ANALYTIC NUMBER THEORY

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Literature

Below is a list of recommended additional literature. Much of the material of this course has been taken from the books of Jameson and the two books of Davenport.

H. Davenport, *Multiplicative Number Theory (2nd ed.)*, Springer Verlag, Graduate Texts in Mathematics 74, 1980.

H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Cambridge University Press, 1963, reissued in 2005 in the Cambridge Mathematical Library series.

A. Granville, *What is the best approach to counting primes*, arXiv:1406.3754 [math.NT].

A.E. Ingham, *The distribution of prime numbers,* Cambridge University Press, 1932 (reissued in 1990).

H. Iwaniec, E. Kowalski, *Analytic Number Theory,* American Mathematical Society Colloquium Publications 53, American Mathematical Society, 2004.

G.J.O. Jameson, *The Prime Number Theorem,* London Mathematical Society, Student Texts 53, Cambridde University Press, 2003.

S. Lang, *Algebraic Number Theory,* Addison-Wesley, 1970. S. Lang, *Complex Analysis (4th. ed.),* Springer Verlag, Graduate Texts in Mathematics 103, 1999.

D.J. Newman, *Analytic Number Theory,* Springer Verlag, Graduate Texts in Mathematics 177, 1998.

E.C. Titchmarsh, *The theory of the Riemann zeta function (2nd. ed., revised by D.R. Heath-Brown),* Oxford Science Publications, Clarendon Press Oxford, 1986.

R.C. Vaughan, *The Hardy-Littlewood method (2nd ed.)*, Cambridge University Press, 1997.

Notation

- $\limsup\limits_{n\to\infty} x_n$ or $\overline{\lim}_{n\to\infty} x_n$

  For a sequence of reals $\{x_n\}$ we define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\big(\sup_{m\geqslant n} x_m\big)$. We have $\limsup_{n\to\infty} x_n = \infty$ if and only if the sequence $\{x_n\}$ is not bounded from above, i.e., if for every $A > 0$ there is n with $x_n > A$.
  In case that the sequence $\{x_n\}$ is bounded from above, we have $\limsup_{n\to\infty} x_n = \alpha$ where α is the largest limit point ('limes superior') of the sequence $\{x_n\}$, in other words, for every $\varepsilon > 0$ there are infinitely many n such that $x_n \geqslant \alpha - \varepsilon$, while there are only finitely many n such that $x_n \geqslant \alpha + \varepsilon$.

- $\liminf\limits_{n\to\infty} x_n$ or $\underline{\lim}_{n\to\infty} x_n$

  For a sequence of reals $\{x_n\}$ we define $\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\big(\inf_{m\geqslant n} x_m\big)$. We have $\liminf_{n\to\infty} x_n = -\infty$ if the sequence $\{x_n\}$ is not bounded from below, and the smallest limit point ('limes inferior') of the sequence $\{x_n\}$ otherwise.

- $f(x) = g(x) + O(h(x))$ as $x \to \infty$

  there are constants x_0, C such that $|f(x) - g(x)| \leqslant Ch(x)$ for all $x \geqslant x_0$

- $f(x) = g(x) + o(h(x))$ as $x \to \infty$

  $\lim\limits_{x\to\infty} \dfrac{f(x) - g(x)}{h(x)} = 0$

- $f(x) \sim g(x)$ as $x \to \infty$

  $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$

- $f(x) \ll g(x)$, $g(x) \gg f(x)$
  (Vinogradov symbols; used only if $g(x) > 0$ for all sufficiently large x, i.e., there is x_0 such that $g(x) > 0$ for all $x \geqslant x_0$)

  $f(x) = O(g(x))$ as $x \to \infty$, that is, there are constants $x_0 > 0, C > 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \geqslant x_0$

- $f(x) \asymp g(x)$
  (used only if $f(x) > 0$, $g(x) > 0$ for all sufficiently large x)

  there are constants $x_0, C_1, C_2 > 0$ such that $C_1 f(x) \leqslant g(x) \leqslant C_2 f(x)$ for all $x \geqslant x_0$

- $f(x) = \Omega(g(x))$ as $x \to \infty$
  (defined only if $g(x) > 0$ for $x \geqslant x_0$ for some $x_0 > 0$)

  $\limsup\limits_{x\to\infty} \dfrac{|f(x)|}{g(x)} > 0$, that is, there is a sequence $\{x_n\}$ with $x_n \to \infty$ as $n \to \infty$ such that $\lim\limits_{n\to\infty} \dfrac{|f(x_n)|}{g(x_n)} > 0$ (possibly ∞)

- $f(x) = \Omega^{\pm}(g(x))$ as $x \to \infty$
  (defined only if $g(x) > 0$ for $x \geqslant x_0$ for some $x_0 > 0$)

  $\limsup\limits_{x\to\infty} \dfrac{f(x)}{g(x)} > 0$, $\liminf\limits_{x\to\infty} \dfrac{f(x)}{g(x)} < 0$, that is, there are sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to \infty$, $y_n \to \infty$ as $n \to \infty$ such that $\lim\limits_{n\to\infty} \dfrac{f(x_n)}{g(x_n)} > 0$ (possibly ∞) and $\lim\limits_{n\to\infty} \dfrac{f(y_n)}{g(y_n)} < 0$ (possibly $-\infty$)

- $\pi(x)$

  Number of primes $\leqslant x$

- $\pi(x; q, a)$

  Number of primes p with $p \equiv a \pmod q$ and $p \leqslant x$

- $\mathrm{Li}(x)$

  $\mathrm{Li}(x) = \displaystyle\int_2^x \frac{dt}{\log t}$; this is a good approximation for $\pi(x)$.

Chapter 0

Prerequisites

We have collected some facts from algebra and analysis which we will not discuss during our course, which will not be a subject of the examination, but to which we will have to refer quite often. Students are requested to read this through.

0.1 Groups

Literature:
P. Stevenhagen: *Collegedictaat Algebra 1 (Dutch)*, Universiteit Leiden.
S. Lang: *Algebra, 2nd ed.*, Addison-Wesley, 1984.

0.1.1 Definition

A group is a set G, together with an operation $\cdot : G \times G \to G$ satisfying the following axioms:

- $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$;
- there is $e_G \in G$ such that $g \cdot e_G = e_G \cdot g = g$ for all $g \in G$;
- for all $g \in G$ there is $h \in G$ with $g \cdot h = h \cdot g = e_G$.

From these axioms it follows that the unit element e_G is uniquely determined, and that the inverse h defined by the last axiom is uniquely determined; henceforth we write g^{-1} for this h.

If moreover, $g_1 \cdot g_2 = g_2 \cdot g_1$, we say that the group G is *abelian* or *commutative*.

Remark. For $n \in \mathbb{Z}_{>0}$, $g \in G$ we write g^n for g multiplied with itself n times. Further, $g^0 := e_G$ and $g^n := (g^{-1})^{|n|}$ for $n \in \mathbb{Z}_{<0}$. This is well-defined by the associative axiom, and we have $(g^m)(g^n) = g^{m+n}$, $(g^m)^n = g^{mn}$ for $m, n \in \mathbb{Z}$.

0.1.2 Subgroups

Let G be a group with group operation $\cdot$. A subgroup of G is a subset H of G that is a group with the group operation of G. This means that $g_1 \cdot g_2 \in H$ for all $g_1, g_2 \in H$; $e_G \in H$; and $g^{-1} \in H$ for all $g \in H$. It is easy to see that H is a subgroup of G if and only if $g_1 \cdot g_2^{-1} \in H$ for all $g_1, g_2 \in H$. We write $H \leqslant G$ if H is a subgroup of G.

0.1.3 Cosets, order, index

Let G be a group and H a subgroup of G. The left cosets of G with respect to H are the sets $gH = \{g \cdot h : h \in H\}$. Two left cosets g_1H, g_2H are equal if and only if $g_1^{-1}g_2 \in H$.

The right cosets of G with respect to H are the sets $Hg = \{h \cdot g : h \in H\}$. Two right cosets Hg_1, Hg_2 are equal if and only if $g_2g_1^{-1} \in H$.

There is a one-to-one correspondence between the left cosets and right cosets of G with respect to H, given by $gH \leftrightarrow Hg^{-1}$. Thus, the collection of left cosets has the same cardinality as the collection of right cosets. This cardinality is called the *index* of H in G, notation $(G : H)$.

The order of a group G is its cardinality, notation $|G|$. Assume that $|G|$ is finite. Let again H be a subgroup of G. Since the left cosets w.r.t. H are pairwise disjoint and have the same number of elements as H, and likewise for right cosets, we have

$$(G : H) = \frac{|G|}{|H|}.$$

An important consequence of this is, that $|H|$ divides $|G|$.

0.1.4 Normal subgroup, factor group

Let G be a group, and H a subgroup of G. We call H a normal subgroup of G if $gH = Hg$, that is, if $gHg^{-1} = H$ for every $g \in G$.

Let H be a normal subgroup of G. Then the cosets of G with respect to H form a group with group operation $(g_1H) \cdot (g_2H) = (g_1g_2) \cdot H$. This operation is well-defined. We denote this group by G/H; it is called the factor group of G with respect to H. Notice that the unit element of G/H is $e_GH = H$. If G is finite, we have $|G/H| = (G : H) = |G|/|H|$.

0.1.5 Order of an element

Let G be a group, and $g \in G$. The order of g, notation $\mathrm{ord}(g)$, is the smallest positive integer n such that $g^n = e_G$; if such an integer n does not exist we say that g has infinite order.

We recall some properties of orders of group elements. Suppose that $g \in G$ has finite order n.

- $g^a = g^b \Longleftrightarrow a \equiv b \pmod n$.
- Let $k \in \mathbb{Z}$. Then $\mathrm{ord}(g^k) = n/\gcd(k, n)$.
- $\{e_G, g, g^2, \dots, g^{n-1}\}$ is a subgroup of G of cardinality $n = \mathrm{ord}(g)$. Hence if G is finite, then $\mathrm{ord}(g)$ divides $|G|$. Consequently, $g^{|G|} = e_G$.

Example. Let q be a positive integer. A prime residue class modulo q is a residue class of the type $a \bmod q$, where $\gcd(a, q) = 1$. The prime residue classes form a group under multiplication, which is denoted by $(\mathbb{Z}/q\mathbb{Z})^*$. The unit element of this group is $1 \bmod q$, and the order of this group is $\varphi(q)$, that is the number of positive integers $\leqslant q$ that are coprime with q. It follows that if $\gcd(a,q)=1$, then $a^{\varphi(q)} \equiv 1 \pmod{q}$.

0.1.6 Cyclic groups

The cyclic group generated by g, denoted by $\langle g \rangle$, is given by $\{g^k : k \in \mathbb{Z}\}$. In case that $G = \langle g \rangle$ is finite, say of order $n \geqslant 2$, we have

$$\langle g \rangle = \{e_G = g^0, g, g^2, \ldots, g^{n-1}\}, \quad g^n = e_G.$$

So g has order n.

Example 1. $\mu_n = \{\rho \in \mathbb{C}^* : \rho^n = 1\}$, that is the group of roots of unity of order n is a cyclic group of order n. For a generator of μ_n one may take any primitive root of unity of order n, i.e., $e^{2\pi i k/n}$ with $k \in \mathbb{Z}$, $\gcd(k,n)=1$.

Example 2. Let p be a prime number, and $(\mathbb{Z}/p\mathbb{Z})^* = \{a \bmod p, \gcd(a,p)=1\}$ the group of prime residue classes modulo p with multiplication. This is a cyclic group of order $p-1$.

Let $G = \langle g \rangle$ be a cyclic group and H a subgroup of G. Let k be the smallest positive integer such that $g^k \in H$. Using, e.g., division with remainder, one shows that $g^r \in H$ if and only if $r \equiv 0 \pmod{k}$. Hence $H = \langle g^k \rangle$ and $(G:H) = k$.

0.1.7 Homomorphisms and isomorphisms

Let G_1, G_2 be two groups. A homomorphism from G_1 to G_2 is a map $f : G_1 \to G_2$ such that $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$ and $f(e_{G_1}) = e_{G_2}$. This implies that $f(g^{-1}) = f(g)^{-1}$ for $g \in G_1$.

Let $f : G_1 \to G_2$ be a homomorphism. The kernel and image of f are given by

$$\mathrm{Ker}(f) := \{g \in G_1 : f(g) = e_{G_2}\}, \quad f(G_1) = \{f(g) : g \in G_1\},$$

respectively. Notice that $\mathrm{Ker}(f)$ is a normal subgroup of G_1. It is easy to check that f is injective if and only if $\mathrm{Ker}(f) = \{e_{G_1}\}$.

Let G be a group and H a normal subgroup of G. Then

$$f : G \to G/H : \ g \mapsto gH$$

is a surjective homomorphism from G to G/H, the canonical homomorphism from G to G/H. Notice that the kernel of this homomorphism is H. Thus, every normal subgroup of G occurs as the kernel of some homomorphism.

A homomorphism $f : G_1 \to G_2$ which is bijective is called an isomorphism from G_1 to G_2. In case that there is an isomorphism from G_1 to G_2 we say that G_1, G_2 are isomorphic, notation $G_1 \cong G_2$. Notice that a homomorphism $f : G_1 \to G_2$ is an isomorphism if and only if $\mathrm{Ker}(f) = \{e_{G_1}\}$ and $f(G_1) = G_2$. Further, in this case the inverse map $f^{-1} : G_2 \to G_1$ is also an isomorphism.

Let $f : G_1 \to G_2$ be a homomorphism of groups and $H = \mathrm{Ker}(f)$. This yields an isomorphism

$$\overline{f} : G_1/H \to f(G_1) : \ \overline{f}(gH) = f(g).$$

Proposition 0.1. *Let C be a cyclic group. If C is infinite, then it is isomorphic to $\mathbb{Z}^+$ (the additive group of $\mathbb{Z}$). If C has finite order n, then it is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^+$ (the additive group of residue classes modulo n).*

Proof. Let $C = \langle g \rangle$. Define $f : \mathbb{Z}^+ \to C$ by $n \mapsto g^n$. This is a surjective homomorphism; let H denote its kernel. Thus, $\mathbb{Z}^+/H \cong C$. We have $H = \{0\}$ if C is infinite, and $H = n\mathbb{Z}^+$ if C has order n. This implies the proposition. □

0.1.8 Direct products

Let $G_1, \dots, G_r$ be groups. Denote by e_{G_i} the unit element of G_i. The *direct product* $G_1 \times \cdots \times G_r$ is the set of tuples $(g_1, \dots, g_r)$ with $g_i \in G_i$ for $i = 1, \dots, r$, endowed with the group operation

$$(g_1, \dots, g_r) \cdot (h_1, \dots, h_r) = (g_1 h_1, \dots, g_r h_r).$$

This is obviously a group, with unit element $(e_{G_1}, \dots, e_{G_r})$ and inverse $(g_1, \dots, g_r)^{-1} = (g_1^{-1}, \dots, g_r^{-1})$.

Proposition 0.2. *Let G, $G_1, \dots, G_r$ be groups. Then the following two assertions are equivalent:*

(i) $G \cong G_1 \times \cdots \times G_r$;

(ii) there are subgroups $H_1, \dots, H_r$ of G satisfying the following properties:
(a) $H_i \cong G_i$ for $i = 1, \dots, r$;

(b) $H_1, \ldots, H_r$ *commute, that is,* $H_iH_j = H_jH_i$ *for* $i, j = 1, \ldots, r$*;*
(c) $G = H_1 \cdots H_r$*, i.e., every element of* G *can be expressed as* $g_1 \cdots g_r$ *with* $g_i \in H_i$ *for* $i = 1, \ldots, r$*;*
(d) $H_1, \ldots, H_r$ *are independent, i.e., if* $g_i \in H_i$ $(i = 1, \ldots, r)$ *are any elements such that* $g_1 \cdots g_r = e_G$*, then* $g_i = e_G$ *for* $i = 1, \ldots, r$*.*

Proof. $(ii) \Rightarrow (i)$. Properties (b),(c),(d) imply that

$$H_1 \times \cdots \times H_r \to G : \, (g_1, \ldots, g_r) \mapsto g_1 \cdots g_r$$

is a group isomorphism. Together with (a) this implies (i).

$(i) \Rightarrow (ii)$. Let $G' := G_1 \times \cdots \times G_r$ and for $i = 1, \ldots, r$, define the group

$$G_i' := \{(e_{G_1}, \ldots, g_i, \ldots, e_{G_r}) : \, g_i \in G_i\}$$

where the i-th coordinate is g_i and the other components are the unit elements of the respective groups. Clearly, $G_i' \cong G_i$ for $i = 1, \ldots, r$, $G_1', \ldots, G_r'$ commute, $G' = G_1' \cdots G_r'$ and $G_1', \ldots, G_r'$ are independent. Let $f : G \to G_1 \times \cdots \times G_r$ be an isomorphism and $H_i := f^{-1}(G_i')$ for $i = 1, \ldots, r$. Then $H_1, \ldots, H_r$ satisfy (a)–(d). □

Notice that (b),(c),(d) imply that every element of G can be expressed *uniquely* as a product $g_1 \cdots g_r$ with $g_i \in H_i$ for $i = 1, \ldots, r$.

In what follows, if a group G has subgroups $H_1, \ldots, H_r$ satisfying (b),(c),(d), we say that G is the direct product of $H_1, \ldots, H_r$, and denote this by $G = H_1 \times \cdots \times H_r$.

0.1.9 Abelian groups

The group operation of an abelian group is often denoted by $+$, but in this course we stick to the multiplicative notation. The unit element of an abelian group A is denoted by 1 or 1_A. It is obvious that every subgroup of an abelian group is a normal subgroup. In Proposition 0.2, the condition that $H_1, \ldots, H_r$ commute holds automatically so it can be dropped.

The following important theorem, which we state without proof, implies that the finite cyclic groups are the building blocks of the finite abelian groups.

Theorem 0.3. *Every finite abelian group is a direct product of finite cyclic groups.*

Proof. See S. Lang, Algebra, 2nd ed. Addison-Wesley, 1984, Ch.1, §10. $\square$

Let A be a finite, multiplicatively written abelian group of order $\geqslant 2$ with unit element 1. Theorem 0.3 implies that A is a direct product of cyclic subgroups, say $C_1, \dots, C_r$. Assume that C_i has order $n_i \geqslant 2$; then $C_i = \langle h_i \rangle$, where $h_i \in A$ is an element of order n_i. We call $\{h_1, \dots, h_r\}$ a *basis* for A.

Every element of A can be expressed uniquely as $g_1 \cdots g_r$, where $g_i \in C_i$ for $i = 1, \dots, r$. Further, every element of C_i can be expressed as a power h_i^k, and $h_i^k = 1$ if and only if $k \equiv 0 \pmod{n_i}$. Together with Proposition 0.2 this implies the following characterization of a basis for A:

$$\left\{ \begin{array}{l} A = \{h_1^{k_1} \cdots h_r^{k_r} : \quad k_i \in \mathbb{Z} \text{ for } i = 1, \dots, r\}, \\ \text{there are integers } n_1, \dots, n_r \geqslant 2 \text{ such that} \\ h_1^{k_1} \cdots h_r^{k_r} = 1 \Longleftrightarrow k_i \equiv 0 \pmod{n_i} \text{ for } i = 1, \dots, r. \end{array} \right. \tag{0.1}$$

0.2 Infinite products in analysis

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We define

$$\prod_{n=1}^{\infty} A_n := \lim_{N \to \infty} \prod_{n=1}^{N} A_n$$

provided the limit exists.

In applications it will be important that $\prod_{n=1}^{\infty} A_n \neq 0$. It is not sufficient to assume that all $A_n \neq 0$, for instance $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) = 0$. In general, we have

$$\prod_{n=1}^{\infty} A_n \text{ exists and is} \neq 0, \pm\infty \iff A_n \neq 0 \text{ for all } n \text{ and } \sum_{n=1}^{\infty} \log A_n \text{ converges,}$$

where we take the principal logarithm, i.e., with imaginary part in $(-\pi, \pi]$. The following criterion is more useful for our purposes.

Proposition 0.4. *Assume that* $\sum_{n=1}^{\infty} |A_n - 1| < \infty$. *Then the following hold:*
(i) $\prod_{n=1}^{\infty} A_n$ *exists and is* $\neq \pm\infty$, *and* $\prod_{n=1}^{\infty} A_n \neq 0$ *if* $A_n \neq 0$ *for all* n.
(ii) $\prod_{n=1}^{\infty} A_n$ *is invariant under rearrangements of the* A_n, *i.e., if* σ *is any bijection of* $\mathbb{Z}_{>0}$, *then* $\prod_{n=1}^{\infty} A_{\sigma(n)}$ *exists and is equal to* $\prod_{n=1}^{\infty} A_n$.

Proof. (i) Let $a_n := |A_n - 1|$ for $n = 1, 2, \ldots$. Let M, N be integers with $N > M > 0$. Then, using $|1+z| \leqslant e^{|z|}$ for $z \in \mathbb{C}$ and

$$\Big|\prod_{i=1}^{r}(1+z_i) - 1\Big| \leqslant \prod_{i=1}^{r}(1+|z_i|) - 1 \leqslant \exp\Big(\sum_{i=1}^{r}|z_i|\Big) - 1 \quad \text{for } z_1, \ldots, z_r \in \mathbb{C},$$

we get

$$\begin{aligned} (0.2) \qquad \left|\prod_{n=1}^{N} A_n - \prod_{n=1}^{M} A_n\right| &= \prod_{n=1}^{M}|A_n| \cdot \left|\prod_{n=M+1}^{N} A_n - 1\right| \\ &\leqslant \exp\Big(\sum_{n=1}^{M} a_n\Big) \cdot \Big(\exp\Big(\sum_{n=M+1}^{N} a_n\Big) - 1\Big) \end{aligned}$$

which tends to 0 as $M, N \to \infty$. Hence $\prod_{n=1}^{\infty} A_n = \lim_{N\to\infty} \prod_{n=1}^{N} A_n$ exists and is finite.

Assume that $A_n \neq 0$ for all n. Choose M such that $\sum_{n=M}^{\infty} a_n < \frac{1}{2}$. Then for $N > M$ we have

$$\begin{aligned} \Big|\prod_{n=1}^{N} A_n\Big| &= \prod_{n=1}^{M}|A_n| \cdot \prod_{n=M+1}^{N}|A_n| \\ &\geqslant \prod_{n=1}^{M}|A_n| \cdot \Big(1 - \sum_{n=M+1}^{N} a_n\Big) \geqslant \tfrac{1}{2}\prod_{n=1}^{M}|A_n| =: C > 0, \end{aligned}$$

hence $\left|\prod_{n=1}^{\infty} A_n\right| \geqslant C > 0$. This proves (i).

(ii) Let M, N be positive integers such that $N > M$ and $\{\sigma(1), \ldots, \sigma(N)\}$ contains $\{1, \ldots, M\}$. Similarly to (0.2) we get

$$\left|\prod_{n=1}^{N} A_{\sigma(n)} - \prod_{n=1}^{M} A_n\right| \leqslant \exp\Big(\sum_{n=1}^{M} a_n\Big) \cdot \Bigg(\exp\Big(\sum_{n\leqslant N, \sigma(n)>M} a_{\sigma(n)}\Big) - 1\Bigg).$$

If for fixed M we let first $N \to \infty$ and then let $M \to \infty$, the right-hand side tends to 0. Hence $\prod_{n=1}^{\infty} A_{\sigma(n)} = \prod_{n=1}^{\infty} A_n$. $\square$

0.3 Uniform convergence

We consider functions $f : D \to \mathbb{C}$ where D can be any set. We can express each such function as $g + ih$ where g, h are functions from D to $\mathbb{R}$. We write $g = \operatorname{Re} f$ and $h = \operatorname{Im} f$.

We recall that if D is a topological space (in this course mostly a subset of $\mathbb{R}^n$ with the usual topology), then f is continuous if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous.

In case that $D \subseteq \mathbb{R}$, we say that f is differentiable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentiable, and in that case we define the derivative of f by $f' = (\operatorname{Re} f)' + i(\operatorname{Im} f)'$.

In what follows, let D be any set and $\{F_n\} = \{F_n\}_{n=1}^{\infty}$ a sequence of functions from D to $\mathbb{C}$.

Definition. We say that $\{F_n\}$ converges pointwise on D if $F(z) := \lim_{n\to\infty} F_n(z)$ exists for all $z \in D$, and that $\{F_n\}$ converges uniformly on D if moreover,

$$\lim_{n\to\infty} \left(\sup_{z\in D} |F_n(z) - F(z)| \right) = 0.$$

Facts:

- $\{F_n\}$ converges uniformly on D if and only if $\displaystyle\lim_{M,N\to\infty} \left(\sup_{z\in D} |F_M(z) - F_N(z)| \right) = 0.$
- Let D be a topological space, assume that all functions F_n are continuous, and that $\{F_n\}$ converges to a function F uniformly on D. Then F is continuous on D.

Let again D be any set and $\{F_n\}_{n=1}^{\infty}$ a sequence of functions from D to $\mathbb{C}$. We say that the series $\sum_{n=1}^{\infty} F_n$ converges pointwise/uniformly on D if the partial sums $\sum_{n=1}^{k} F_n$ converge pointwise/uniformly on D. Further, we say that $\sum_{n=1}^{\infty} F_n$ is pointwise absolutely convergent on D if $\sum_{n=1}^{\infty} |F_n(z)|$ converges for every $z \in D$.

Proposition 0.5 (Weierstrass criterion for series)**.** *Assume that there are finite real numbers M_n such that*

$$|F_n(z)| \leqslant M_n \textit{ for } z \in D,\ n \geqslant 1, \quad \sum_{n=1}^{\infty} M_n \textit{ converges.}$$

Then $\sum_{n=1}^{\infty} F_n$ *is both uniformly convergent, and pointwise absolutely convergent on* D.

We need a similar result for infinite products of functions. Let again D be any set and $\{F_n : D \to \mathbb{C}\}_{n=1}^{\infty}$ a sequence of functions. We define the limit function $\prod_{n=1}^{\infty} F_n$ by

$$\prod_{n=1}^{\infty} F_n(z) := \lim_{N\to\infty} \prod_{n=1}^{N} F_n(z) \quad (z \in D),$$

provided that for every $z \in D$ the limit exists.

We say that $\prod_{n=1}^{\infty} F_n$ converges uniformly on D if the limit function $F := \prod_{n=1}^{\infty} F_n$ exists for every $z \in D$, and

$$\lim_{N\to\infty} \left(\sup_{z\in D} \left| F(z) - \prod_{n=1}^{N} F_n(z) \right| \right) = 0.$$

Proposition 0.6 (Weierstrass criterion for infinite products)**.** *Assume that there are finite real numbers* M_n *such that*

$$|F_n(z) - 1| \leqslant M_n \text{ for } z \in D,\ n \geqslant 1, \quad \sum_{n=1}^{\infty} M_n \text{ converges.}$$

Then $\prod_{n=1}^{\infty} F_n$ *is uniformly convergent on* D *and moreover, if* $z \in D$ *is such that* $F_n(z) \neq 0$ *for all* n*, then also* $F(z) \neq 0$.

Proof. Applying (0.2) with $A_n = F_n(z)$ and using $|F_n(z) - 1| \leqslant M_n$ for $z \in D$, we obtain that for any two integers M, N with $N > M > 0$, and all $z \in D$,

$$\left| \prod_{n=1}^{N} F_n(z) - \prod_{n=1}^{M} F_n(z) \right| \leqslant \exp\Big(\sum_{n=1}^{M} M_n \Big) \cdot \left(\exp\Big(\sum_{n=M+1}^{N} M_n \Big) - 1 \right).$$

Since the right-hand side is independent of z and tends to 0 as $M, N \to \infty$, the uniform convergence follows. Further, if $F_n(z) \neq 0$ for all n then $\prod_{n=1}^{\infty} F_n(z) \neq 0$ by Proposition 0.4. $\square$

0.4 Integration

In this course, all integrals will be Lebesgue integrals of real or complex measurable functions on $\mathbb{R}^n$ (always with respect to the Lebesgue measure on $\mathbb{R}^n$). It is not really necessary to know what these are, and you will be perfectly able to follow the course without any knowledge of Lebesgue theory. But we will often have to deal with infinite integrals of infinite series of functions, and to handle these, Lebesgue theory is much more convenient than the theory of Riemann integrals.

It is important to mention here that Lebesgue integrals are equal to Riemann integrals whenever the latter are defined. However, Lebesgue integrals can be defined for a much larger class of functions. Further, in Lebesgue theory there are some very powerful convergence theorems for sequences of functions, theorems on interchanging multiple integrals, etc., which we will frequently apply. If you are willing to take for granted that all functions appearing in this course are measurable, there will be no problem to understand or apply these theorems.

In this subsection we have collected a few useful facts, which are amply sufficient for our course.

0.4.1 Measurable sets

The length of a bounded interval $I = [a, b], [a, b), (a, b]$ or (a, b), where $a, b \in \mathbb{R}$, $a < b$, is given by $l(I) := b - a$. Let $n \in \mathbb{Z}_{\geqslant 1}$. An *interval* in $\mathbb{R}^n$ is a cartesian product of bounded intervals $I = \prod_{i=1}^n I_i$. We define the volume of I by $l(I) := \prod_{i=1}^n l(I_i)$.

Let A be an arbitrary subset of $\mathbb{R}^n$. We define the *outer measure* of A by

$$\lambda^*(A) := \inf \sum_{i=1}^{\infty} l(I_i),$$

where the infimum is taken over all countable unions of intervals $\bigcup_{i=1}^{\infty} I_i \supset A$. We say that a set A is *measurable* if

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \cap A^c) \quad \text{for every } S \subseteq \mathbb{R}^n,$$

where $A^c = \mathbb{R}^n \setminus A$ is the complement of A. In this case we define the (Lebesgue) measure of A by $\lambda(A) := \lambda^*(A)$. This measure may be finite or infinite. It can be shown that intervals are measurable, and that $\lambda(I) = l(I)$ for any interval I in $\mathbb{R}^n$.

Facts:

- A countable union $\bigcup_{i=1}^{\infty} A_i$ of measurable sets A_i is measurable. Further, the complement of a measurable set is measurable. Hence a countable intersection of measurable sets is measurable.

- All open and closed subsets of $\mathbb{R}^n$ are measurable.

- Let $A = \cup_{i=1}^{\infty} A_i$ be a countable union of pairwise disjoint measurable sets. Then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_i)$, where we agree that $\lambda(A) = 0$ if $\lambda(A_i) = 0$ for all i.

- Under the assumption of the axiom of choice, one can construct non-measurable subsets of $\mathbb{R}^n$.

Let A be a measurable subset of $\mathbb{R}^n$. We say that a particular condition holds for *almost all* $x \in A$, it if holds for all $x \in A$ with the exception of a subset of Lebesgue measure 0. If the condition holds for almost all $x \in \mathbb{R}^n$, we say that it holds *almost everywhere*.

All sets occurring in this course will be measurable; we will never bother about the verification in individual cases.

0.4.2 Measurable functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called measurable if for every $a \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : f(x) > a\}$ is measurable.

A function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable.

Facts:

- If $A \subset \mathbb{R}^n$ is measurable then its characteristic function, given by $I_A(x) = 1$ if $x \in A$, $I_A(x) = 0$ otherwise is measurable.

- Every continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable. More generally, f is measurable if its set of discontinuities has Lebesgue measure 0.

- If $f, g : \mathbb{R}^n \to \mathbb{C}$ are measurable then $f + g$ and fg are measurable. Further, the function given by $x \mapsto f(x)/g(x)$ if $g(x) \neq 0$ and $x \mapsto 0$ if $g(x) = 0$ is measurable.
- If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable, then so are $\max(f, g)$ and $\min(f, g)$.
- If $\{f_k : \mathbb{R}^n \to \mathbb{C}\}$ is a sequence of measurable functions and $f_k \to f$ pointwise on $\mathbb{R}^n$, then f is measurable.

All functions occurring in our course can be proved to be measurable by combining the above facts. We will always omit such nasty verifications, and take the measurability of the functions for granted.

0.4.3 Lebesgue integrals

The Lebesgue integral is defined in various steps.

1) An elementary function on $\mathbb{R}^n$ is a function of the type $f = \sum_{i=1}^r c_i I_{D_i}$, where $D_1, \dots, D_r$ are pairwise disjoint measurable subsets of $\mathbb{R}^n$, and $c_1, \dots, c_r$ positive reals. Then we define $\int f dx := \sum_{i=1}^r c_i \lambda(D_i)$.

2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and $f \geqslant 0$ on $\mathbb{R}^n$. Then we define $\int f dx := \sup \int g dx$ where the supremum is taken over all elementary functions $g \leqslant f$. Thus, $\int f dx$ is defined and $\geqslant 0$ but it may be infinite.

3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary measurable function. Then we define

$$\int f dx := \int \max(f, 0) dx - \int \max(-f, 0) dx,$$

provided that at least one of the integrals is finite. If both integrals are finite, we say that f is *integrable* or *summable*.

4) Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that f is integrable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable, and in that case we define

$$\int f dx := \int (\operatorname{Re} f) dx + i \int (\operatorname{Im} f) dx.$$

5) Let D be a measurable subset of $\mathbb{R}^n$. Let f be a complex function defined on a set containing D. We define $f \cdot I_D$ by defining it to be equal to f on D and equal to 0 outside D. We say that f is measurable on D if $f \cdot I_D$ is measurable. Further, we say that f is integrable over D if $f \cdot I_D$ is integrable, and in that case we define $\int_D f dx := \int f \cdot I_D dx$.

Facts:

- Let D be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a measurable function. Then f is integrable over D if and only if $\int_D |f| dx < \infty$ and in that case, $|\int_D f dx| \leqslant \int_D |f| dx$.
- Let again D be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$, $g : D \to \mathbb{R}_{\geqslant 0}$ measurable functions, such that $\int_D g dx < \infty$ and $|f| \leqslant g$ on D. Then f is integrable over D, and $|\int_D f dx| \leqslant \int_D g dx$.
- Let D be a closed interval in $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a bounded function which is Riemann integrable over D. Then f is Lebesgue integrable over D and the Lebesgue integral $\int_D f dx$ is equal to the Riemann integral $\int_D f(x) dx$.
- Let $f : [0, \infty) \to \mathbb{C}$ be such that the improper Riemann integral $\int_0^\infty |f(x)| dx$ converges. Then the improper Riemann integral $\int_0^\infty f(x) dx$ converges as well, and it is equal to the Lebesgue integral $\int_{[0,\infty)} f dx$. However, an improper Riemann integral $\int_0^\infty f(x) dx$ which is convergent, but for which $\int_0^\infty |f(x)| dx = \infty$ can not be interpreted as a Lebesgue integral. The same applies to the other types of improper Riemann integrals, e.g., $\int_a^b f(x) dx$ where f is unbounded on (a, b).
- An absolutely convergent series of complex terms $\sum_{n=0}^\infty a_n$ may be interpreted as a Lebesgue integral. Define the function A by $A(x) := a_n$ for $x \in \mathbb{R}$ with $n \leqslant x < n+1$ and $A(x) := 0$ for $x < 0$. Then A is measurable and integrable, and $\sum_{n=0}^\infty a_n = \int A dx$.

0.4.4 Important theorems

Theorem 0.7 (Dominated Convergence Theorem). *Let $D \subseteq \mathbb{R}^n$ be a measurable set and $\{f_k : D \to \mathbb{C}\}_{k \geqslant 0}$ a sequence of functions that are all integrable over D, and such that $f_k \to f$ pointwise on D. Assume that there is an integrable function $g : D \to \mathbb{R}_{\geqslant 0}$ such that $|f_k(x)| \leqslant g(x)$ for all $x \in D$, $k \geqslant 0$. Then f is integrable over D, and $\int_D f_k dx \to \int_D f dx$.*

Corollary 0.8. *let $D \subset \mathbb{R}^n$ be a measurable set of finite measure and $\{f_k : D \to \mathbb{C}\}_{k\geqslant 0}$ a sequence of functions that are all integrable over D, and such that $f_k \to f$ uniformly on D. Then f is integrable over D, and $\int_D f_k dx \to \int_D f dx$.*

Proof. Let $\varepsilon > 0$. There is k_0 such that $|f(x) - f_k(x)| < \varepsilon$ for all $x \in D$, $k > k_0$. The constant function $x \mapsto \varepsilon$ is integrable over D since D has finite measure. Hence for $k > k_0$, $f - f_k$ is integrable over D, and so f is integrable over D. Consequently, $|f|$ is integrable over D. Now $|f_k| < \varepsilon + |f|$ for $k > k_0$. So by the Dominated Convergence Theorem, $\int_D f_k dx \to \int_D f dx$. □

In the theorem below, we write points of $\mathbb{R}^{m+n}$ as (x, y) with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Further, dx, dy, $d(x, y)$ denote the Lebesgue measures on $\mathbb{R}^m$, $\mathbb{R}^n$, $\mathbb{R}^{m+n}$, respectively.

Theorem 0.9 (Fubini-Tonelli)**.** *Let D_1, D_2 be measurable subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively, and $f : D_1 \times D_2 \to \mathbb{C}$ a measurable function. Assume that at least one of the integrals*

$$\int_{D_1\times D_2} |f(x,y)|d(x,y), \quad \int_{D_1} \left(\int_{D_2} |f(x,y)|dy \right) dx, \quad \int_{D_2} \left(\int_{D_1} |f(x,y)|dx \right) dy$$

is finite. Then they are all finite and equal.
Further, f is integrable over $D_1 \times D_2$, $x \mapsto f(x, y)$ is integrable over D_1 for almost all $y \in D_2$, $y \mapsto f(x, y)$ is integrable over D_2 for almost all $x \in D_1$, and

$$\int_{D_1\times D_2} f(x,y)d(x,y) = \int_{D_1} \left(\int_{D_2} f(x,y)dy \right) dx = \int_{D_2} \left(\int_{D_1} f(x,y)dx \right) dy.$$

Corollary 0.10. *Let D be a measurable subset of $\mathbb{R}^m$ and $\{f_k : D \to \mathbb{C}\}_{k\geqslant 0}$ a sequence of functions that are all integrable over D and such that $\sum_{k=0}^{\infty} |f_k|$ converges pointwise on D. Assume that at least one of the quantities*

$$\sum_{k=0}^{\infty} \int_D |f_k(x)|dx, \quad \int_D \Big(\sum_{k=0}^{\infty} |f_k(x)| \Big) dx$$

is finite. Then $\sum_{k=0}^{\infty} f_k$ is integrable over D and

$$\sum_{k=0}^{\infty} \int_D f_k(x)dx = \int_D \Big(\sum_{k=0}^{\infty} f_k(x) \Big) dx.$$

Proof. Apply the Theorem of Fubini-Tonelli with $n = 1$, $D_1 = D$, $D_2 = [0, \infty)$, $F(x, y) = f_k(x)$ where k is the integer with $k \leqslant y < k + 1$. $\square$

Corollary 0.11. *Let $\{a_{kl}\}_{k,l=0}^{\infty}$ be a double sequence of complex numbers such that at least one of*

$$\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty}|a_{kl}|\Big),\quad \sum_{l=0}^{\infty}\Big(\sum_{k=0}^{\infty}|a_{kl}|\Big)$$

converges. Then both

$$\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty}a_{kl}\Big),\quad \sum_{l=0}^{\infty}\Big(\sum_{k=0}^{\infty}a_{kl}\Big)$$

converge and are equal.

Proof. Apply the Theorem of Fubini-Tonelli with $m = n = 1$, $D_1 = D_2 = [0, \infty)$, $F(x, y) = a_{kl}$ where k, l are the integers with $k \leqslant x < k + 1$, $l \leqslant y < l + 1$. $\square$

0.4.5 Useful inequalities

We have collected some inequalities, stated without proof, which frequently show up in analytic number theory. The proofs belong to a course in measure theory or functional analysis.

Proposition 0.12. *Let D be a measurable subset of $\mathbb{R}^n$ and $f, g : D \to \mathbb{C}$ measurable functions. Let p, q be reals > 1 with $\frac{1}{p} + \frac{1}{q} = 1$. Then if all integrals are defined,*

$$\Big|\int_D fg \cdot dx\Big| \leqslant \Big(\int_D |f|^p dx\Big)^{1/p} \cdot \Big(\int_D |g|^q dx\Big)^{1/q} \quad \textit{(Hölder's Inequality).}$$

In particular,

$$\Big|\int_D fgdx\Big| \leqslant \Big(\int_D |f|^2 dx\Big)^{1/2} \cdot \Big(\int_D |g|^2 dx\Big)^{1/2} \quad \textit{(Cauchy-Schwarz' Inequality).}$$

Corollary 0.13. *Let $a_1, \ldots, a_r$, $b_1, \ldots, b_r$ be complex numbers and p, q reals > 1 with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Big|\sum_{n=1}^{r} a_n b_n\Big| \leqslant \Big(\sum_{n=1}^{r}|a_n|^p\Big)^{1/p} \cdot \Big(\sum_{n=1}^{r}|b_n|^q\Big)^{1/q} \quad \textit{(Hölder).}$$

In particular,

$$\Big|\sum_{n=1}^{r} a_n b_n\Big| \leqslant \Big(\sum_{n=1}^{r} |a_n|^2\Big)^{1/2} \cdot \Big(\sum_{n=1}^{r} |b_n|^2\Big)^{1/2} \quad \textit{(Cauchy-Schwarz).}$$

This follows from Proposition 0.12 by taking $D = [0, r)$, $f(x) = a_n$, $g(x) = b_n$ for $n - 1 \leqslant x < n$, $n = 1, \ldots, r$.

A function φ from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ is called *convex* if $\varphi((1-t)x + ty) \leqslant (1-t)\varphi(x) + t\varphi(y)$ holds for all $x, y \in I$ and all $t \in [0, 1]$. In particular, φ is convex on I if φ is twice differentiable and $\varphi'' \geqslant 0$ on I.

Proposition 0.14. *Let D be a measurable subset of $\mathbb{R}^n$ with $0 < \lambda(D) < \infty$, let $f : D \to \mathbb{R}_{>0}$ be a Lebesgue integrable function and let $\varphi : \mathbb{R}_{>0} \to \mathbb{R}$ be a convex function. Then*

$$\varphi\Big(\frac{1}{\lambda(D)} \int_D f \cdot dx\Big) \leqslant \frac{1}{\lambda(D)} \int_D (\varphi \circ f) dx \quad \textit{(Jensen's Inequality).}$$

Corollary 0.15. *Let $a_1, \ldots, a_r$ be positive reals, and let $\varphi : \mathbb{R}_{>0} \to \mathbb{R}$ be a convex function. Then*

$$\varphi\Big(\frac{1}{r} \sum_{n=1}^{r} a_n\Big) \leqslant \frac{1}{r} \sum_{n=1}^{r} \varphi(a_n).$$

In particular,

$$\frac{1}{r} \sum_{n=1}^{r} a_n \geqslant \sqrt[r]{a_1 \cdots a_n} \quad \textit{(arithmetic mean} \geqslant \textit{geometric mean).}$$

The first assertion follows by applying Proposition 0.14 with $D = [0, r)$ and $f(x) = a_n$ for $x \in [n-1, n)$. The second assertion follows by applying the first with $\varphi(x) = -\log x$.

0.5 Line integrals

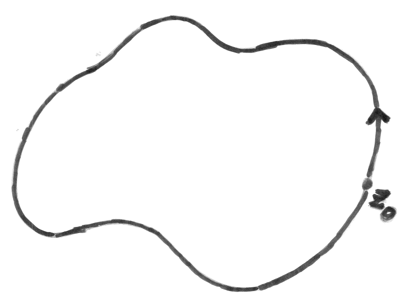
0.5.1 Paths in $\mathbb{C}$

We consider continuous functions $g : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$ and $a < b$. Two continuous functions $g_1 : [a, b] \to \mathbb{C}$, $g_2 : [c, d] \to \mathbb{C}$ are called equivalent if there is a continuous monotone increasing function $\varphi : [a, b] \to [c, d]$ such that $g_1 = g_2 \circ \varphi$. The equivalence classes of this relation are called *paths* (in $\mathbb{C}$), and a function $g : [a, b] \to \mathbb{C}$ representing a path is called a *parametrization* of the path. Roughly speaking, a path is a curve in $\mathbb{C}$, together with a direction in which it is traversed.

A *(continuously) differentiable path* is a path represented by a (continuously) differentiable function $g : [a, b] \to \mathbb{C}$.

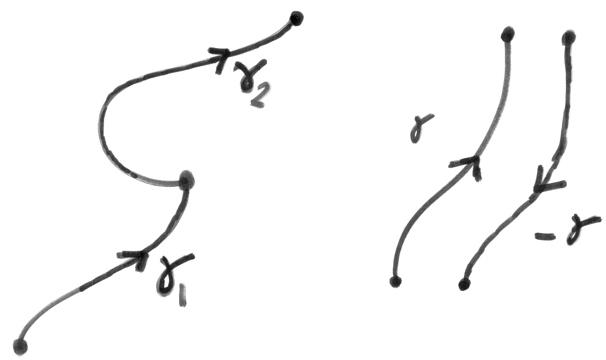
Let γ be a path. Choose a parametrization $g : [a, b] \to \mathbb{C}$ of γ. We call $g(a)$ the *start point* and $g(b)$ the *end point* of γ. Further, $g([a, b])$ is called the *support* of γ. By saying that a function is continuous on γ, or that γ is contained in a particular set, etc., we mean the support of γ.

The path γ is said to be *closed* if its end point is equal to its start point, i.e., if $g(a) = g(b)$. The path γ is called a *contour* if it is closed, has no self-intersections, and is traversed counterclockwise (we will not give the cumbersome formal definition of this intuitively obvious notion).



Let γ_1, γ_2 be paths, such that the end point of γ_1 is equal to the start point of γ_2. We define $\gamma_1 + \gamma_2$ to be the path obtained by first traversing γ_1 and then γ_2. For instance, if $g_1 : [a, b] \to \mathbb{C}$ is a parametrization of γ_1 then we may choose a parametrization $g_2 : [b, c] \to \mathbb{C}$ of γ_2; then $g : [a, c] \to \mathbb{C}$ defined by $g(t) := g_1(t)$ if $a \leqslant t \leqslant b$, $g(t) := g_2(t)$ if $b \leqslant t \leqslant c$ is a parametrization of $\gamma_1 + \gamma_2$.

Given a path γ, we define $-\gamma$ to be the path traversed in the opposite direction, i.e., the start point of $-\gamma$ is the end point of γ and conversely.



Let γ be a path and $F : \gamma \to \mathbb{C}$ a continuous function on (the support of) γ. Then $F(\gamma)$ is the path such that if $g : [a, b] \to \mathbb{C}$ is a parametrization of γ then $F \circ g : [a, b] \to \mathbb{C}$ is a parametrization of $F(\gamma)$.

0.5.2 Line integrals

All paths occurring in our course will be built up from circle segments and line segments. So for our purposes, it suffices to define integrals of continuous functions along *piecewise continuously differentiable paths*, these are paths of the shape $\gamma_1 + \cdots + \gamma_r$, where $\gamma_1, \ldots, \gamma_r$ are continuously differentiable paths, and for $i = 1, \ldots, r-1$, the end point of γ_i coincides with the start point of γ_{i+1}.

let γ be a continuously differentiable path, and $f : \gamma \to \mathbb{C}$ a continuous function. Choose a continuously differentiable parametrization $g : [a, b] \to \mathbb{C}$ of γ. Then we define

$$\int_\gamma f(z)dz := \int_a^b f(g(t))g'(t)dt.$$

Further, we define the *length* of γ by

$$L(\gamma) := \int_a^b |g'(t)|dt.$$

These notions do not depend on the choice of g.

If $\gamma = \gamma_1 + \cdots + \gamma_r$ is a piecewise continuously differentiable path with continuously differentiable pieces $\gamma_1, \ldots, \gamma_r$ and $f : \gamma \to \mathbb{C}$ is continuous, we define

$$\int_\gamma f(z)dz := \sum_{i=1}^{r} \int_{\gamma_i} f(z)dz$$

and

$$L(\gamma) := \sum_{i=1}^{r} L(\gamma_i).$$

In case that γ is closed, we write $\oint_\gamma f(z)dz$. It can be shown that the value of this integral is independent of the choice of the common start point and end point of γ.

We mention here that line integrals $\int_\gamma f(z)dz$ can be defined also for paths γ that are not necessarily piecewise continuously differentiable. For piecewise continuously differentiable paths, this new definition coincides with the one given above.

Let γ be any path and choose a parametrization $g : [a, b] \to \mathbb{C}$ of γ. A partition of $[a, b]$ is a tuple $P = (t_0, \ldots, t_s)$ where $a = t_0 < t_1 < \cdots < t_s = b$. We define the length of γ by

$$L(\gamma) := \sup_P \sum_{i=1}^{s} |g(t_i) - g(t_{i-1})|,$$

where the supremum is taken over all partitions P of $[a, b]$. This does not depend on the choice of g. We call γ *rectifiable* if $L(\gamma) < \infty$ (in another language, this means that the function g is of *bounded variation*).

Let γ be a rectifiable path, and $g : [a, b] \to \mathbb{C}$ a parametrization of γ. Given a partition $P = (t_0, \ldots, t_s)$ of $[a, b]$, we define the mesh of P by

$$\delta(P) := \max_{1 \leqslant i \leqslant s} |t_i - t_{i-1}|.$$

A sequence of intermediate points of P is a tuple $W = (w_1, \ldots, w_s)$ such that $t_0 < w_1 < t_1 < w_2 < t_2 < \cdots < t_s$.

Let $f : \gamma \to \mathbb{C}$ be a continuous function. For a partition P of $[a, b]$ and a tuple of intermediate points W of P we define

$$S(f, g, P, W) := \sum_{i=1}^{s} f(g(w_i))(g(t_i) - g(t_{i-1})).$$

One can show that there is a finite number, denoted $\int_\gamma f(z)dz$, such that for any choice of parametrization $g : [a,b] \to \mathbb{C}$ of γ and any sequence $(P_n, W_n)_{n\geqslant 0}$ of partitions P_n of $[a,b]$ and sequences of intermediate points W_n of P_n with $\delta(P_n) \to 0$,

$$\int_\gamma f(z)dz = \lim_{n\to\infty} S(f, g, P_n, W_n).$$

In another language, $\int_\gamma f(z)dz$ is equal to the *Riemann-Stieltjes integral* $\int_a^b f(g(t))dg(t)$.

0.5.3 Properties of line integrals

Below (and in the remainder of the course), by a path we will mean a piecewise continuously differentiable path. In fact, all properties below hold for line integrals over rectifiable paths, but in textbooks on complex analysis, these properties are never proved in this generality.

- Let γ be a path, and $f : \gamma \to \mathbb{C}$ continuous. Then
$$\left|\int_\gamma f(z)dz\right| \leqslant L(\gamma) \cdot \sup_{z\in\gamma} |f(z)|.$$
- Let γ_1, γ_2 be two paths such that the end point of γ_1 and the start point of γ_2 coincide. Let $f : \gamma_1 + \gamma_2 \to \mathbb{C}$ continuous. Then
$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$
- Let γ be a path and $f : \gamma \to \mathbb{C}$ continuous. Then
$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$
- Let γ be a path and $\{f_n : \gamma \to \mathbb{C}\}_{n=1}^\infty$ a sequence of continuous functions. Suppose that $f_n \to f$ uniformly on γ, i.e., $\sup_{z\in\gamma} |f_n(z) - f(z)| \to 0$ as $n \to \infty$. Then f is continuous on γ, and $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$ as $n \to \infty$.
- Call a function $F : U \to \mathbb{C}$ on an open subset U of $\mathbb{C}$ analytic if for every $z \in U$ the limit
$$F'(z) = \lim_{h\in\mathbb{C},\, h\to 0} \frac{F(z+h) - F(z)}{h}$$

exists. Let γ be a path with start point z_0 and end point z_1, and let F be an analytic function defined on an open set $U \subset \mathbb{C}$ that contains γ. Then

$$\int_\gamma F'(z)dz = F(z_1) - F(z_0).$$

- Let γ be a path and F an analytic function defined on some open set containing γ. Further, let $f : F(\gamma) \to \mathbb{C}$ be continuous. Then

$$\int_{F(\gamma)} f(w)dw = \int_\gamma f(F(z))F'(z)dz.$$

Examples. **1.** Let $\gamma_{a,r}$ denote the circle with center a and radius r, traversed counterclockwise. For $\gamma_{a,r}$ we may choose a parametrization $t \mapsto a + re^{2\pi it}$, $t \in [0, 1]$. Let $n \in \mathbb{Z}$. Then

$$\begin{aligned}\oint_{\gamma_{a,r}} (z-a)^n dz &= \int_0^1 r^n e^{2n\pi it} \cdot 2\pi i \cdot re^{2\pi it} dt \\ &= 2\pi i r^{n+1} \int_0^1 e^{2(n+1)\pi it} dt = \begin{cases} 2\pi i & \text{if } n = -1; \\ 0 & \text{if } n \neq -1. \end{cases}\end{aligned}$$

2. For $z_0, z_1 \in \mathbb{C}$, denote by $[z_0, z_1]$ the line segment from z_0 to z_1. For $[z_0, z_1]$ we may choose a parametrization $t \mapsto z_0 + t(z_1 - z_0)$, $t \in [0, 1]$. Let $f : [z_0, z_1] \to \mathbb{C}$ be continuous. Then

$$\int_{[z_0,z_1]} f(z)d(z) = \int_0^1 f(z_0 + t(z_1 - z_0))(z_1 - z_0)dt.$$

0.6 Topology

We recall some facts about the topology of $\mathbb{C}$.

0.6.1 Basic facts

Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$. We define the open disk and closed disk with center a and radius r,

$$D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}, \quad \overline{D}(a, r) = \{z \in \mathbb{C} : |z - a| \leqslant r\}.$$

Recall that a subset U of $\mathbb{C}$ is called open if either $U = \emptyset$, or for every $a \in U$ there is $\delta > 0$ with $D(a, \delta) \subset U$. A subset U of $\mathbb{C}$ is closed if its complement $U^c = \mathbb{C} \setminus U$ is open. It is easy to verify that the union of any possibly infinite collection of open subsets of $\mathbb{C}$ is open. Further, the intersection of finitely many open subsets is open. Consequently, the intersection of any possibly infinite collection of closed sets is closed, and the union of finitely many closed subsets is closed.

A subset S of $\mathbb{C}$ is called *compact*, if for every collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{C}$ with $S \subset \bigcup_{\alpha \in I} U_\alpha$ there is a finite subset F of I such that $S \subset \bigcup_{\alpha \in F} U_\alpha$, in other words, every open cover of S has a finite subcover.

By the *Heine-Borel Theorem*, a subset of $\mathbb{C}$ is compact if and only if it is closed and bounded.

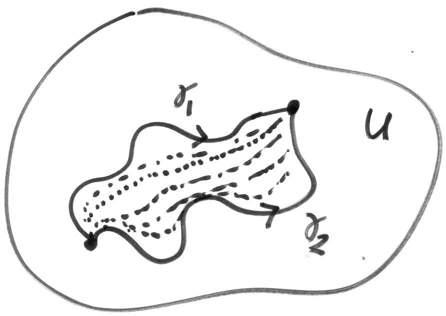
Let U be a non-empty subset of $\mathbb{C}$. A point $z_0 \in \mathbb{C}$ is called a *limit point* of U if there is a sequence $\{z_n\}$ in U such that all z_n are distinct and $z_n \to z_0$ as $n \to \infty$. Recall that a non-empty subset U of $\mathbb{C}$ is closed if and only if each of its limit points belongs to U.

Let U be a non-empty subset of $\mathbb{C}$, and $S \subset U$. Then S is called *discrete in* U if it has no limit points in U. Recall that by the *Bolzano-Weierstrass Theorem*, every infinite subset of a compact set K has a limit point in K. This implies that S is discrete in U if and only if for every compact set K with $K \subset U$, the intersection $K \cap S$ is finite.

Let U be a non-empty, open subset of $\mathbb{C}$. We say that U is *connected* if there are no non-empty open sets U_1, U_2 with $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. We say that U is *pathwise connected* if for every $z_0, z_1 \in U$ there is a path $\gamma \subset U$ with start point z_0 and end point z_1. A fact (typical for the topological space $\mathbb{C}$) is that a non-empty open subset U of $\mathbb{C}$ is connected if and only if it is pathwise connected.

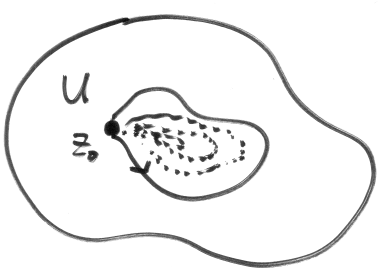
Let U be any, non-empty open subset of $\mathbb{C}$. We can express U as a disjoint union $\bigcup_{\alpha \in I} U_\alpha$, with I some index set, such that two points of U belong to the same set U_α if and only if they are connected by a path contained in U. The sets U_α are open, connected, and pairwise disjoint. We call these sets U_α the connected components of U.

0.6.2 Homotopy



Let $U \subseteq \mathbb{C}$ and γ_1, γ_2 two paths in U with start point z_0 and end point z_1. Then γ_1, γ_2 are homotopic in U if one can be continuously deformed into the other within U. More precisely this means the following: there are parametrizations $f : [0,1] \to \mathbb{C}$ of γ_1, $g : [0,1] \to \mathbb{C}$ of γ_2 and a continuous map $H : [0,1] \times [0,1] \to U$ with the following properties:

$$H(0,t) = f(t), \quad H(1,t) = g(t) \quad \text{for } 0 \leqslant t \leqslant 1;$$
$$H(s,0) = z_0, \quad H(s,1) = z_1 \quad \text{for } 0 \leqslant s \leqslant 1.$$



Let $U \subseteq \mathbb{C}$ be open and non-empty. We call U *simply connected* ('without holes') if it is connected and if every closed path in U can be contracted to a point in U, that is, if z_0 is any point in U and γ is any closed path in U containing z_0, then γ is homotopic in U to z_0.

A map $f : D_1 \to D_2$, where D_1, D_2 are subsets of $\mathbb{C}$, is called a *homeomorphism* if f is a bijection, and both f and f^{-1} are continuous. Homeomorphisms preserve topological properties of sets such as openness, closedness, boundedness, (simple) connectedness, etc.

Theorem 0.16 (Schoenflies Theorem for curves)**.** *Let γ be a contour in $\mathbb{C}$. Then there is a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\gamma_{0,1}) = \gamma$, where $\gamma_{0,1}$ is the unit circle with center 0 and radius 1, traversed counterclockwise.*

Corollary 0.17 (Jordan Curve Theorem)**.** *Let γ be a contour in $\mathbb{C}$. Then $\mathbb{C} \setminus \gamma$ has* *two connected components, U_1 and U_2. The component U_1 is bounded and simply connected, while U_2 is unbounded.*

The component U_1 is called the *interior* of γ, notation $\mathrm{int}(\gamma)$, and U_2 the *exterior* of γ, notation $\mathrm{ext}(\gamma)$.

0.7 Complex analysis

0.7.1 Basics

In what follows, U is a non-empty open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. We say that f is *holomorphic* or *analytic* in $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

In that case, the limit is denoted by $f'(z_0)$. We say that f is analytic on U if f is analytic in every $z \in U$; in that case, the derivative $f'(z)$ is defined for every $z \in U$. We say that f is analytic around z_0 if it is analytic on some open disk $D(z_0, \delta)$ for some $\delta > 0$. Finally, given a not necessarily open subset A of $\mathbb{C}$ and a function $f : A \to \mathbb{C}$, we say that f is analytic on A if there is an open set $U \supseteq A$ such that f is defined on U and analytic on U. An everywhere analytic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.

For any two analytic functions f, g on some open set $U \subseteq \mathbb{C}$,we have the usual rules for differentiation $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$ and $(f/g)' = (gf' - fg')/g^2$ (the latter is defined for any z with $g(z) \neq 0$). Further, given a non-empty set $U \subseteq \mathbb{C}$, and analytic functions $f : U \to \mathbb{C}$, $g : f(U) \to \mathbb{C}$, the composition $g \circ f$ is analytic on U and $(g \circ f)' = (g' \circ f) \cdot f'$.

Recall that a power series around $z_0 \in \mathbb{C}$ is an infinite sum

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. The radius of convergence of this series is given by

$$R = R_f = \left(\limsup_{n \to \infty} \sqrt[n]{|a_n|} \right)^{-1}.$$

We state without proof the following fact.

Theorem 0.18. *Let $z_0 \in \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ a power series around $z_0 \in \mathbb{C}$ with radius of convergence $R > 0$. Then f defines a function on $D(z_0, R)$ which is analytic infinitely often. For $k \geqslant 0$ the k-th derivative $f^{(k)}$ of f has a power series expansion with radius of convergence R given by*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}.$$

In each of the examples below, R denotes the radius of convergence of the given power series.

$$\begin{aligned}
&e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, && R = \infty, \quad (e^z)' = e^z. \\
&\cos z = (e^{iz} + e^{-iz})/2 = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, && R = \infty, \quad \cos' z = -\sin z. \\
&\sin z = (e^{iz} - e^{-iz})/2i = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, && R = \infty, \quad \sin' z = \cos z. \\
&(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, && R = 1, \quad ((1+z)^\alpha)' = \alpha(1+z)^{\alpha-1} \\
&\text{where } \alpha \in \mathbb{C}, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}. && \\
&\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot z^n, && R = 1, \quad \log'(1+z) = (1+z)^{-1}.
\end{aligned}$$

0.7.2 Cauchy's Theorem and some applications

In the remainder of this course, a path will always be a piecewise continuously differentiable path. Recall that for a piecewise continuously differentiable path γ, say $\gamma = \gamma_1 + \cdots + \gamma_r$ where $\gamma_1, \ldots, \gamma_r$ are paths with continuously differentiable parametrizations $g_i : [a_i, b_i] \to \mathbb{C}$, and for a continuous function $f : \gamma \to \mathbb{C}$ we have $\int_\gamma f(z)dz = \sum_{i=1}^r \int_{a_i}^{b_i} f(g_i(t))g_i'(t)dt$.

Theorem 0.19 (Cauchy)**.** *Let $U \subseteq \mathbb{C}$ be a non-empty open set and $f : U \to \mathbb{C}$ an analytic function. Further, let γ_1, γ_2 be two paths in U with the same start point and end point that are homotopic in U. Then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. Any textbook on complex analysis. □

Corollary 0.20. *Let $U \subseteq \mathbb{C}$ be a non-empty, open, simply connected set, and $f : U \to \mathbb{C}$ an analytic function. Then for any closed path γ in U,*

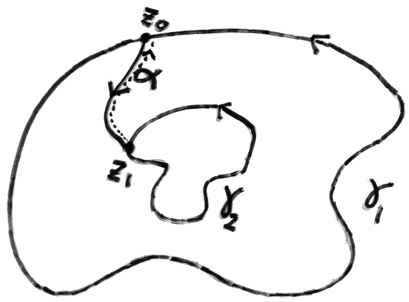
$$\oint_\gamma f(z)dz = 0.$$

Proof. The path γ is homotopic in U to a point, and a line integral along a point is 0. □

Corollary 0.21. *Let γ_1, γ_2 be two contours (closed paths without self-intersections traversed counterclockwise), such that γ_2 is contained in the interior of γ_1. Let $U \subset \mathbb{C}$ be an open set which contains γ_1, γ_2 and the region between γ_1 and γ_2. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then*

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

Proof.



Let z_0, z_1 be points on γ_1, γ_2 respectively, and let α be a path from z_0 to z_1 lying inside the region between γ_1 and γ_2 without self-intersections.

Then γ_1 is homotopic in U to the path $\alpha + \gamma_2 - \alpha$, which consists of first traversing α, then γ_2, and then α in the opposite direction. Hence

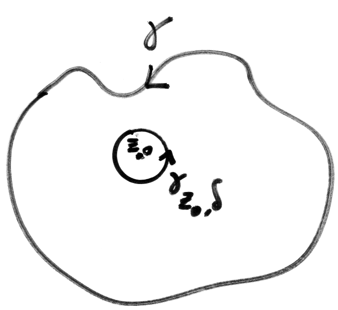
$$\oint_{\gamma_1} f(z)dz = \left(\int_\alpha + \oint_{\gamma_2} - \int_\alpha\right) f(z)dz = \oint_{\gamma_2} f(z)dz.$$

□

Corollary 0.22 (Cauchy's Integral Formula). *Let γ be a contour in $\mathbb{C}$, $U \subset \mathbb{C}$ an open set containing γ and its interior, z_0 a point in the interior of γ, and $f : U \to \mathbb{C}$ an analytic function. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \cdot dz = f(z_0).$$

Proof.



Let $\gamma_{z_0,\delta}$ be the circle with center z_0 and radius δ, traversed counterclockwise. Then by Corollary 0.21 we have for any sufficiently small $\delta > 0$,

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \cdot dz = \frac{1}{2\pi i} \oint_{\gamma_{z_0,\delta}} \frac{f(z)}{z - z_0} \cdot dz.$$

Now, since $f(z)$ is continuous, hence uniformly continuous on any sufficiently small compact set containing z_0,

$$\left|\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\cdot dz\,-\,f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}} \frac{f(z)}{z-z_0}\cdot dz\,-\,f(z_0)\right|$$

$$\begin{aligned}
&= \left|\int_0^1 \frac{f(z_0+\delta e^{2\pi it})}{\delta e^{2\pi it}}\cdot \delta e^{2\pi it}dt\,-\,f(z_0)\right| \\
&= \left|\int_0^1 \left\{f(z_0+\delta e^{2\pi it}) - f(z_0)\right\}dt\right| \leqslant \sup_{0\leqslant t\leqslant 1} |f(z_0+\delta e^{2\pi it}) - f(z_0)| \\
&\to 0 \;\text{ as } \delta \downarrow 0.
\end{aligned}$$

This completes our proof. □

We now show that every analytic function f on a simply connected set has an anti-derivative. We first prove a simple lemma.

Lemma 0.23. *Let $U \subseteq \mathbb{C}$ be a non-empty, open, connected set, and let $f : U \to \mathbb{C}$ be an analytic function such that $f' = 0$ on U. Then f is constant on U.*

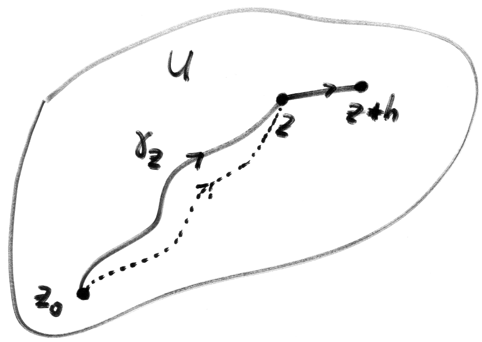
Proof. Fix a point $z_0 \in U$ and let $z \in U$ be arbitrary. Take a path γ_z in U from z_0 to z which exists since U is (pathwise) connected. Then

$$f(z) - f(z_0) = \int_{\gamma_z} f'(w)dw = 0.$$

□

Corollary 0.24. *Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected set, and $f : U \to \mathbb{C}$ an analytic function. Then there exists an analytic function $F : U \to \mathbb{C}$ with $F' = f$. Further, F is determined uniquely up to addition with a constant.*

Proof (sketch). If F_1, F_2 are any two analytic functions on U with $F_1' = F_2' = f$, then $F_1' - F_2'$ is constant on U since U is connected. This shows that an anti-derivative of f is determined uniquely up to addition with a constant. It thus suffices to prove the existence of an analytic function F on U with $F' = f$.



Fix $z_0 \in U$. Given $z \in U$, we define $F(z)$ by

$$F(z) := \int_{\gamma_z} f(w)dw,$$

where γ_z is any path in U from z_0 to z. This does not depend on the choice of γ_z. For let γ_1, γ_2 be any two paths in U from z_0 to z. Then $\gamma_1 - \gamma_2$ (the path consisting of first traversing γ_1 and then γ_2 in the opposite direction) is homotopic to z_0 since U is simply connected, hence

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \oint_{\gamma_1-\gamma_2} f(z)dz = 0.$$

To prove that $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$, take a path γ_z from z_0 to z and then the line segment $[z, z+h]$ from z to $z+h$. Then since f is uniformly continuous on any sufficiently small compact set around z,

$$\begin{aligned} F(z+h) - F(z) &= \Big(\int_{\gamma_z+[z,z+h]} - \int_{\gamma_z}\Big) f(w)dw = \int_{[z,z+h]} f(w)dw \\ &= \int_0^1 f(z+th)hdt = h\left(f(z) + \int_0^1 (f(z+th) - f(z))dt\right). \end{aligned}$$

So

$$\begin{aligned} \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\int_0^1 (f(z+th) - f(z))dt\right| \\ &\leqslant \sup_{0\leqslant t\leqslant 1} |f(z+th) - f(z)| \to 0 \quad \text{as } h \to 0. \end{aligned}$$

This completes our proof. □

Example. Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected subset of $\mathbb{C}$ with $0 \notin U$. Then $1/z$ has an anti-derivative on U.

For instance, if $U = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0\}$ we may take as anti-derivative of $1/z$,

$$\operatorname{Log} z := \log|z| + i\operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the argument of z in the interval $(-\pi, \pi)$ (this is called the *principal value* of the logarithm).

On $\{z \in \mathbb{C} : |z-1| < 1\}$ we may take as anti-derivative of $1/z$ the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}.$$

0.7.3 Taylor series

Theorem 0.25. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in U$ and $R > 0$ be such that $D(z_0, R) \subseteq U$. Then f has a Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{converging for } z \in D(z_0, R).$$

Further, we have for $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z-z_0)^{n+1}} \cdot dz \quad \textit{for any } r \textit{ with } 0 < r < R. \tag{0.3}$$

Proof. We fix $z \in D(z_0, R)$ and use w to indicate a complex variable. Choose r with $|z - z_0| < r < R$. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \cdot dw.$$

We rewrite the integrand. We have

$$\begin{aligned}
\frac{f(w)}{w-z} &= \frac{f(w)}{(w-z_0) - (z-z_0)} = \frac{f(w)}{w-z_0} \cdot \left(1 - \frac{z-z_0}{w-z_0}\right)^{-1} \\
&= \frac{f(w)}{w-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty} \frac{f(w)}{(w-z_0)^{n+1}} \cdot (z-z_0)^n.
\end{aligned}$$

The latter series converges uniformly on $\gamma_{z_0,r}$. For let $M := \sup_{w \in \gamma_{z_0,r}} |f(w)|$. Then

$$\sup_{w \in \gamma_{z_0,r}} \left| \frac{f(w)}{(w-z_0)^{n+1}} \cdot (z-z_0)^n \right| \leqslant \frac{M}{r} \left(\frac{|z-z_0|}{r}\right)^n =: M_n$$

and $\sum_{n=0}^{\infty} M_n$ converges since $|z - z_0| < r$. Consequently,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w-z} \cdot dw \\ &= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \sum_{n=0}^{\infty} \left(\frac{f(w)}{(w-z_0)^{n+1}} \cdot (z-z_0)^n \right) dw \\ &= \sum_{n=0}^{\infty} (z-z_0)^n \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{(w-z_0)^{n+1}} \cdot dw \right\}. \end{aligned}$$

Now Theorem 0.25 follows since by Corollary 0.21 the integral in (0.3) is independent of r. □

Corollary 0.26. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set, and $f : U \to \mathbb{C}$ an analytic function. Then f is analytic on U infinitely often, that is, for every $k \geqslant 0$ the k-the derivative $f^{(k)}$ exists, and is analytic on U.*

Proof. Pick $z \in U$. Choose $\delta > 0$ such that $D(z, \delta) \subset U$. Then for $w \in D(z, \delta)$ we have

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n \quad \text{with } a_n = \frac{1}{2\pi i} \oint_{\gamma_{z,r}} \frac{f(w)}{(w-z)^{n+1}} \cdot dw \text{ for } 0 < r < \delta.$$

Now for every $k \geqslant 0$, the k-th derivative $f^{(k)}(z)$ exists and is equal to $k! a_k$. □

Corollary 0.27. *Let γ be a contour in $\mathbb{C}$, and U an open subset of $\mathbb{C}$ containing γ and its interior. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for every z in the interior of γ and every $k \geqslant 0$ we have*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$

Proof. Choose $\delta > 0$ such that $\gamma_{z,\delta}$ lies in the interior of γ. By Corollary 0.21,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw = \frac{1}{2\pi i} \oint_{\gamma_{z,\delta}} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$

By the argument in Corollary 0.26, this is equal to $f^{(k)}(z)/k!$. □

We prove a generalization of Cauchy's integral formula.

Corollary 0.28. *Let γ_1, γ_2 be two contours such that γ_1 is lying in the interior of γ_2. Let $U \subset \mathbb{C}$ be an open set which contains γ_1, γ_2 and the region between γ_1, γ_2. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for any z_0 in the region between γ_1 and γ_2 we have*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} dz.$$

Proof. We have seen that around z_0 the function f has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Define the function on U,

$$g(z) := \frac{f(z) - a_0}{z - z_0} \quad (z \neq z_0); \qquad g(z_0) := a_1.$$

The function g is clearly analytic on $U \setminus \{z_0\}$. Further,

$$\frac{g(z) - g(z_0)}{z - z_0} = \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2} \to a_2 \text{ as } z \to z_0.$$

Hence g is also analytic at $z = z_0$. In particular, g is analytic in the region between γ_1 and γ_2. So by Corollary 0.21,

$$\oint_{\gamma_1} g(z) dz = \oint_{\gamma_2} g(z) dz.$$

Together with Corollaries 0.22, 0.21 this implies

$$\begin{aligned} f(z_0) = a_0 &= \frac{1}{2\pi i} \oint_{\gamma_2} \frac{a_0}{z - z_0} \cdot dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{a_0}{z - z_0} \cdot dz \\ &= \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} \cdot dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} \cdot dz. \end{aligned}$$

□

0.7.4 Isolated singularities, Laurent series, meromorphic functions

We define the *punctured disk* with center $z_0 \in \mathbb{C}$ and radius $r > 0$ by

$$D^0(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.$$

If f is an analytic function defined on $D^0(z_0, r)$ for some $r > 0$, we call z_0 an *isolated singularity* of f. In case that there exists an analytic function g on the non-punctured disk $D(z_0, r)$ such that $g(z) = f(z)$ for $z \in D^0(z_0, r)$, we call z_0 a *removable* singularity of f.

Theorem 0.29. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in U$, and let $R > 0$ be such that $D^0(z_0, R) \subseteq U$. Then f has a Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{converging for } z \in D^0(z_0, R).$$

Further, we have for $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz \quad \text{for any } r \text{ with } 0 < r < R. \tag{0.4}$$

Proof. We fix $z \in D^0(z_0, R)$ and use w to denote a complex variable. Choose r_1, r_2 with $0 < r_1 < |z - z_0| < r_2 < R$.

By Corollary 0.28 we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r_2}} \frac{f(w)}{w - z} \cdot dw - \frac{1}{2\pi i} \oint_{\gamma_{z_0,r_1}} \frac{f(w)}{w - z} \cdot dw =: I_1 - I_2, \tag{0.5}$$

say. Completely similarly to Theorem 0.25, one shows that

$$I_1 = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{with } a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r_2}} \frac{f(w)}{(w - z_0)^{n+1}} \cdot dw.$$

Notice that for w on the inner circle γ_{z_0,r_1} we have

$$\begin{aligned} \frac{f(w)}{w - z} &= \frac{f(w)}{(w - z_0) - (z - z_0)} = -\frac{f(w)}{z - z_0} \cdot \left(1 - \frac{w - z_0}{z - z_0}\right)^{-1} \\ &= -\sum_{m=0}^{\infty} f(w)(z - z_0)^{-m-1}(w - z_0)^m. \end{aligned}$$

Further, one easily shows that the latter series converges uniformly to $f(w)/(w - z)$

on γ_{z_0,r_1}. After a substitution $n = -m-1$, it follows that

$$\begin{aligned} I_2 &= \frac{-1}{2\pi i}\oint_{\gamma_{z_0,r_2}} \left(\sum_{m=0}^{\infty} f(w)(w-w_0)^m (z-z_0)^{-m-1}\right)\cdot dw \\ &= -\sum_{n=-\infty}^{-1} a_n(z-z_0)^n, \quad \text{with } a_n = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r_1}} \frac{f(w)}{(w-z_0)^{n+1}}\cdot dw. \end{aligned}$$

By substituting the expressions for I_1, I_2 obtained above into (0.5), we obtain

$$f(z) = I_1 - I_2 = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$

Further, by Corollary 0.21 we have

$$a_n = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r}} \frac{f(w)}{(w-z_0)^{n+1}}\cdot dw$$

for any $n \in \mathbb{Z}$ and any r with $0 < r < R$. This completes our proof. $\square$

Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ an analytic function. Then z_0 is an isolated singularity of f, and there is $R > 0$ such that f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

converging for $z \in D^0(z_0, R)$. Notice that z_0 is a removable singularity of f if $a_n = 0$ for all $n < 0$.

The point z_0 is called

- *an essential singularity of f* if there are infinitely many $n < 0$ with $a_n \neq 0$;
- *a pole of order k of f* for some $k > 0$ if $a_{-k} \neq 0$ and $a_n = 0$ for $n < -k$; a pole of order 1 is called *simple*;
- *a zero of order k of f for some $k > 0$* if $a_k \neq 0$ and $a_n = 0$ for $n < k$; a zero of order 1 is called simple.

Notice that if f has a zero of order k at z_0 then in particular, z_0 is a removable singularity of f and so we may assume that f is defined and analytic at z_0. Moreover, z_0 is a zero of order k of f if and only if $f^{(j)}(z_0) = 0$ for $j = 0, \ldots, k-1$, and $f^{(k)}(z_0) \neq 0$.

For f, z_0 as above, we define

$$\mathrm{ord}_{z_0}(f) := \text{smallest } k \in \mathbb{Z} \text{ such that } a_k \neq 0.$$

Thus,

z_0 essential singularity of $f \iff \mathrm{ord}_{z_0}(f) = -\infty$;
z_0 pole of order k of $f \iff \mathrm{ord}_{z_0}(f) = -k$;
z_0 zero of order k of $f \iff \mathrm{ord}_{z_0}(f) = k$.

Further, $\mathrm{ord}_{z_0}(f) = k$ if and only if there is a function g that is analytic around z_0 such that $f(z) = (z - z_0)^k g(z)$ for $z \neq z_0$ and $g(z_0) \neq 0$.

Lemma 0.30. *Let $R > 0$ and let $f, g : D^0(z_0, R) \to \mathbb{C}$ be two analytic functions. Assume that $g \neq 0$ on $D^0(z_0, R)$, and that z_0 is not an essential singularity of f or g. Then*

$$\begin{aligned} &\mathrm{ord}_{z_0}(f+g) \geqslant \min\big(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\big); \\ &\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g); \\ &\mathrm{ord}_{z_0}(f/g) = \mathrm{ord}_{z_0}(f) - \mathrm{ord}_{z_0}(g). \end{aligned}$$

Proof. Exercise.
The function ord_{z_0} is an example of a *discrete valuation.* A discrete valuation on a field K is a surjective map $v : K \to \mathbb{Z} \cup \{\infty\}$ such that $v(0) = \infty$; $v(x) \in \mathbb{Z}$ for $x \in K$, $x \neq 0$; $v(xy) = v(x) + v(y)$ for $x, y \in K$; and $v(x+y) \geqslant \min(v(x), v(y))$ for $x, y \in K$. □

A *meromorphic* function on U is a complex function f with the following properties:
(i) there is a set S discrete in U such that f is defined and analytic on $U \setminus S$;
(ii) all elements of S are poles of f.

We say that a complex function f is meromorphic around z_0 if f is analytic on $D^0(z_0, r)$ for some $r > 0$, and z_0 is a pole of f.

It is easy to verify that if f, g are meromorphic functions on U then so are $f+g$ and $f \cdot g$. It can be shown as well (less trivial) that if U is connected and g is a non-zero meromorphic function on U, then the set of zeros of f is discrete in U. The zeros of g are poles of $1/g$, and the poles of g are zeros of $1/g$. Hence $1/g$ is meromorphic on U. Consequently, if U is an open, connected subset of $\mathbb{C}$, then the functions meromorphic on U form a *field*.

0.7.5 Residues, logarithmic derivatives

Let $z_0 \in \mathbb{C}$, $R > 0$ and let $f : D^0(z_0, R) \to \mathbb{C}$ be an analytic function. Then f has a Laurent series expansion converging on $D^0(z_0, R)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We define the *residue of f at z_0* by

$$\text{res}(z_0, f) := a_{-1}.$$

In particular, if f is analytic at z_0 then $\text{res}(z_0, f) = 0$. By Theorem 0.29 we have

$$\text{res}(z_0, f) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} f(z)dz$$

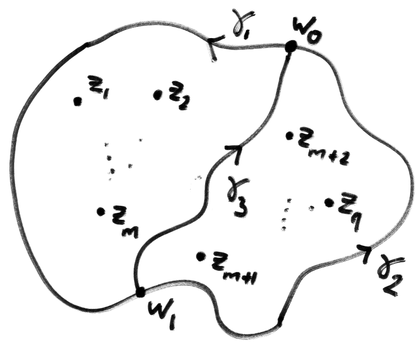
for any r with $0 < r < R$.

Theorem 0.31 (Residue Theorem)**.** *let γ be a contour in $\mathbb{C}$. let $z_1, \ldots, z_q$ be in the interior of γ. Let f be a complex function that is analytic on an open set containing γ and the interior of γ minus $\{z_1, \ldots, z_q\}$. Then*

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)dz = \sum_{i=1}^{q} \text{res}(z_i, f).$$

Proof. We proceed by induction on q. First let $q = 1$. Choose $r > 0$ such that $\gamma_{z_1,r}$ lies in the interior of γ. Then by Corollary 0.21,

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_{z_1,r}} f(z)dz = \text{res}(z_1, f).$$



Now let $q > 1$ and assume the Residue Theorem is true for fewer than q points. We cut γ into two pieces, the piece γ_1 from a point w_0 to w_1 and the piece γ_2 from w_1 to w_0 so that $\gamma = \gamma_1 + \gamma_2$. Then we take a path γ_3 from w_1 to w_0 inside the interior of γ without self-intersections; this gives two contours $\gamma_1 + \gamma_3$ and $-\gamma_3 + \gamma_2$.

We choose γ_3 in such a way that it does not hit any of the points $z_1, \ldots, z_q$ and both the interiors of these contours contain points from $z_1, \ldots, z_q$. Without loss of generality, we assume that the interior of $\gamma_1+\gamma_3$ contains $z_1, \ldots, z_m$ with $0 < m < q$, while the interior of $-\gamma_3 + \gamma_2$ contains $z_{m+1}, \ldots, z_q$. Then by the induction hypothesis,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\gamma f(z)dz &= \frac{1}{2\pi i}\oint_{\gamma_1} f(z)dz + \frac{1}{2\pi i}\oint_{\gamma_2} f(z)dz \\
&= \frac{1}{2\pi i}\oint_{\gamma_1+\gamma_3} f(z)dz + \frac{1}{2\pi i}\oint_{-\gamma_3+\gamma_2} f(z)dz \\
&= \sum_{i=1}^{m}\operatorname{res}(z_i, f) + \sum_{i=m+1}^{q}\operatorname{res}(z_i, f) = \sum_{i=1}^{q}\operatorname{res}(z_i, f),
\end{aligned}$$

completing our proof. □

We have collected some useful facts about residues. Both f, g are analytic functions on $D^0(z_0, r)$ for some $r > 0$.

Lemma 0.32. *(i) f has a simple pole or removable singularity at z_0 with residue α*

$$\Longleftrightarrow \lim_{z\to z_0}(z - z_0)f(z) = \alpha \Longleftrightarrow f(z) - \frac{\alpha}{z - z_0} \text{ is analytic around } z_0.$$

(ii) Suppose f has a pole of order 1 at z_0 and g is analytic and non-zero at z_0. Then

$$\operatorname{res}(z_0, fg) = g(z_0)\operatorname{res}(z_0, f).$$

(iii) Suppose that f is analytic and non-zero at z_0 and g has a zero of order 1 at z_0. Then f/g has a pole of order 1 at z_0, and

$$\operatorname{res}(z_0, f/g) = f(z_0)/g'(z_0).$$

Proof. Exercise. $\square$

Let U be a non-empty, open subset of $\mathbb{C}$ and f a meromorphic function on U which is not identically zero. We define the *logarithmic derivative* of f by

$$f'/f.$$

Suppose that U is simply connected and f is analytic and has no zeros on U. Then f'/f has an anti-derivative $h : U \to \mathbb{C}$. One easily verifies that $(e^h/f)' = 0$. Hence e^h/f is constant on U. By adding a suitable constant to h we can achieve that $e^h = f$. That is, we may view h as the logarithm of f, and f'/f as the derivative of this logarithm. But we will work with f'/f also if U is not simply connected and/or f has zeros or poles on U. In that case, we call f'/f also the logarithmic derivative of f, although it need not be the derivative of some function.

The following facts are easy to prove: if f, g are two meromorphic functions on U that are not identically zero, then

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}, \quad \frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}.$$

Further, if U is connected, then

$$\frac{f'}{f} = \frac{g'}{g} \Longleftrightarrow f = cg \text{ for some constant } c.$$

Lemma 0.33. *Let $z_0 \in \mathbb{C}$, $r > 0$ and let $f : D^0(z_0, r) \to \mathbb{C}$ be analytic. Assume that z_0 is either a removable singularity or a pole of f. Then z_0 is a simple pole or (if z_0 is neither a zero nor a pole of f) a removable singularity of f'/f, and*

$$\operatorname{res}(z_0, f'/f) = \operatorname{ord}_{z_0}(f).$$

Proof. Let $\operatorname{ord}_{z_0}(f) = k$. This means that $f(z) = (z - z_0)^k g(z)$ with g analytic around z_0 and $g(z_0) \neq 0$. Consequently,

$$\frac{f'}{f} = k\frac{(z-z_0)'}{z-z_0} + \frac{g'}{g} = \frac{k}{z-z_0} + \frac{g'}{g}.$$

The function g'/g is analytic around z_0 since $g(z_0) \neq 0$. So by Lemma 0.32, $\operatorname{res}(z_0, f'/f) = k$. $\square$

Corollary 0.34. *Let γ be a contour in $\mathbb{C}$, U an open subset of $\mathbb{C}$ containing γ and its interior, and f a meromorphic function on U. Assume that f has no zeros or poles on γ and let $z_1, \dots, z_q$ be the zeros and poles of f inside γ. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \operatorname{ord}_{z_i}(f) = Z - P,$$

where Z, P denote the number of zeros and poles of f inside γ, counted with their multiplicities.

Proof. By Theorem 0.31 and Lemma 0.33 we have

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \operatorname{res}(z_i, f'/f) = \sum_{i=1}^{q} \operatorname{ord}_{z_i}(f) = Z - P.$$

□