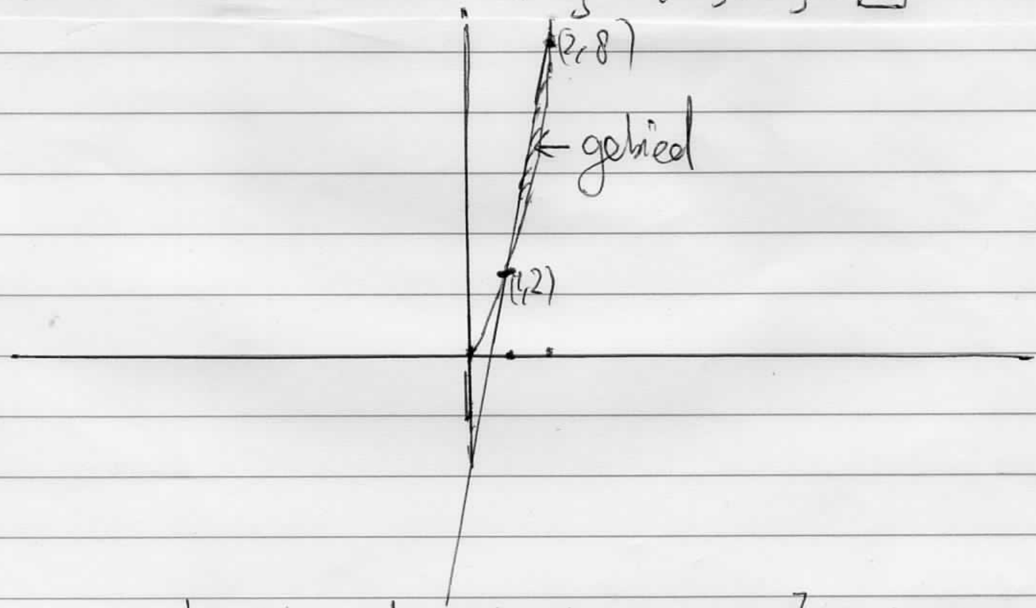
UITWERKING TENTAMEN OW2 (VERSIE 1), 12-6-2020

1 a) We bepalen eerst de snijpunten van de grafieken van f en g.
Er geldt: $f(x)=g(x) \Leftrightarrow 6x-4=2x^2 \Leftrightarrow 2x^2-6x+4=0$
$\Leftrightarrow 2(x^2-3x+2)=0 \Leftrightarrow 2(x-1)(x-2)=0 \Leftrightarrow x=1,\ x=2.$

$$\text{oppervlakte} = \int_1^2 (f(x)-g(x))\,dx = \int_1^2 (6x-4-2x^2)\,dx$$
$$= \left[3x^2-4x-\tfrac{2}{3}x^3\right]_1^2 = \left(12-8-\tfrac{8}{3}\right)-\left(3-4-\tfrac{2}{3}\right)$$
$$= \tfrac{4}{3}-\left(-\tfrac{5}{3}\right)=\tfrac{9}{3}=\boxed{3}$$




b) Gebruik de substitutieregel. Substitueer $u=x^7+2$. Dan is $du=7x^6dx$, $x^6dx=\tfrac{1}{7}du$. Dus
$$\int\sqrt{x^7+2}\,x^6\,dx = \int u^{1/2}\cdot\tfrac{1}{7}\,du = \tfrac{1}{7}\cdot\tfrac{2}{3}\cdot u^{3/2}+C = \boxed{\tfrac{2}{21}(x^7+2)^{3/2}+C}$$

c) Gebruik partiële integratie.
$$\int(2x+1)e^{-3x}dx \underset{\uparrow}{=} (2x+1)\cdot\left(-\tfrac{1}{3}e^{-3x}\right)-\int\left(-\tfrac{1}{3}e^{-3x}\right)\cdot(2x+1)'\,dx$$

$f(x)=2x+1$
$g'(x)=e^{-3x}$
$g(x)=-\tfrac{1}{3}e^{-3x}$

$$= -\tfrac{1}{3}(2x+1)e^{-3x}+\int\tfrac{1}{3}e^{-3x}\cdot 2\,dx$$
$$= \left(-\tfrac{2}{3}x-\tfrac{1}{3}\right)e^{-3x}+\tfrac{2}{3}\cdot\left(-\tfrac{1}{3}\right)e^{-3x}+C$$
$$= \left(-\tfrac{2}{3}x-\tfrac{1}{3}-\tfrac{2}{9}\right)e^{-3x}+C = \left(-\tfrac{2}{3}x-\tfrac{5}{9}\right)e^{-3x}+C$$

c) (vervolg) $\int_0^\infty (2x+1)e^{-3x}\,dx = \lim_{B\to\infty}\int_0^B (2x+1)e^{-3x}\,dx = \lim_{B\to\infty}\left[\left(-\frac{2}{3}x-\frac{5}{9}\right)e^{-3x}\right]_0^B$

$= \lim_{B\to\infty}\left(\left(-\frac{2}{3}B-\frac{5}{9}\right)e^{-3B} - \left(-\frac{5}{9}\right)\right) = \boxed{\frac{5}{9}}$ $\qquad Be^{-3B}\to 0,\ e^{-3B}\to 0$ als $B\to\infty$

2 a) $\frac{\partial f}{\partial x} = 2(x^2-1).2x + 2(x-y)$, $\frac{\partial f}{\partial y} = -2(x-y)$.

stationaire punten: $\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0 \Leftrightarrow \begin{Bmatrix} 2(x^2-1)2x+2(x-y)=0 \\ -2(x-y)=0 \end{Bmatrix}$

$\Leftrightarrow \begin{Bmatrix} 2(x^2-1).2x=0 \\ x=y \end{Bmatrix} \Leftrightarrow (x,y)=(0,0),(1,1),(-1,-1).$

b) $\frac{\partial^2 f}{\partial x^2} = 4.2x.x + 4(x^2-1)+2 = 16x^2-2 = A$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = -2 = B$, $\frac{\partial^2 f}{\partial y^2} = 2 = C$, $H = AC-B^2$

| | A | B | C | H | |
|---|---|---|---|---|---|
| (0,0) | −2 | −2 | 2 | −8 < 0 | zadelpunt |
| (1,1) | [illegible] > 0 | −2 | 2 | 24 > 0 | minimum |
| (−1,−1) | [illegible] > 0 | −2 | 2 | 24 > 0 | minimum |

Er geldt: $f(1,1)=f(-1,-1)=0$, $f(x,y)\geq 0$ voor alle $x,y$
(som van twee kwadraten)
Dus de minima zijn <u>absoluut</u>

c) vergelijking raakvlak

$z = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$

$f(1,2)=1,\ \frac{\partial f}{\partial x}(1,2)=-2,\ \frac{\partial f}{\partial y}(1,2)=2$

$\boxed{z = 1-2(x-1)+2(y-2)}$

3 a) $\left|\frac{z^{10}}{w^9}\right| = \frac{|z|^{10}}{|w|^9}$, $|z| = \sqrt{5^2+12^2} = \sqrt{169} = 13$,
$|w| = \sqrt{5^2+(-12)^2} = \sqrt{169} = 13$

$= \frac{13^{10}}{13^9} = \boxed{13}$

b) We berekenen eerst $(1-i)^{100}$.
Schrijf $1-i = r(\cos\varphi + i\sin\varphi)$ $r = |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$,
$\cos\varphi = \frac{1}{\sqrt{2}}$, $\sin\varphi = -\frac{1}{\sqrt{2}}$, neem $\varphi = -\frac{1}{4}\pi$.
Dus $(1-i)^{100} = (\sqrt{2})^{100} \cdot (\cos -\frac{100}{4}\pi + i\sin -\frac{100}{4}\pi) = 2^{50}(\cos -\frac{1}{2}\pi + i\sin -\frac{1}{2}\pi)$
$= 2^{50}\cdot(-i) = -2^{50}i$.

Dit geeft $\frac{2+i}{(1-i)^{100}} = \frac{2+i}{-2^{50}i} = \frac{(2+i)(-i)}{-2^{50}} = \frac{-2i-i^2}{-2^{50}} = \boxed{2^{-50}(-1+2i)}$

c) Schrijf $3^5(\frac{\sqrt{3}}{2} + \frac{i}{2})$ in de vorm $r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{(3^5 \cdot \frac{\sqrt{3}}{2})^2 + (3^5\cdot\frac{1}{2})^2} = 3^5\sqrt{\frac{3}{4}+\frac{1}{4}} = 3^5$, $\cos\varphi = \frac{1}{2}\sqrt{3}$, $\sin\varphi = \frac{1}{2}$
Neem $\varphi = \frac{1}{6}\pi$.

Dus de oplossingen van $z^{10} = 3^5(\frac{1}{2}\sqrt{3} + \frac{1}{2}i)$ zijn
$z_k = \sqrt[10]{3^5}\cdot\left(\cos\left(\frac{1}{60}\pi + \frac{2k\pi}{10}\right) + i\sin\left(\frac{1}{60}\pi + \frac{2k\pi}{10}\right)\right)$

$= \boxed{\sqrt{3}\left(\cos\left(\frac{1}{60}\pi + \frac{k\pi}{5}\right) + i\sin\left(\frac{1}{60}\pi + \frac{k\pi}{5}\right)\right)}$ $(k = 0, 1, \ldots, 9)$

d) $e^{2z} + 4e^z + 3 = 0$ Stel $w = e^z$. Dan is $w^2 + 4w + 3 = 0$
Dit is een kwadratische vergelijking met discriminant $4^2 - 4\cdot 3 = 4$
Dus de oplossingen van de kwadratische vergelijking zijn
$w_{1,2} = \frac{-4 \pm 2}{2} = -1, -3$

Hieruit volgt: $e^z = -1$ of $e^z = -3$. Er geldt: $-1 = (\cos\pi + i\sin\pi)$
$-3 = 3(\cos\pi + i\sin\pi)$. Dus de oplossingen zijn
$z_k = (\pi + 2k\pi)i$ $(k\in\mathbb{Z})$; $z_k' = \ln 3 + (\pi + 2k\pi)i$ $(k\in\mathbb{Z})$

4 a) $\sum_{n=0}^{\infty}\left(\frac{3^n}{8^n}-\frac{2^n}{9^n}\right) = \sum_{n=0}^{\infty}\left(\frac{3}{8}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n$

$= \frac{1}{1-3/8} - \frac{1}{1-2/9} = \frac{8}{5} - \frac{9}{7} = \frac{56-45}{35} = \boxed{\frac{11}{35}}$

b) Gebruik het quotiëntkenmerk met $a_n = \frac{n^7}{n!}$

Dan is $a_{n+1} = \frac{(n+1)^7}{(n+1)!}$, dus

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^7}{(n+1)!} \Big/ \frac{n^7}{n!} = \frac{(n+1)^7}{n^7} \cdot \frac{n!}{(n+1)!} = \left(1+\frac{1}{n}\right)^7 \cdot \frac{1 \cdot 2 \cdots n}{1 \cdot 2 \cdots (n+1)} = \left(1+\frac{1}{n}\right)^7 \cdot \frac{1}{n+1}$,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$

Dus $\sum_{n=1}^{\infty}\frac{n^7}{n!}$ convergeert.