Analytic Number Theory Fall 2016, Assignment 1
Deadline: Monday October 17

- Don’t forget to write your name and student number on your homework. To simplify the grading, it is preferable that you submit your homework in latex.
- You may either submit your homework at the course, or to Marc Paul Noordman, or send him an electronic version of it by email.
- The number of points for each exercise is indicated in the left margin.

The total number of points is 70. Grade=(number of points)/7.

5 1.a) Let $k$ be a positive integer. Prove that
\[
\int_2^x \frac{dt}{(\log t)^k} = O\left(\frac{x}{(\log x)^k}\right) \text{ as } x \to \infty.
\]

**Hint.** Split the integral into \( \int_2^{f(x)} + \int_{f(x)}^x \) for a well-chosen function \( f(x) \) with \( 2 \leq f(x) < x \) and estimate both parts from above.

5 b) Using integration by parts, prove that for every integer \( n > 0 \),
\[
\text{Li}(x) := \int_2^x \frac{dt}{\log t} = \sum_{i=1}^n (i-1)! \frac{x}{(\log x)^i} + O\left(\frac{x}{(\log x)^{n+1}}\right) \text{ as } x \to \infty.
\]

**Remark.** The error term will increase with \( n \). So the finite sum cannot be expanded into an infinite series.

2. Euclid’s proof that there are infinitely many primes runs as follows. Suppose there are only finitely many primes, \( p_1, p_2, \ldots, p_n \), say. Consider the number \( P := p_1 p_2 \cdots p_n + 1 \). Then either \( P \) itself is a prime or \( P \) is divisible by a prime but in both cases, this prime must be different from \( p_1, \ldots, p_n \). Thus we arrive at a contradiction.

In certain cases, it is possible to give a similar proof for the fact that there are infinitely many primes \( p \) with \( p \equiv a \) (mod \( q \)). Assume there are only finitely many such primes, \( p_1, \ldots, p_n \), say. Construct a function \( P(p_1, \ldots, p_n) \) which is divisible by a prime which is congruent to \( a \) modulo \( q \) but which is different from \( p_1, \ldots, p_n \).
3. a) Let \( p \) be a prime with \( p \equiv 3 \pmod{4} \). Show that there is no integer \( x \) with \( x^2 \equiv -1 \pmod{p} \).

**Hint.** Suppose there does exist such an integer \( x \). Consider the order of \( x \pmod{p} \) in the multiplicative group \((\mathbb{Z}/p\mathbb{Z})^*\) of non-zero residue classes modulo \( p \).

3. b) Show that there are infinitely many primes \( p \) with \( p \equiv 1 \pmod{4} \).

**Hint.** Take \( P(p_1, \ldots, p_n) = 4(p_1p_2 \cdots p_n)^2 + 1 \).

4. c) Show that there are infinitely many primes \( p \) with \( p \equiv 3 \pmod{4} \).
   (You have to find yourself a suitable expression \( P(p_1, \ldots, p_n) \).)

5. d) Let \( p, q \) be distinct prime numbers with \( q \geq 3, p \neq 1 \pmod{q} \). Prove that there is no integer \( x \) with \( 1 + x + x^2 + \cdots + x^{q-1} \equiv 0 \pmod{p} \).

5. e) Let \( q \) be a prime number \( \geq 3 \). Prove that there are infinitely many primes \( p \) with \( p \equiv 1 \pmod{q} \).

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3. In this exercise you are asked to prove Bertrand’s postulate: for every positive integer \( n \) there is a prime number \( p \) with \( n < p \leq 2n \). You have to use the theorems and lemmas proved in Chapter 1 of the lecture notes.

4. a) Prove that for every real \( x \geq 2 \) we have \( \prod_{p \leq x} p \leq 4^x \) (product taken over all prime numbers \( \leq x \)).
   **Hint.** Let \( m := \lfloor x \rfloor \), and proceed by induction on \( m \). If \( m \) is even, you can immediately apply the induction hypothesis. Assume that \( m = 2k + 1 \) is odd and consider \( \prod_{k+1 < p \leq 2k+1} p \).
   It suffices to prove Bertrand’s postulate for \( n \geq 1000 \) since the remaining cases can be verified by straightforward computation. In b),c),d) below let \( n \) be an integer \( \geq 1000 \), and assume that there is no prime \( p \) with \( n < p \leq 2n \).

b) Prove that the binomial coefficient \( \binom{2n}{n} \) is not divisible by any prime \( p \) with \( \frac{2}{3}n < p \leq n \).
   **Hint.** Compute \( \text{ord}_p(\binom{2n}{n}) \).

4. c) Prove that \( \binom{2n}{n} \leq (2n)^{2n} \frac{\pi(\sqrt{2n})}{2^{2n/3}} \cdot 4^{2n/3} \).
   **Hint.** Write \( \binom{2n}{n} = p_1^{k_1} \cdots p_t^{k_t} \) with \( p_i \) distinct primes and \( k_i > 0 \) and split into primes \( p_i \) with \( p_i \leq \sqrt{2n} \) and \( p_i > \sqrt{2n} \); for the latter, \( k_i = 1 \).

4. d) Derive a contradiction.
4. We describe a general method to compute series \( \sum_{n=1}^{\infty} f(n) \), where \( f \) is an even meromorphic function on \( \mathbb{C} \), i.e., \( f(z) = f(-z) \) for \( z \in \mathbb{C} \) minus the poles of \( f \).

Let \( N \) be an integer \( \geq 1 \) and let \( S_N \) be the square through the four points \( \pm(N + \frac{1}{2}) \pm (N + \frac{1}{2})i \), traversed counterclockwise. Assume that \( f \) has only finitely many poles, and that none are lying at the non-zero integers.

1) Compute \( \oint_{S_N} \frac{2\pi if(z)}{e^{2\pi iz} - 1} \cdot dz \), using the Residue Theorem.

2) Prove that \( \lim_{N \to \infty} \oint_{S_N} \frac{2\pi if(z)}{e^{2\pi iz} - 1} \cdot dz = 0 \). Here, you have to use the general inequality

\[
\left| \int_{\gamma} g(z)dz \right| \leq L(\gamma) \cdot \sup_{z \in \gamma} |g(z)|,
\]

where \( \gamma \) is a path in \( \mathbb{C} \), \( g : \gamma \to \mathbb{C} \) is a continuous function, and \( L(\gamma) \) denotes the length of \( \gamma \). Applying this estimate with \( \gamma = S_N \), one has to show that the upper bounds converges to 0 as \( N \to \infty \).

The following lemma, of which we have included a proof here, is crucial in 2).

**Lemma.** There is a constant \( c > 0 \), independent of \( N \), such that \( |e^{2\pi iz} - 1| \geq c \) holds for all integers \( N \geq 1 \) and all \( z \in S_N \).

**Proof.** We consider the four edges of the square separately. First consider the edge from \( (N + \frac{1}{2})(-1 - i) \) to \( (N + \frac{1}{2})(1 - i) \). This can be parametrized by \( (N + \frac{1}{2})(t - i) \) with \(-1 \leq t \leq 1 \). So for the points \( z \) on this edge we have

\[
|e^{2\pi iz} - 1| = |e^{2\pi i(N + \frac{1}{2})(t - i)} - 1| = |e^{2\pi i(N + \frac{1}{2})} - 1| \\
\geq e^{2\pi(N + \frac{1}{2})} - 1 \geq e^{3\pi} - 1.
\]

Next, consider the edge from \( (N + \frac{1}{2})(1 - i) \) to \( (N + \frac{1}{2})(1 + i) \). This can be parametrized by \( (N + \frac{1}{2})(1 + it) \) with \(-1 \leq t \leq 1 \). So for the points \( z \) on this edge we have

\[
|e^{2\pi iz} - 1| = |e^{2\pi i(N + \frac{1}{2})} - 1| = |e^{-2\pi(N + \frac{1}{2})} - 1| \geq 1.
\]

Here we have used that \( e^{2\pi i(N + \frac{1}{2})} = -1 \). The other two edges can be treated in the same manner. \( \square \)

3 a) Let \( f \) be a meromorphic function on \( \mathbb{C} \) that has no poles or zeros at the non-zero integers. Prove that the function \( \frac{2\pi if(z)}{e^{2\pi iz} - 1} \) has residue \( f(k) \) at \( z = k \) for every non-zero integer \( k \).
7 b) The Bernoulli numbers $B_n$ are given by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \quad (z \in \mathbb{C}, |z| < 2\pi).$$

Using the method sketched above, prove that

$$\zeta(2k) = (-1)^{k-1}2^{2k-1}\frac{B_{2k}}{(2k)!} \cdot \pi^{2k} \quad \text{for } k = 1, 2, \ldots.$$  

5. Consider the Dirichlet series

$$F(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots,$$

$$G(s) = 1^{-s} + 2^{-s} - 2 \times 3^{-s} + 4^{-s} + 5^{-s} - 2 \times 6^{-s} + \cdots$$

4 a) Prove that $F(s), G(s)$ converge, and are analytic on $\{s \in \mathbb{C} : \text{Re } s > 0\}$.

4 b) Prove that for $s \in \mathbb{C}$ with $\text{Re } s > 1$ we have

$$F(s) = (1 - 2^{-s}) \sum_{n=1}^{\infty} n^{-s}, \quad G(s) = (1 - 3^{-s}) \sum_{n=1}^{\infty} n^{-s}.$$  

7 c) Use a) and b) to prove that $\sum_{n=1}^{\infty} n^{-s}$ can be continued to an analytic function $\zeta(s)$ on $\{s \in \mathbb{C} : \text{Re } s > 0\} \setminus \{1\}$, with a simple pole with residue 1 at $s = 1$, i.e., $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ if $\text{Re } s > 1$, and $\lim_{s \to 1} (s - 1) \zeta(s) = 1$.

**Hint.** Both functions $1 - 2^{-s}, 1 - 3^{-s}$ have infinitely many zeros in $\mathbb{C}$. Which zeros do they have in common?