1. a) Let \( (a_m)_{m=1}^N \) be a sequence of complex numbers and \( g : [1, N] \to \mathbb{C} \) a continuously differentiable function. Put \( A(x) := \sum_{m \leq x} a_m \) for \( 1 \leq x \leq N \). Prove that \( \sum_{m=1}^N a_m g(m) = A(N)g(N) - \int_1^N A(x)g'(x)dx \). You are not allowed to use the general result on partial summation from the lecture notes.

4 b) Let \( f : \mathbb{Z}_{>0} \to \mathbb{C} \) be an arithmetic function, and suppose there are \( C > 0, \sigma \geq 0 \) such that \( |\sum_{n \leq x} f(n)| \leq C x^\sigma \) for all \( x \geq 1 \). Prove that \( L_f(s) = \sum_{n=1}^\infty f(n)n^{-s} \) has abscissa of convergence \( \leq \sigma \).

3 c) The Möbius function \( \mu \) is given by \( \mu(1) = 1 \) and \( \sum_{d|n} \mu(d) = 0 \) for every integer \( n \geq 2 \). Assume that for every \( \epsilon > 0 \) there is \( C_\epsilon > 0 \) such that \( |\sum_{n \leq x} \mu(n)| \leq C_\epsilon x^{(1/2)+\epsilon} \) for all \( x \geq 1 \). Deduce that \( \zeta(s) \neq 0 \) for all \( s \in \mathbb{C} \) with \( \text{Re} \, s > \frac{1}{2}, \ s \neq 1 \).

(You don’t have to prove this, but together with the functional equation for \( \zeta(s) \) this implies the Riemann Hypothesis).
2. a) Formulate a Tauberian theorem for Dirichlet series \( L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \).

b) Let \( q \) be an integer \( \geq 2 \) and \( \chi \) a real, non-principal character mod \( q \).
Express \( L_{\mu\chi}(s) \) in terms of an \( L \)-function, and prove that
\[
\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \mu(n)\chi(n) = 0.
\]

3. Let \( f \) be a strongly multiplicative function such that \( f(p) \in \{-1, 0, 1\} \) for every prime number \( p \) and there is \( C > 0 \) such that \( |\sum_{n \leq x} f(n)| \leq C \) for all \( x \). Prove that \( L_f(1) = \sum_{n=1}^{\infty} f(n)/n \neq 0 \). To this end, perform the following steps:

a) Assume that \( L_f(1) = 0 \). Show that \( \zeta(s)L_f(s) \) has an analytic continuation to \( \{s \in \mathbb{C} : \text{Re } s > 0\} \).

b) Show that there is a multiplicative arithmetic function \( g \) such that \( \zeta(s)L_f(s) = L_g(s) \) holds for \( \text{Re } s > 1 \) and compute \( g(p^k) \) for every prime power \( p^k \).

c) Deduce a contradiction, using a suitable theorem from the lecture notes.

4. This exercise is related to the last part of our course regarding the circle method. The three parts are independent of each other. Recall that for \( z \in \mathbb{C} \) we use the notation
\[
e(z) := e^{2\pi i z}.
e
3 a) Fix a positive integer \( n \) and denote by \( T(n) \) the number of solutions in positive integers \( x_1, x_2, y \) of the equation
\[n = x_1^2 + x_2^2 + y^3.\]
Define for any $\alpha \in \mathbb{R}$ the functions

$$Q(\alpha) := \sum_{x \in \mathbb{Z}, 1 \leq x \leq n^{1/2}} e(\alpha x^2) \quad \text{and} \quad C(\alpha) := \sum_{y \in \mathbb{Z}, 1 \leq y \leq n^{1/3}} e(\alpha y^3).$$

Prove that

$$T(n) = \int_0^1 Q(\alpha)^2 C(\alpha) e(-\alpha n) d\alpha.$$

3) **b)** Prove that there is a positive constant $c > 0$ such that for all positive integers $b$ and all real numbers $\theta$ in the interval $(0, \frac{1}{4})$ we have

$$\left| \sum_{m \in \mathbb{Z}, 1 \leq m \leq b} e(\theta m) \right| \leq \frac{c}{|\theta|}.$$

4) **c)** For any odd positive integer $q$ define the sum

$$K(q) := \sum_{m \in \mathbb{Z}, 1 \leq m \leq q} e\left(\frac{m^2 + m + 7}{q}\right).$$

Prove the equality

$$|K(q)|^2 = \sum_{h_1 \in \mathbb{Z}, 1 \leq h_1 \leq q} e\left(\frac{h_1^2 + h_1}{q}\right) \sum_{m_2 \in \mathbb{Z}, 1 \leq m_2 \leq q} e\left(\frac{2h_1 m_2}{q}\right)$$

and deduce from it that $|K(q)|^2 = q$. 