ANALYTIC NUMBER THEORY

Thursday January 26, 14:00-17:00

- Write down your name and student number on each sheet. Take care that these are **VERY WELL READABLE**.
- Indicate whether you are doing Bachelor wiskunde, Master mathematics, Algant, or any other program and if not from Leiden, from which other university you are coming.
- There are four exercises on three pages. In each exercise you are allowed to use all theorems from the lecture notes, unless otherwise stated. Formulate the theorems you are using.
- To facilitate the grading, please give your answers in English.
- The maximal number of points for each part of an exercise is indicated in the left margin. Grade is (number of points)/4.

---

3 **1.** *a)* Let $(a_m)_{m=1}^N$ be a sequence of complex numbers and $g : [1, N] \to \mathbb{C}$ a continuously differentiable function. Put $A(x) := \sum_{m \le x} a_m$ for $1 \le x \le N$. Prove that $\sum_{m=1}^N a_m g(m) = A(N)g(N) - \int_1^N A(x)g'(x)dx$. You are not allowed to use the general result on partial summation from the lecture notes.

4 *b)* Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arithmetic function, and suppose there are $C > 0$, $\sigma \ge 0$ such that $|\sum_{n \le x} f(n)| \le Cx^\sigma$ for all $x \ge 1$. Prove that $L_f(s) = \sum_{n=1}^\infty f(n)n^{-s}$ has abscissa of convergence $\le \sigma$.

3 *c)* The Möbius function μ is given by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for every integer $n \ge 2$. Assume that for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $|\sum_{n \le x} \mu(n)| \le C_\epsilon x^{(1/2)+\epsilon}$ for all $x \ge 1$. Deduce that $\zeta(s) \ne 0$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{2}$, $s \ne 1$.
(You don't have to prove this, but together with the functional equation for $\zeta(s)$ this implies the Riemann Hypothesis).

3 **2.** *a)* Formulate a Tauberian theorem for Dirichlet series $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$.

7 *b)* Let q be an integer ≥ 2 and χ a real, non-principal character mod q. Express $L_{\mu\chi}(s)$ in terms of an L-function, and prove that $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \mu(n)\chi(n) = 0$.

3. Let f be a strongly multiplicative function such that $f(p) \in \{-1, 0, 1\}$ for every prime number p and there is $C > 0$ such that $|\sum_{n\leq x} f(n)| \leq C$ for all x. Prove that $L_f(1) = \sum_{n=1}^{\infty} f(n)/n \neq 0$. To this end, perform the following steps:

3 *a)* Assume that $L_f(1) = 0$. Show that $\zeta(s)L_f(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

4 *b)* Show that there is a multiplicative arithmetic function g such that $\zeta(s)L_f(s) = L_g(s)$ holds for $\operatorname{Re} s > 1$ and compute $g(p^k)$ for every prime power p^k.

3 *c)* Deduce a contradiction, using a suitable theorem from the lecture notes.

4. This exercise is related to the last part of our course regarding the *circle method*. The three parts are independent of each other. Recall that for $z \in \mathbb{C}$ we use the notation

$$\mathrm{e}(z) := \mathrm{e}^{2\pi i z}.$$

3 *a)* Fix a positive integer n and denote by $T(n)$ the number of solutions in positive integers x_1, x_2, y of the equation

$$n = x_1^2 + x_2^2 + y^3.$$

Define for any $\alpha \in \mathbb{R}$ the functions

$$Q(\alpha) := \sum_{\substack{x\in\mathbb{Z}\\ 1\le x\le n^{1/2}}} \mathrm{e}(\alpha x^2) \quad \text{and} \quad C(\alpha) := \sum_{\substack{y\in\mathbb{Z}\\ 1\le y\le n^{1/3}}} \mathrm{e}(\alpha y^3).$$

Prove that

$$T(n) = \int_0^1 Q(\alpha)^2 C(\alpha)\mathrm{e}(-\alpha n)\mathrm{d}\alpha.$$

3 *b)* Prove that there is a positive constant $c > 0$ such that for all positive integers b and all real numbers θ in the interval $(0, \frac{1}{4})$ we have

$$\Big| \sum_{\substack{m\in\mathbb{Z}\\ 1\le m\le b}} \mathrm{e}(\theta m) \Big| \le \frac{c}{|\theta|}.$$

4 *c)* For any odd positive integer q define the sum

$$K(q) := \sum_{\substack{m\in\mathbb{Z}\\ 1\le m\le q}} \mathrm{e}\left(\frac{m^2+m+7}{q}\right).$$

Prove the equality

$$|K(q)|^2 = \sum_{\substack{h_1\in\mathbb{Z}\\ 1\le h_1\le q}} \mathrm{e}\left(\frac{h_1^2+h_1}{q}\right) \sum_{\substack{m_2\in\mathbb{Z}\\ 1\le m_2\le q}} \mathrm{e}\left(\frac{2h_1 m_2}{q}\right)$$

and deduce from it that $|K(q)|^2 = q$.