}{2}$, neem $\varphi = -\frac{1}{6}\pi$.

Dus

$$(\sqrt{3}-i)^{50} = 2^{50}\left(\cos\left(-\frac{50}{6}\pi\right) + i\sin\left(-\frac{50}{6}\pi\right)\right) = 2^{50}\left(\cos\left(-8\tfrac{1}{3}\pi\right) + i\sin\left(-8\tfrac{1}{3}\pi\right)\right)$$
$$= 2^{50}\left(\cos\left(-\tfrac{1}{3}\pi\right) + i\sin\left(-\tfrac{1}{3}\pi\right)\right) = 2^{50}\left(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}\,i\right)$$

Dus $(\sqrt{3}+i)^{50} + (\sqrt{3}-i)^{50} = 2^{50}\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i + \frac{1}{2} - \frac{1}{2}\sqrt{3}\,i\right) = \boxed{2^{50}}$

c) Stel $w = z^3$. Dan krijgen we $w^2 - 4w + 8 = 0$, discriminant $D = 4^2 - 4\cdot 8 = -16$, oplossingen $w_{1,2} = \frac{4 \pm \sqrt{16}\cdot i}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$

Los op: $z^3 = 2+2i$ $\quad 2+2i = r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{2^2+2^2} = 2\sqrt{2}$, $\cos\varphi = \frac{2}{2\sqrt{2}} = \frac{1}{2}\sqrt{2}$, $\sin\varphi = \frac{2}{2\sqrt{2}} = \frac{1}{2}\sqrt{2}$

neem $\varphi = \frac{1}{4}\pi$.

oplossingen: $z_k = \sqrt[3]{2\sqrt{2}}\cdot\left(\cos\left(\frac{\varphi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\varphi}{3} + \frac{2k\pi}{3}\right)\right)$

$= \sqrt{2}\cdot\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$ $\quad k=0,1,2$

Los op: $z^3 = 2-2i$ $\quad 2-2i = r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{2^2+(-2)^2} = 2\sqrt{2}$, $\cos\varphi = \frac{2}{2\sqrt{2}} = \frac{1}{2}\sqrt{2}$, $\sin\varphi = -\frac{1}{2}\sqrt{2}$

neem $\varphi = -\frac{1}{4}\pi$.

oplossingen: $z_k' = \sqrt[3]{2\sqrt{2}}\cdot\left(\cos\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$ $\quad k=0,1,2.$

d) $3i = 3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

oplossingen van $e^{2z} = 3i$:

$2z = \ln 3 + (\frac{\pi}{2} + 2k\pi)i \quad (k \in \mathbb{Z})$, dus $\boxed{z = \frac{1}{2}\ln 3 + (\frac{\pi}{4} + k\pi)i \quad (k \in \mathbb{Z})}$

4 a) $\sum_{n=2}^{\infty} \frac{(-2)^n + 5^n}{10^n} = \sum_{n=2}^{\infty} \left(\frac{-2}{10}\right)^n + \sum_{n=2}^{\infty} \left(\frac{5}{10}\right)^n = \sum_{n=2}^{\infty} \left(-\frac{1}{5}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$

(gebruik $\sum_{n=p}^{\infty} r^n = \frac{r^p}{1-r}$ voor $-1<r<1$)

$= \frac{(-1/5)^2}{1-(-1/5)} + \frac{(1/2)^2}{1-\frac{1}{2}} = \frac{1/25}{6/5} + \frac{1/4}{1/2} = \frac{1}{25}\cdot\frac{5}{6} + \frac{1}{2} = \frac{1}{30} + \frac{1}{2} = \frac{16}{30} = \boxed{\frac{8}{15}}$

b) vergelijkingskenmerk: $\sum_{n=q}^{\infty} b_n$ convergent, $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$

$\Rightarrow \sum_{n=p}^{\infty} a_n$ convergent.

$\sum_{n=q}^{\infty} b_n$ divergent, $\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \Rightarrow \sum_{n=p}^{\infty} a_n$ divergent

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Neem $a_n = \frac{n^{1/2}+2}{n^2+1}$. Teller is van orde van grootte $n^{1/2}$, noemer van orde van grootte n^2,

dus a_n is van orde van grootte $n^{-3/2}$.

Vergelijk a_n met $b_n = n^{-3/2}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2}+2}{n^2+1} / n^{-3/2} = \lim_{n\to\infty} \frac{(n^{1/2}+2)n^{3/2}}{n^2+1} = \lim_{n\to\infty} \frac{n^2+2n^{3/2}}{n^2+1}$

$= \lim_{n\to\infty} \frac{1+2n^{-1/2}}{1+n^{-2}} = 1.$

$\sum_{n=1}^{\infty} n^{-3/2}$ is convergent

Volgens het vergelijkingskenmerk is $\sum_{n=1}^{\infty} \frac{n^{1/2}+2}{n^2+1}$ $\boxed{\text{convergent}}$.