UITWERKING TENTAMEN CONTINUE WISKUNDE 2
18-4-2017

1 a) $\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int e^u\, 3\,du = 3e^u + C = 3e^{\sqrt[3]{x}} + C$

$u = \sqrt[3]{x} = x^{1/3}$

$du = \frac{1}{3}x^{-2/3}\,dx$

$\sqrt[3]{x^2}^{-1}\,dx = x^{-2/3}\,dx = 3\,du$

$$\int_0^1 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \lim_{a\downarrow 0}\int_a^1 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \lim_{a\downarrow 0}\left[3e^{\sqrt[3]{x}}\right]_a^1 = \lim_{a\downarrow 0}\left(3e^1 - 3e^{\sqrt[3]{a}}\right)$$

$$= \boxed{3e-3}$$

b) $\int x\cos x\,dx = x\sin x - \int \sin x \cdot x'\,dx = x\sin x - \int \sin x\,dx$

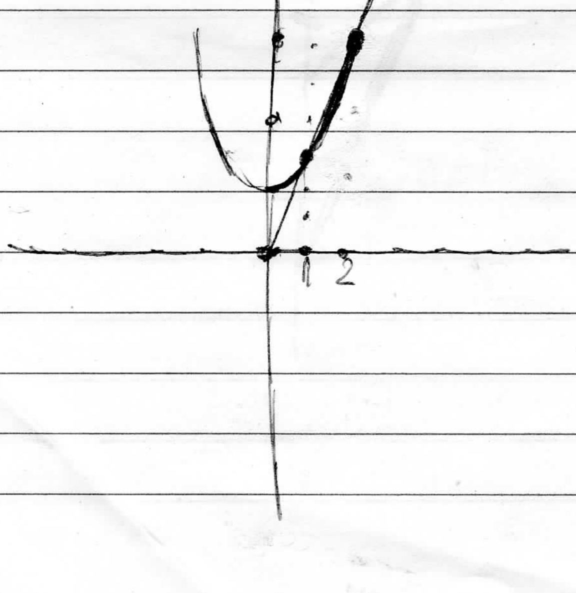
$f(x) = x$
$g'(x) = \cos x$
$g(x) = \sin x$

$= x\sin x + \cos x + C$

$$\int_0^{\frac{2}{3}\pi} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\frac{2}{3}\pi} = \tfrac{2}{3}\pi\cdot\sin\tfrac{2}{3}\pi + \cos\tfrac{2}{3}\pi - 0\cdot\sin 0 - \cos 0$$

$$= \tfrac{2}{3}\pi\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2} - 1 = \boxed{\tfrac{1}{3}\pi\sqrt{3} - \tfrac{3}{2}}$$

c) Snijpunten van de grafieken van $f(x)$ en $g(x)$ bepalen:
$f(x) = g(x) \Leftrightarrow x^2+2 = 3x \Leftrightarrow x^2-3x+2=0 \Leftrightarrow (x-1)(x-2)=0 \Leftrightarrow x=1$ of $x=2$



oppervlakte van gebied:

$$\int_1^2 (g(x)-f(x))\,dx = \int_1^2 (3x - x^2 - 2)\,dx$$

$$= \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 - 2x\right]_1^2$$

$$= \left(\tfrac{3}{2}\times 4 - \tfrac{1}{3}\times 8 - 2\times 2\right) - \left(\tfrac{3}{2} - \tfrac{1}{3} - 2\right)$$

$$= 6 - \tfrac{8}{3} - 4 - \tfrac{3}{2} + \tfrac{1}{3} + 2 = 4 - \tfrac{14}{6} - \tfrac{9}{6} = \boxed{\tfrac{1}{6}}$$

2 a) $(x^2-\frac{1}{2})^2+(x-y)^2+\frac{3}{4} = x^4 - 2\cdot\frac{1}{2}x^2+\frac{1}{4}+x^2-2xy+y^2+\frac{3}{4} = x^4-2xy+y^2+1$
$= f(x,y)$

De kwadraten zijn altijd ≥ 0, dus $f(x,y)\geq \frac{3}{4}$ voor alle x,y.

b) $\frac{\partial f}{\partial x} = 4x^3-2y$, $\frac{\partial f}{\partial y} = 2y-2x$

stationaire punten bepalen:

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Leftrightarrow 4x^3=2y$ en $y=x \Leftrightarrow 4x^3=2x$ en $y=x$

$\Leftrightarrow 4x^3-2x=0$ en $y=x$

$\Leftrightarrow 2x(2x^2-1)=0$ en $y=x \Leftrightarrow (x=0$ of $2x^2=1)$ en $y=x$

$\Leftrightarrow x=0$ of $x=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{2}\sqrt{2}$ en $y=x$

$\Leftrightarrow (x,y)=(0,0)$ of $(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})$ of $(-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})$

c) $\frac{\partial^2 f}{\partial x^2}=12x^2=A$, $\frac{\partial^2 f}{\partial y\partial x}=-2=B$, $\frac{\partial^2 f}{\partial y^2}=2=C$ $\quad H=AC-B^2$

	A	B	C	H	conclusie
$(0,0)$	0	-2	2	$-4<0$	zadelpunt
$(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})$	$6>0$	-2	2	$8>0$	minimum (absoluut)
$(-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})$	$6>0$	-2	2	$8>0$	minimum (absoluut)

Er geldt $f(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})=f(-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})=\frac{3}{4}$ wegens a),
en $f(x,y)\geq\frac{3}{4}$ voor alle (x,y) omdat de kwadraten ≥ 0 zijn
Dus f neemt in $(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})$ absolute minima aan.

d) $f(1,2)=1^4+2^2-2\times1\times2+1=2$, $\frac{\partial f}{\partial x}(1,2)=4-2\times2=0$, $\frac{\partial f}{\partial y}=2\times2-2\times1=2$,

vergelijking raakvlak:

$$\boxed{\begin{aligned} z &= f(1,2)+\frac{\partial f}{\partial x}(1,2)(x-1)+\frac{\partial f}{\partial y}(1,2)(y-2) \\ &= 2+0\cdot(x-1)+2(y-2) = 2y-2 \end{aligned}}$$

③

3a) $\left|\frac{3+2i}{5+i}\right| = \frac{|3+2i|}{|5+i|} = \frac{\sqrt{3^2+2^2}}{\sqrt{5^2+1^2}} = \frac{\sqrt{13}}{\sqrt{26}} = \sqrt{\frac{13}{26}} = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{2}\sqrt{2}}$

b) Schrijf eerst $\sqrt{3}+i = r.(\cos\varphi + i\sin\varphi)$ met $r>0$.

$r = |\sqrt{3}+i| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{4} = 2.$

$\cos\varphi = \frac{\sqrt{3}}{2}$, $\sin\varphi = \frac{1}{2} \Rightarrow \sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$(\sqrt{3}+i)^{15} = 2^{15}\left(\cos\frac{15}{6}\pi + i\sin\frac{15}{6}\pi\right) = 2^{15}\left(\cos\frac{5}{2}\pi + i\sin\frac{5}{2}\pi\right)$

$= 2^{15}(0+1.i) = \boxed{2^{15}i}$

c) Stel eerst $z^3 = w$. Dan $w^2 + w - 6 = 0$ of wel $(w-2)(w+3)=0$

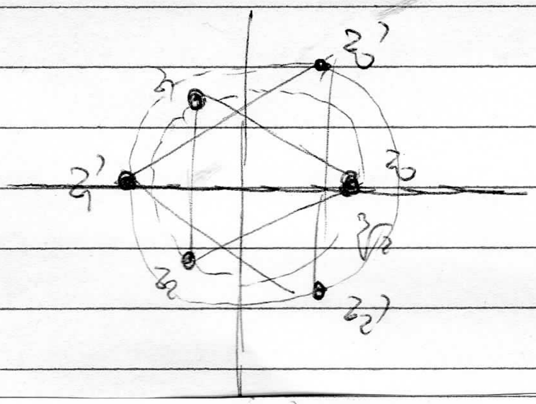
dus wel $w=2$ of $w=-3$. Op te lossen: $z^3 = 2$, $z^3 = -3$

$z^3 = 2$: $\quad 2 = 2(\cos 0 + i\sin 0)$

$z_k = \sqrt[3]{2}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$ $\quad (k=0,1,2)$

$z^3 = -3$: $\quad -3 = 3(\cos\pi + i\sin\pi)$

$z'_k = \sqrt[3]{3}\left(\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right)$ $\quad (k=0,1,2)$



④

④ a) $\sum_{k=0}^{\infty} \frac{3\times 4^k + 5\times(-2)^k}{7^k} = \sum_{k=0}^{\infty} \frac{3\times 4^k}{7^k} + \sum_{k=0}^{\infty} 5\times\frac{(-2)^k}{7^k} = 3\sum_{k=0}^{\infty}\left(\frac{4}{7}\right)^k + 5\sum_{k=0}^{\infty}\left(\frac{-2}{7}\right)^k$

$= \frac{3}{1-\frac{4}{7}} + \frac{5}{1-(-\frac{2}{7})} = \frac{3}{3/7} + \frac{5}{9/7} = 3\times\frac{7}{3} + \frac{5\times 7}{9} = 7 + \frac{35}{9} = \boxed{\frac{98}{9}}$

b) Quotiëntkenmerk: als $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$ dan is $\sum_{k=0}^{\infty} a_k$ convergent,

als $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$ dan is $\sum_{k=0}^{\infty} a_k$ divergent.

Pas dit toe met $a_k = \frac{k+1}{3^k}$. Dan $a_{k+1} = \frac{k+2}{3^{k+1}}$, dus

$\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+2}{3^{k+1}} \Big/ \frac{k+1}{3^k} = \lim_{k\to\infty} \frac{k+2}{3^{k+1}}\cdot\frac{3^k}{k+1} = \lim_{k\to\infty} \frac{1}{3}\,\frac{1+2/k}{1+1/k} = \frac{1}{3} < 1.$

Dus $\sum_{k=0}^{\infty} \frac{k+1}{3^k}$ is $\boxed{\text{convergent.}}$

c) Vergelijkingskenmerk: als $a_k \geq 0$, $b_k \geq 0$ en $\lim_{k\to\infty} \frac{a_k}{b_k} = l$

met $0<l<\infty$, dan $\sum_{k=0}^{\infty} a_k$ convergent $\Leftrightarrow \sum_{k=0}^{\infty} b_k$ convergent

Pas dit toe met $a_k = \frac{\sqrt{k}}{2k-1}$. Voor k groot is $a_k \approx \frac{\sqrt{k}}{2k} \approx \frac{1}{2}k^{-\frac{1}{2}}$.

Neem $b_k = k^{-1/2}$. Dan $\frac{a_k}{b_k} = \frac{\sqrt{k}}{2k-1} / k^{-\frac{1}{2}} = \frac{\sqrt{k}\cdot k^{1/2}}{2k-1} = \frac{k}{2k-1}$

dus $\lim_{k\to\infty} \frac{1}{2-\frac{1}{k}} = \frac{1}{2}$, $0<\frac{1}{2}<\infty$

Er geldt: $\sum_{k=0}^{\infty} k^{-1/2}$ is divergent. Dus $\sum_{k=0}^{\infty} \frac{\sqrt{k}}{2k-1}$ is $\boxed{\text{divergent}}$