Chapter 10

The singular integral

Our aim in this chapter is to replace the functions $S^*(n)$ and $J^*(n)$ by more convenient expressions; these will be called the singular series $S(n)$ and the singular integral $J(n)$. This will be done in section 10.1. We shall show that the order of magnitude of the singular integral is $n^2$ in section 10.2.

10.1 Introducing $S(n)$ and $J(n)$

Define

$$S(q) := \sum_{\substack{1 \leq a \leq q \\gcd(a,q)=1}} S(q,a) \theta(-an/q),$$

where

$$S(q,a) = \sum_{m=1}^{q} e(am^3/q).$$

Thus,

$$S^*(n) = \sum_{q \leq n^{1/100}} \frac{S(q)}{q^9}.$$

Lemma 10.1. Let $a, q$ be coprime integers and $\epsilon$ any positive real. Then

$$|S(a,q)| \ll \epsilon q^{3/4 + \epsilon}.$$
Proof. The argument is an analogue of the differencing process (see pages 151-153 of Chapter 8). We have

\[ |S(q, a)|^2 = \sum_{m_2(\text{mod } q)} \sum_{m_1(\text{mod } q)} e \left( \frac{a}{q} (m_1^3 - m_2^3) \right). \]

The transformation \( m_1 \mapsto h_1 \) given by \( m_1 \equiv h_1 + m_2(\text{mod } q) \) shows that

\[ |S(q, a)|^2 = \sum_{h_1(\text{mod } q)} \sum_{m_2(\text{mod } q)} e \left( \frac{a}{q} (3h_1(m_1m_2 + m_2^2)) \right), \]

hence the triangle inequality gives

\[ |S(q, a)|^2 \lesssim \sum_{h_1(\text{mod } q)} \left| \sum_{m_2(\text{mod } q)} e \left( \frac{a}{q} (3h_1(m_1m_2 + m_2^2)) \right) \right|. \]

Now Cauchy’s inequality reveals that

\[ |S(q, a)|^4 \lesssim q \sum_{h_1(\text{mod } q)} \left| \sum_{m_2(\text{mod } q)} e \left( \frac{a}{q} (3h_1(m_1m_2 + m_2^2)) \right) \right|^2. \]

The inner term is

\[ \left| \sum_{m_2(\text{mod } q)} e \left( \frac{a}{q} (3h_1(m_1m_2 + m_2^2)) \right) \right|^2 = \sum_{m_2, m_3(\text{mod } q)} e \left( \frac{a}{q} (3h_1(m_1m_2 - m_3) + (m_2^2 - m_3^3)) \right), \]

and the substitution \( m_2 \mapsto h_2 \) given by \( m_2 \equiv hm_3 + h_2(\text{mod } q) \) leads to

\[ \sum_{h_2(\text{mod } q)} e \left( \frac{3a}{q} (h_1^2h_2 + h_1h_2^2) \right) \sum_{m_3(\text{mod } q)} e \left( \frac{6a}{q} h_1h_2m_3 \right). \]

Note that the coprimality of \( a, q \) shows that the sum over \( m_3 \) equals \( q \) when \( q \) divides \( 6h_1h_2 \) and vanishes otherwise. We obtain that

\[ |S(q, a)|^4 \lesssim q^2 \# \{ 1 \leq h_1, h_2 \leq q : 6h_1h_2 \}. \]

The integers \( 6h_1h_2 \) lie in the range \([1, 6q^2]\) and are divisible by \( q \). Hence there exists \( i \in [1, 6q] \) such that \( 6h_1h_2 = iq \). Therefore

\[ |S(q, a)|^4 \lesssim q^2 \sum_{i=1}^{6q} \# \{ 1 \leq h_1, h_2 \leq q : 6h_1h_2 = iq \}. \]
In order to have $6h_1h_2 = iq$ both integers $h_1, h_2$ must divide $iq$ and there are only

$$\tau(iq)^2 \ll \epsilon (iq)^{\epsilon/2} \ll q^\epsilon$$

such pairs, where $\epsilon$ is any positive real. This concludes our proof.

The last lemma shows that

$$\frac{|S(q)|}{q^9} \ll \frac{1}{q^{1 + \frac{1}{4} + \epsilon}},$$

therefore the following series, usually referred to as the *singular series*,

(10.1) $$\mathcal{G}(n) := \sum_{q=1}^{\infty} \frac{S(q)}{q^9}$$

converges absolutely and satisfies

$$\mathcal{G}(n) - \mathcal{G}^*(n) \ll \epsilon \sum_{q > n^{1/300}} \frac{1}{q^{1 + \frac{1}{4} + \epsilon}} \ll \int_{n^{1/300}}^{\infty} \frac{dt}{t^{1 + \frac{1}{4} + \epsilon}} \ll \epsilon n^{-\frac{1}{1200} + \epsilon}.$$

This shows that

(10.2) $$\mathcal{G}(n) \ll 1$$

and by (9.6) we obtain

(10.3) $$R^*(n) = (\mathcal{G}(n) + O_\epsilon(n^{-\frac{1}{1200} + \epsilon}))J^*(n).$$

We next replace $J^*(n)$ by a more suitable integral. For this we shall need to need the behaviour of $v(\beta) = \frac{1}{3} \sum_{m=1}^{n} e(\beta m)/m^{2/3}$ in the range $|\beta| \leq \frac{1}{2}$.

**Lemma 10.2.** Let $\beta \in \mathbb{R}$ with $|\beta| \leq \frac{1}{2}$. Then

$$|v(\beta)| \ll \min\{n^{1/3}, |\beta|^{-1/3}\}.$$

**Proof.** If $\beta$ is close to 0 then the terms $e(\beta m)$ in the definition of $v(\beta)$ remain close to 1. Hence using the triangle inequality one does not lose much information,

$$|v(\beta)| \leq \frac{1}{3} \sum_{1 \leq m \leq n} \frac{1}{m^{2/3}} \leq \frac{1}{3} \int_{1}^{n} \frac{dt}{t^{2/3}} + O(1) \ll n^{1/3}.$$
If $|\beta| \leq 1/n$ then $|\beta|^{-1/3} > n^{1/3}$, hence the claim of our lemma is evident.

In the remaining case $|\beta| > 1/n$ we see that

$$\left| \sum_{m \leq 1/|\beta|} \frac{e(\beta m)}{m^{2/3}} \right| \leq \sum_{m \leq 1/|\beta|} \frac{1}{m^{2/3}} \ll (1/|\beta|)^{1/3},$$

which is acceptable. We use partial summation to estimate the remaining sum

$$\sum_{1/|\beta| < m \leq n} \frac{e(\beta m)}{m^{2/3}}.$$

For this purpose we define for $t \in \mathbb{R}$,

$$A(t) := \sum_{1 \leq m \leq t} e(\beta m) = e(\beta m) \frac{e(\beta [t]) - 1}{e(\beta) - 1},$$

and observe that the inequality $|e(\beta) - 1| \gg |\beta|$, valid for $|\beta| < 1/2$, yields

$$A(t) \ll \frac{1}{|\beta|},$$

with an implied constant that is independent of $t$. Partial summation now gives

$$\sum_{1/|\beta| < m \leq n} \frac{e(\beta m)}{m^{2/3}} = \frac{A(n)}{n^{2/3}} - \frac{A(1/|\beta|)}{|\beta|^{-2/3}} + \int_{1/|\beta|}^{n} A(t) \frac{dt}{t^{5/3}},$$

which is

$$\ll \frac{1/|\beta|}{n^{2/3}} + \frac{|\beta|^{2/3}}{|\beta|} + \frac{1}{|\beta|} \frac{1}{|\beta|^{-2/3}} \ll |\beta|^{-1/3}.$$

Define the following integral (which is usually called *singular integral*),

$$J(n) := \int_{-1/2}^{1/2} v(\beta)^0 e(-\beta n) d\beta$$

(10.4)

and observe that Lemma 10.2 shows that

$$J(n) \ll \int_{0}^{1/n} n^{0/3} d\beta + \int_{1/n}^{1/2} \frac{d\beta}{|\beta|^{3/3}}.$$
hence

\[(10.5) \quad J(n) \ll n^2.\]

Now recall the definition of $J^*(n)$ in (9.8). We have

\[
J(n) - J^*(n) = \int_{n^{-1+1/300} \leq |\beta| \leq 1/2} v(\beta)^9 e(-\beta n) d\beta,
\]

which according to Lemma 10.2 is

\[
\ll \int_{1/2}^{1/2} \beta^{-3} d\beta \ll n^{2-\frac{1}{150}}.
\]

Using (10.2), (10.3) and (10.5) we find an absolute constant $\delta > 0$ such that

\[
R^*(n) = \mathcal{S}(n)J(n) + O(n^{2-\delta}),
\]

which when combined with (8.8), (8.10) and (9.5) yields the following theorem.

**Theorem 10.3.** We have

\[
\lim_{n \to +\infty} \left| \frac{R(n)}{n^2} - \frac{\mathcal{S}(n)J(n)}{n^2} \right| = 0.
\]

### 10.2 The singular integral

The *Beta function* is defined as

\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \text{for } x, y > 0.
\]

Before relating the singular integral $J(n)$ to the Beta function we need some information on sums of monotone arithmetic functions. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be any monotonic function. Comparing the sum $\sum_{y < n \leq x} f(n)$ with the integral $\int_y^x f(t) dt$ we see that

\[
\sum_{y < n \leq x} f(n) = \int_y^x f(t) dt + O(1 + |f(y)| + |f(x)|).
\]

Therefore if $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is monotonic in each interval

\[(0, x_1), (x_1, x_2), \ldots, (x_k, x)\]

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then
\[ \sum_{0 \leq m \leq n} f(m) = \int_0^n f(t)dt + O(1 + |f(0)| + |f(n)| + \sum_{i=1}^k |f(x_i)|). \]

The function \( f(x) := x^{\beta-1}(n-x)^{\alpha-1}, \) defined for 
\( \beta \in (0, 1], \ \alpha \geq \beta, \) has derivative \( x^{\beta-2}(n-x)^{\alpha-2}((\beta-1)n-x(\alpha+\beta-2)), \) which vanishes at 0 and \( n \) and \( X = \frac{\beta-1}{\alpha+\beta-2}. \) If \( X \in (0, n) \) then
\[ \sum_{0 \leq m \leq n} f(m) = \int_0^n f(t)dt + O(1 + |f(X)|) \]
and if \( X \notin (0, n) \) then the error term is \( O(1). \) The substitution \( t \mapsto y \) given by 
\( t = ny \) shows that 
\[ \int_0^n f(t)dt = n^{\alpha+\beta-1}B(\beta, \alpha) \]
and the error term is \( \ll n^{\alpha-1}, \) therefore
\[ \sum_{m=1}^{n-1} m^{\beta-1}(n-m)^{\alpha-1} = n^{\alpha+\beta-1} \left( B(\beta, \alpha) + O(n^{-\beta}) \right). \]

Before proceeding we need to recall a few standard facts about the Gamma function. It is defined as 
\[ \Gamma(t) := \int_0^\infty t^{x-1}e^{-t}dt \ for \ x > 0 \]
and satisfies
\[ \Gamma(1) = 1, \]
\[ \Gamma(t+1) = t\Gamma(t) \ for \ t > 0, \]
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ for \ x, y > 0. \]

Observe that \( \Gamma(x) = (x-1)! \) for every positive integer \( x. \) So \( B(x, y) = \frac{x+y}{xy} \cdot \left( \frac{x+y}{x} \right)^{-1} \)
for all positive integers \( x, y. \)

We have the following theorem.
Theorem 10.4. We have
\[ J(n) = \Gamma \left( \frac{4}{3} \right)^9 \frac{n^2}{2} \left( 1 + O(n^{-1/3}) \right). \]

Proof. We begin by proving by induction that for every integer \( s \geq 2 \), one has
\[ (10.10) \quad \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{s}{3}}} = \Gamma \left( \frac{4}{3} \right)^s \Gamma \left( \frac{s}{3} \right)^{-1} n^{\frac{s}{3} - 1} \left( 1 + O(n^{-1/3}) \right). \]

For \( s = 2 \) this is valid due to (10.6) with \( \alpha = \beta = \frac{1}{3} \), as well as (10.8) and (10.9). Assuming that (10.10) is valid for some integer \( s \geq 2 \), then
\[ \sum_{1 \leq m_{s+1} \leq n-1} \frac{1}{3m_{s+1}^2} \left( \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{s}{3}}} \right), \]

equals
\[ \sum_{1 \leq m_{s+1} \leq n-1} \frac{1}{3m_{s+1}^2} \left( \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{s}{3}}} \right), \]

which is
\[ \Gamma \left( \frac{4}{3} \right)^s \Gamma \left( \frac{s}{3} \right)^{-1} \frac{1}{3} \sum_{1 \leq m \leq n-1} m^{\frac{1}{3}} \left( n - m \right)^{\frac{s}{3} - 1} + O \left( \sum_{1 \leq m \leq n-1} m^{1/3-1} \left( n - m \right)^{(s-1)/3-1} \right), \]

due to the induction hypothesis. Using (10.6) with \( \beta = \frac{1}{3} \) and \( \alpha = \frac{s}{3}; \frac{(s-1)}{3} - 1 \) respectively for the main and the error term, we conclude the proof of (10.10).

Combining (10.4) and the definition of \( v \),
\[ v(\beta) = \frac{1}{3} \sum_{1 \leq m \leq n} e(\beta m) \frac{1}{m^{\frac{2}{3}}}, \]

gives
\[ J(n) = \frac{1}{3^9} \sum_{1 \leq m_1, \ldots, m_9 \leq n} \frac{1}{(m_1 \cdots m_9)^{\frac{2}{3}}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e(\beta(n - m_1 - \cdots - m_9)) \, d\beta. \]
The integral vanishes except when \( n - m_1 - \cdots - m_9 = 0 \), thus obtaining

\[
J(n) = \frac{1}{3^9} \sum_{1 \leq m_1, \ldots, m_9 \leq n} \frac{1}{(m_1 \cdots m_9)^{\frac{2}{3}}}
\]

and according to (10.10) our theorem is valid.