Below is a list of recommended additional literature.


Notation

• \( \limsup_{n \to \infty} x_n \) or \( \lim_{n \to \infty} x_n \)

For a sequence of reals \( \{x_n\} \) we define \( \limsup_{n \to \infty} x_n := \lim_{n \to \infty} (\sup_{m \geq n} x_m) \).
We have \( \limsup_{n \to \infty} x_n = \infty \) if and only if the sequence \( \{x_n\} \) is not bounded from above, i.e., if for every \( A > 0 \) there is \( n \) with \( x_n > A \).
In case that the sequence \( \{x_n\} \) is bounded from above, we have \( \limsup_{n \to \infty} x_n = \alpha \) where \( \alpha \) is the largest limit point (‘limes superior’) of the sequence \( \{x_n\} \), in other words, for every \( \varepsilon > 0 \) there are infinitely many \( n \) such that \( x_n \geq \alpha - \varepsilon \), while there are only finitely many \( n \) such that \( x_n \geq \alpha + \varepsilon \).

• \( \liminf_{n \to \infty} x_n \) or \( \lim_{n \to \infty} x_n \)

For a sequence of reals \( \{x_n\} \) we define \( \liminf_{n \to \infty} x_n := \lim_{n \to \infty} (\inf_{m \geq n} x_m) \).
We have \( \liminf_{n \to \infty} x_n = -\infty \) if the sequence \( \{x_n\} \) is not bounded from below, and the smallest limit point (‘limes inferior’) of the sequence \( \{x_n\} \) otherwise.

• \( f(x) = g(x) + o(h(x)) \) as \( x \to \infty \)

\[ \lim_{x \to \infty} \frac{f(x) - g(x)}{h(x)} = 0, \text{ i.e., } f(x) - g(x) \text{ is of smaller order of magnitude than } h(x). \]

Examples: \( f(x) = g(x) + o(1) \) as \( x \to \infty \) means that \( \lim_{x \to \infty} (f(x) - g(x)) = 0 \); \( \log x = o(x^\varepsilon) \) as \( x \to \infty \) for every \( \varepsilon > 0 \).

• \( f(x) = g(x) + O(h(x)) \) as \( x \to \infty \)

There are constants \( x_0 > 0, C > 0 \) such that \( |f(x) - g(x)| \leq Ch(x) \) for all \( x \geq x_0, \) i.e., \( f(x) - g(x) \) is of order of magnitude at most \( h(x) \).
We call \( g(x) + O(h(x)) \) as \( x \to \infty \) an asymptotic formula for \( f(x) \), with main term \( g(x) \) and error term \( O(h(x)) \). Of course, such an asymptotic formula is interesting only if the error term is of smaller order of magnitude than the main term, i.e., \( h(x) = o(|g(x)|) \) as \( x \to \infty \). If \( g(x) \) is of order of magnitude at most \( h(x) \), i.e., \( g(x) = O(h(x)) \) as \( x \to \infty \), we can just as well write
\( f(x) = O(h(x)) \) as \( x \to \infty \).
Likewise, if \( f(x) = g(x) + o(f(x)) \) as \( x \to \infty \), we call \( g(x) \) the main term and \( o(h(x)) \) the error term.

**Examples:** \( f(x) = g(x) + O(1) \) as \( x \to \infty \) means that \( |f(x) - g(x)| \) is bounded; 
\[ \log(1 + x^{-1}) = x^{-1} + O(x^{-2}) \] as \( x \to \infty \) (from the expansion \( \log(1 + x^{-1}) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{-n} / n \) for \( |x| > 1 \));
\[ (1 + x^{-1})^\alpha = 1 + \alpha x^{-1} + O(x^{-2}) \] as \( x \to \infty \) for every \( \alpha \in \mathbb{R} \) (from the expansion \( (1 + x^{-1})^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^{-n} \) for \( |x| > 1 \), where \( \binom{\alpha}{n} = \frac{\alpha(-1)\cdots(-\alpha+n+1)}{n!} \));
\[ e^{1/x} = 1 + x^{-1} + O(x^{-2}) \] as \( x \to \infty \) (from the expansion \( e^{1/x} = \sum_{n=0}^{\infty} x^{-n}/n! \)).

- \( f(x) \sim g(x) \) as \( x \to \infty \)
  \[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \]

- \( f(x) \ll g(x), g(x) \gg f(x) \) as \( x \to \infty \)
  (Vinogradov symbols; used only if \( g(x) > 0 \) for all sufficiently large \( x \), i.e., there is \( x_0 \) such that \( g(x) > 0 \) for all \( x \geq x_0 \)).
  \( f(x) = O(g(x)) \) as \( x \to \infty \), that is, there are constants \( x_0 > 0, C > 0 \) such that \( |f(x)| \leq Cg(x) \) for all \( x \geq x_0 \).

- \( f(x) \asymp g(x) \) as \( x \to \infty \)
  (used only if \( f(x) > 0, g(x) > 0 \) for all sufficiently large \( x \))
  there are constants \( x_0, C_1, C_2 > 0 \) such that \( C_1 f(x) \leq g(x) \leq C_2 f(x) \) for all \( x \geq x_0 \).

- \( f(x) = \Omega(g(x)) \) as \( x \to \infty \)
  (defined only if \( g(x) > 0 \) for \( x \geq x_0 \) for some \( x_0 > 0 \))
  \[ \limsup_{x \to \infty} \frac{|f(x)|}{g(x)} > 0, \] that is, there is a sequence \( \{x_n\} \) with \( x_n \to \infty \) as \( n \to \infty \)
  such that \( \lim_{n \to \infty} \frac{|f(x_n)|}{g(x_n)} > 0 \) (possibly \( \infty \)).
• \( f(x) = \Omega^\pm(g(x)) \) as \( x \to \infty \)  
(defined only if \( g(x) > 0 \) for \( x \geq x_0 \) for some \( x_0 > 0 \))  

\[
\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0, \quad \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0, \text{ that is, there are sequences } \{x_n\} \text{ and } \{y_n\} \n
\text{with } x_n \to \infty, y_n \to \infty \text{ as } n \to \infty \text{ such that } \lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} > 0 \text{ (possibly } \infty) \text{ and } \\
\lim_{n \to \infty} \frac{f(y_n)}{g(y_n)} < 0 \text{ (possibly } -\infty) \n
• \( \gamma \) (Euler-Mascheroni constant)  
\[
\lim_{N \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N \right) 
\]

• \( |\mathcal{A}| \)  
Cardinality of a set \( \mathcal{A} \).

• \( \sum_{n \leq x}, \sum_{p \leq x}, \sum_{d \mid n}, \sum_{p \mid n} \)  
Summations over all positive integers \( \leq x \), all primes \( \leq x \), all positive divisors of \( n \) (including \( n \) itself), all primes dividing \( n \); there is a similar notation for products \( \prod \ldots \). In general, in summations or products, \( n \) will be used to denote a positive integer, \( p \) to denote a prime, and \( d \) to denote a positive divisor of a given integer.

• \( \sum_{p}, \prod_{p} \)  
Infinite sum, infinite product over all primes.

• \( \pi(x) \)  
Number of primes \( \leq x \).
• \(\theta(x)\), \(\psi(x)\)

\[ \sum_{p \leq x} \log p, \quad \sum_{p^k \leq x} \log p, \] where the summations are over all primes \(\leq x\), respectively all prime powers \(\leq x\).

• \(\pi(x; q, a)\)

Number of primes \(p\) with \(p \equiv a \pmod{q}\) and \(p \leq x\); here \(q\) is any integer \(\geq 2\) and \(a\) is any integer coprime with \(q\).

• \(\theta(x; q, a), \psi(x; q, a)\)

\[ \sum_{p \leq x, p \equiv a \pmod{q}} \log p, \quad \sum_{p^k \leq x, p^k \equiv a \pmod{q}} \log p, \] where the summations are over all primes \(\leq x\) that are congruent to \(a\) modulo \(q\), respectively all prime powers \(\leq x\) that are congruent to \(a\) modulo \(q\).

• \(\operatorname{Li}(x)\)

\(\operatorname{Li}(x) = \int_{2}^{x} \frac{dt}{\log t}\); this is a good approximation for \(\pi(x)\).

• \(\Lambda(n)\)

Von Mangoldt function; it is given by \(\Lambda(n) = \log p\) if \(n = p^k\) for some prime \(p\) and exponent \(k \geq 1\), and \(\Lambda(n) = 0\) if \(n = 1\) or \(n\) is not a prime power; it should be verified that \(\psi(x) = \sum_{n \leq x} \Lambda(n)\), where the summation is over all positive integers \(n \leq x\).

• \(\varphi(n)\)

Euler’s totient function, given by \(\varphi(n) := |\{a \in \mathbb{Z} : 1 \leq a < n, \gcd(a, n) = 1\}|\).
• $\mu(n)$
Möbius function, given by $\mu(1) = 1$, $\mu(n) = (-1)^t$ if $n$ is a product $p_1 \cdots p_t$ of distinct primes, and $\mu(n) = 0$ if $n$ is not square-free, i.e., divisible by $p^2$ for some prime number $p$.

• $\omega(n)$, $\Omega(n)$
number of primes dividing $n$, number of prime powers dividing $n$, i.e., if $n = p_1^{k_1} \cdots p_t^{k_t}$ with $p_1, \ldots, p_t$ distinct primes and $k_1, \ldots, k_t$ distinct integers, then $\omega(n) = t$ and $\Omega(n) = k_1 + \cdots + k_t$; in particular, $\omega(1) = \Omega(1) = 0$.

• $E(n)$
$E(n) = 1$ for every positive integer $n$.

• $e(n)$
e(1) = 1 and $e(n) = 0$ for all integers $n > 1$.

• $\tau(n)$ (or $\sigma_0(n)$)
number of positive divisors of $n$, including $n$ itself, i.e., $\sum_{d|n} 1$, for instance $\tau(6) = 4$, since 1, 2, 3, 6 are the divisors of 6.

• $\sigma(n)$ (or $\sigma_1(n)$)
sum of the positive divisors of $n$ including $n$ itself, i.e., $\sum_{d|n} d$, for instance $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

• $\sigma_\alpha(n)$
$\sum_{d|n} d^\alpha$
Chapter 0

Prerequisites

We have collected some facts from algebra and analysis which we will not discuss during our course, which will not be a subject of the examination, but to which we will have to refer quite often.

We need only a little bit of algebra, basically elementary group theory. As for analysis, most of the facts we mention are covered by standard courses on analysis, Lebesgue integration and complex analysis, with the exception maybe of (sub)sections 0.2, 0.7.6, 0.7.7.

Students are expected to be familiar with the results that we mention so that we can use them in our course without much explanation. In some cases we have provided proofs, either since they may help to gain some confidence with the material, or since we couldn’t find a good reference for them. These proofs will not be used in our course, nor will they be examined.

Apart from what is mentioned in these prerequisites, nothing else from Lebesgue integration theory or complex analysis is used, so also students who did not follow courses on these topics should be able to follow Analytic Number Theory after having read these prerequisites.
0.1 Groups

Literature:
P. Stevenhagen: *Collegedictaat Algebra 1* (Dutch), Universiteit Leiden.

0.1.1 Definition

A group is a set $G$, together with an operation $\cdot : G \times G \to G$ satisfying the following axioms:

- $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$;
- there is $e_G \in G$ such that $g \cdot e_G = e_G \cdot g = g$ for all $g \in G$;
- for all $g \in G$ there is $h \in G$ with $g \cdot h = h \cdot g = e_G$.

From these axioms it follows that the unit element $e_G$ is uniquely determined, and that the inverse $h$ defined by the last axiom is uniquely determined; henceforth we write $g^{-1}$ for this $h$.

If moreover, $g_1 \cdot g_2 = g_2 \cdot g_1$, we say that the group $G$ is *abelian* or *commutative*.

**Remark.** For $n \in \mathbb{Z}_{\geq 0}$, $g \in G$ we write $g^n$ for $g$ multiplied with itself $n$ times. Further, $g^0 := e_G$ and $g^n := (g^{-1})^{\lceil n \rceil}$ for $n \in \mathbb{Z}_{< 0}$. This is well-defined by the associative axiom, and we have $(g^m)(g^n) = g^{m+n}$, $(g^m)^n = g^{mn}$ for $m, n \in \mathbb{Z}$.

0.1.2 Subgroups

Let $G$ be a group with group operation $\cdot$. A subgroup of $G$ is a subset $H$ of $G$ that is a group with the group operation of $G$. This means that $g_1 \cdot g_2 \in H$ for all $g_1, g_2 \in H$; $e_G \in H$; and $g^{-1} \in H$ for all $g \in H$. It is easy to see that $H$ is a subgroup of $G$ if and only if $g_1 \cdot g_2^{-1} \in H$ for all $g_1, g_2 \in H$. We write $H \leq G$ if $H$ is a subgroup of $G$. 

2
0.1.3 Cosets, order, index

Let $G$ be a group and $H$ a subgroup of $G$. The left cosets of $G$ with respect to $H$ are the sets $gH = \{g \cdot h : h \in H\}$. Two left cosets $g_1H$, $g_2H$ are equal if and only if $g_1^{-1}g_2 \in H$ and otherwise disjoint.

The right cosets of $G$ with respect to $H$ are the sets $Hg = \{h \cdot g : h \in H\}$. Two right cosets $Hg_1$, $Hg_2$ are equal if and only if $g_2g_1^{-1} \in H$ and otherwise disjoint.

There is a one-to-one correspondence between the left cosets and right cosets of $G$ with respect to $H$, given by $gH \leftrightarrow Hg^{-1}$. Thus, the collection of left cosets has the same cardinality as the collection of right cosets. This cardinality is called the index of $H$ in $G$, notation $(G : H)$.

The order of a group $G$ is its cardinality, notation $|G|$. Assume that $|G|$ is finite. Let again $H$ be a subgroup of $G$. Since the left cosets w.r.t. $H$ are pairwise disjoint and have the same number of elements as $H$, and likewise for right cosets, we have

$$(G : H) = \frac{|G|}{|H|}.$$ 

An important consequence of this is, that $|H|$ divides $|G|$.

0.1.4 Normal subgroup, factor group

Let $G$ be a group, and $H$ a subgroup of $G$. We call $H$ a normal subgroup of $G$ if $gH = Hg$, that is, if $gHg^{-1} = H$ for every $g \in G$.

Let $H$ be a normal subgroup of $G$. Then the cosets of $G$ with respect to $H$ form a group with group operation $(g_1H) \cdot (g_2H) = (g_1g_2) \cdot H$. This operation is well-defined. We denote this group by $G/H$; it is called the factor group of $G$ with respect to $H$. Notice that the unit element of $G/H$ is $e_GH = H$. If $G$ is finite, we have $|G/H| = (G : H) = |G|/|H|$.

0.1.5 Order of an element

Let $G$ be a group, and $g \in G$. The order of $g$, notation $\text{ord}(g)$, is the smallest positive integer $n$ such that $g^n = e_G$; if such an integer $n$ does not exist we say that $g$ has infinite order.
We recall some properties of orders of group elements. Suppose that $g \in G$ has finite order $n$.

- $g^a = g^b \iff a \equiv b \pmod{n}$.
- Let $k \in \mathbb{Z}$. Then $\text{ord}(g^k) = n/\gcd(k, n)$.
- $\{e_G, g, g^2, \ldots, g^{n-1}\}$ is a subgroup of $G$ of cardinality $n = \text{ord}(g)$. Hence if $G$ is finite, then $\text{ord}(g)$ divides $|G|$. Consequently, $g^{\lfloor|G|\rfloor} = e_G$.

**Example.** Let $q$ be a positive integer. A prime residue class modulo $q$ is a residue class of the type $a \mod q$, where $\gcd(a, q) = 1$. The prime residue classes form a group under multiplication, which is denoted by $(\mathbb{Z}/q\mathbb{Z})^\ast$. The unit element of this group is $1 \mod q$, and the order of this group is $\varphi(q)$, that is the number of positive integers $\leq q$ that are coprime with $q$. It follows that if $\gcd(a, q) = 1$, then $a^{\varphi(q)} \equiv 1 \pmod{q}$.

### 0.1.6 Cyclic groups

The cyclic group generated by $g$, denoted by $\langle g \rangle$, is given by $\{g^k : k \in \mathbb{Z}\}$. In case that $G = \langle g \rangle$ is finite, say of order $n \geq 2$, we have

$$\langle g \rangle = \{e_G = g^0, g, g^2, \ldots, g^{n-1}\}, \quad g^n = e_G.$$  

So $g$ has order $n$.

**Example 1.** $\mu_n = \{\rho \in \mathbb{C}^\ast : \rho^n = 1\}$, that is the group of roots of unity of order $n$ is a cyclic group of order $n$. For a generator of $\mu_n$ one may take any primitive root of unity of order $n$, i.e., $e^{2\pi ik/n}$ with $k \in \mathbb{Z}$, $\gcd(k, n) = 1$.

**Example 2.** Let $p$ be a prime number, and $(\mathbb{Z}/p\mathbb{Z})^\ast = \{a \mod p, \gcd(a, p) = 1\}$ the group of prime residue classes modulo $p$ with multiplication. This is a cyclic group of order $p - 1$.

Let $G = \langle g \rangle$ be a cyclic group and $H$ a subgroup of $G$. Let $k$ be the smallest positive integer such that $g^k \in H$. Using, e.g., division with remainder, one shows that $g^r \in H$ if and only if $r \equiv 0 \pmod{k}$. Hence $H = \langle g^k \rangle$ and $(G : H) = k$. 

---

4
0.1.7 Homomorphisms and isomorphisms

Let \( G_1, G_2 \) be two groups. A homomorphism from \( G_1 \) to \( G_2 \) is a map \( f : G_1 \to G_2 \) such that \( f(g_1 g_2) = f(g_1)f(g_2) \) for all \( g_1, g_2 \in G \) and \( f(e_{G_1}) = e_{G_2} \). This implies that \( f(g^{-1}) = f(g)^{-1} \) for \( g \in G_1 \).

Let \( f : G_1 \to G_2 \) be a homomorphism. The kernel and image of \( f \) are given by

\[
\text{Ker}(f) := \{ g \in G_1 : f(g) = e_{G_2} \}, \quad f(G_1) = \{ f(g) : g \in G_1 \},
\]

respectively. Notice that \( \text{Ker}(f) \) is a normal subgroup of \( G_1 \). It is easy to check that \( f \) is injective if and only if \( \text{Ker}(f) = \{ e_{G_1} \} \).

Let \( G \) be a group and \( H \) a normal subgroup of \( G \). Then

\[
f : G \to G/H : g \mapsto gH
\]

is a surjective homomorphism from \( G \) to \( G/H \), the canonical homomorphism from \( G \) to \( G/H \). Notice that the kernel of this homomorphism is \( H \). Thus, every normal subgroup of \( G \) occurs as the kernel of some homomorphism.

A homomorphism \( f : G_1 \to G_2 \) which is bijective is called an isomorphism from \( G_1 \) to \( G_2 \). In case that there is an isomorphism from \( G_1 \) to \( G_2 \) we say that \( G_1, G_2 \) are isomorphic, notation \( G_1 \cong G_2 \). Notice that a homomorphism \( f : G_1 \to G_2 \) is an isomorphism if and only if \( \text{Ker}(f) = \{ e_{G_1} \} \) and \( f(G_1) = G_2 \). Further, in this case the inverse map \( f^{-1} : G_2 \to G_1 \) is also an isomorphism.

Let \( f : G_1 \to G_2 \) be a homomorphism of groups and \( H = \text{Ker}(f) \). This yields an isomorphism

\[
\overline{f} : G_1/H \to f(G_1) : \overline{f}(gH) = f(g).
\]

**Proposition 0.1.** Let \( C \) be a cyclic group. If \( C \) is infinite, then it is isomorphic to \( \mathbb{Z}^+ \) (the additive group of \( \mathbb{Z} \)). If \( C \) has finite order \( n \), then it is isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^+ \) (the additive group of residue classes modulo \( n \)).

**Proof.** Let \( C = \langle g \rangle \). Define \( f : \mathbb{Z}^+ \to C \) by \( x \mapsto g^x \). This is a surjective homomorphism; let \( H \) denote its kernel. Thus, \( \mathbb{Z}^+/H \cong C \). We have \( H = \{0\} \) if \( C \) is infinite, and \( H = n\mathbb{Z}^+ \) if \( C \) has order \( n \). This implies the proposition.

\[\square\]
0.1.8 Direct products

Let $G_1, \ldots, G_r$ be groups. Denote by $e_{G_i}$ the unit element of $G_i$. The (external) direct product $G_1 \times \cdots \times G_r$ is the set of tuples $(g_1, \ldots, g_r)$ with $g_i \in G_i$ for $i = 1, \ldots, r$, endowed with the group operation

$$(g_1, \ldots, g_r) \cdot (h_1, \ldots, h_r) = (g_1 h_1, \ldots, g_r h_r).$$

This is obviously a group, with unit element $(e_{G_1}, \ldots, e_{G_r})$ and inverse $(g_1, \ldots, g_r)^{-1} = (g_1^{-1}, \ldots, g_r^{-1})$.

Let $G$ be a group and $G_1, \ldots, G_r$ subgroups of $G$. We say that $G$ is the internal direct product of $G_1, \ldots, G_r$ if:

(a) $G = G_1 \cdots G_r$, i.e., every element of $G$ can be expressed as $g_1 \cdots g_r$ with $g_i \in G_i$ for $i = 1, \ldots, r$;

(b) $G_1, \ldots, G_r$ commute, that is, for all $i, j = 1, \ldots, r$ and all $g_i \in G_i, g_j \in G_j$ we have $g_i g_j = g_j g_i$;

(c) $G_1, \ldots, G_r$ are independent, i.e., if $g_i \in G_i$ ($i = 1, \ldots, r$) are any elements such that $g_1 \cdots g_r = e_G$, then $g_i = e_G$ for $i = 1, \ldots, r$.

A consequence of (a), (b), (c) is that every element of $G$ can be expressed uniquely as a product $g_1 \cdots g_r$ with $g_i \in G_i$ for $i = 1, \ldots, r$.

**Proposition 0.2.** Let $G, G_1, \ldots, G_r$ be groups.

(i) Suppose $G$ is the internal direct product of $G_1, \ldots, G_r$. Then $G \cong G_1 \times \cdots \times G_r$.

(ii) Suppose $G \cong G_1 \times \cdots \times G_r$. Then there are subgroups $H_1, \ldots, H_r$ of $G$ such that $H_i \cong G_i$ for $i = 1, \ldots, r$ and $G$ is the internal direct product of $H_1, \ldots, H_r$.

**Proof.** (i) The map $G_1 \times \cdots \times G_r \to G : (g_1, \ldots, g_r) \mapsto g_1 \cdots g_r$ is easily seen to be an isomorphism.

(ii) Let $G' := G_1 \times \cdots \times G_r$ and for $i = 1, \ldots, r$, define the group

$$G'_i := \{(e_{G_1}, \ldots, e_{G_i}, \ldots, e_{G_r}) : g_i \in G_i\}$$

where the $i$-th coordinate is $g_i$ and the other components are the unit elements of the respective groups. Clearly, $G'$ is the internal direct product of $G'_1, \ldots, G'_r$, and $G'_i \cong G_i$ for $i = 1, \ldots, r$. Let $f : G \to G_1 \times \cdots \times G_r$ be an isomorphism. Then $G$ is the internal direct product of $H_i := f^{-1}(G'_i)$ $(i = 1, \ldots, r)$, and $H_i \cong G'_i \cong G_i$ for $i = 1, \ldots, r$. 

\[\square\]
We will sometimes be sloppy and write \( G = G_1 \times \cdots \times G_r \) if \( G \) is the internal direct product of subgroups \( G_1, \ldots, G_r \).

0.1.9 Abelian groups

The group operation of an abelian group is often denoted by +, but in this course we stick to the multiplicative notation. The unit element of an abelian group \( A \) is denoted by 1 or \( 1_A \). It is obvious that every subgroup of an abelian group is a normal subgroup. In Proposition 0.2, the condition that \( H_1, \ldots, H_r \) commute holds automatically so it can be dropped.

The following important theorem, which we state without proof, implies that the finite cyclic groups are the building blocks of the finite abelian groups.

**Theorem 0.3.** Every finite abelian group is isomorphic to a direct product of finite cyclic groups.

**Proof.** See S. Lang, Algebra, 2nd ed. Addison-Wesley, 1984, Ch.1, §10. \( \square \)

Let \( A \) be a finite, multiplicatively written abelian group of order \( \geq 2 \) with unit element 1. Theorem 0.3 implies that \( A \) is the internal direct product of cyclic subgroups, say \( C_1, \ldots, C_r \). Assume that \( C_i \) has order \( n_i \geq 2 \); then \( C_i = \langle h_i \rangle \), where \( h_i \in A \) is an element of order \( n_i \). We call \( \{h_1, \ldots, h_r\} \) a basis for \( A \).

Every element of \( A \) can be expressed uniquely as \( g_1 \cdots g_r \), where \( g_i \in C_i \) for \( i = 1, \ldots, r \). Further, every element of \( C_i \) can be expressed as a power \( h_i^k \), and \( h_i^k = 1 \) if and only if \( k \equiv 0 \pmod{n_i} \). Together with Proposition 0.2 this implies the following characterization of a basis for \( A \):

\[
A = \left\{ h_1^{k_1} \cdots h_r^{k_r} : \ k_i \in \mathbb{Z} \text{ for } i = 1, \ldots, r \right\},
\]

(0.1) there are integers \( n_1, \ldots, n_r \geq 2 \) such that
\[
h_1^{k_1} \cdots h_r^{k_r} = 1 \iff k_i \equiv 0 \pmod{n_i} \text{ for } i = 1, \ldots, r.
\]

0.2 Infinite products in analysis

Given a sequence \( \{a_n\}_{n=1}^\infty \) of complex numbers, when saying that it converges we mean that there is \( \ell \in \mathbb{C} \) such that \( \lim_{n \to \infty} a_n = \ell \). When saying that the limit
exists, we mean that either the sequence converges or (in case that the \( a_n \) are real) that the limit is \( \pm \infty \); so for instance \( \lim_{n \to \infty} (-1)^n \) does not exist.

Let \( \{A_n\}_{n=1}^\infty \) be a sequence of complex numbers. We define
\[
\prod_{n=1}^\infty A_n := \lim_{N \to \infty} \prod_{n=1}^N A_n
\]
provided the limit exists (so either it is finite or \( \pm \infty \) in case that the \( A_n \) are real).

Clearly, \( \prod_{n=1}^\infty A_n = 0 \) if \( A_n = 0 \) for some \( n \). But if \( A_n \neq 0 \) for all \( n \) then it may still happen that \( \prod_{n=1}^\infty A_n = 0 \), for instance \( \prod_{n=1}^\infty (1 - \frac{1}{n+1}) = 0 \). (It is common practice to say that \( \prod_{n=1}^\infty A_n \) converges if there is non-zero \( \ell \in \mathbb{C} \) such that \( \lim_{N \to \infty} \prod_{n=1}^N A_n = \ell \). We will not use this notion of convergence and say instead that \( \prod_{n=1}^\infty A_n \) exists and is \( \neq 0, \pm \infty \).

In general, we have
\[
\prod_{n=1}^\infty A_n \text{ exists and is } \neq 0, \pm \infty \iff A_n \neq 0 \text{ for all } n \text{ and } \sum_{n=1}^\infty \log A_n \text{ converges},
\]
where we take the principal complex logarithm, i.e., with imaginary part in \((-\pi, \pi]\).

The following criterion is more useful for our purposes.

**Proposition 0.4.** Assume that \( \sum_{n=1}^\infty |A_n - 1| < \infty \). Then the following hold:

(i) \( \prod_{n=1}^\infty A_n \) exists and is \( \neq \pm \infty \), and \( \prod_{n=1}^\infty A_n \neq 0 \) if \( A_n \neq 0 \) for all \( n \).

(ii) \( \prod_{n=1}^\infty A_n \) is invariant under rearrangements of the \( A_n \), i.e., if \( \sigma \) is any bijection of \( \mathbb{Z}_{>0} \), then \( \prod_{n=1}^\infty A_{\sigma(n)} \) exists and is equal to \( \prod_{n=1}^\infty A_n \).

**Proof.** (i) Let \( a_n := |A_n - 1| \) for \( n = 1, 2, \ldots \) Let \( M, N \) be integers with \( N > M > 0 \). Then, using \( |1 + z| \leq e^{|z|} \) for \( z \in \mathbb{C} \) and
\[
\left| \prod_{i=1}^r (1 + z_i) - 1 \right| \leq \prod_{i=1}^r (1 + |z_i|) - 1 \leq \exp \left( \sum_{i=1}^r |z_i| \right) - 1 \text{ for } z_1, \ldots, z_r \in \mathbb{C},
\]
we get
\[
\left| \prod_{n=1}^N A_n - \prod_{n=1}^M A_n \right| = \prod_{n=1}^M |A_n| \cdot \prod_{n=M+1}^N A_n - 1
\]
\[
\leq \exp \left( \sum_{n=1}^M a_n \right) \cdot \left( \exp \left( \sum_{n=M+1}^N a_n \right) - 1 \right)
\]

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which tends to 0 as $M, N \to \infty$. Hence $\prod_{n=1}^{\infty} A_n = \lim_{N \to \infty} \prod_{n=1}^{N} A_n$ exists and is finite.

Assume that $A_n \neq 0$ for all $n$. Choose $M$ such that $\sum_{n=M}^{\infty} a_n < \frac{1}{2}$. Then for $N > M$ we have

$$\left| \prod_{n=1}^{N} A_n \right| = \prod_{n=1}^{M} |A_n| \cdot \prod_{n=M+1}^{N} |A_n| \geq \prod_{n=1}^{M} |A_n| \cdot \left( 1 - \sum_{n=M+1}^{N} a_n \right) \geq \frac{1}{2} \prod_{n=1}^{M} |A_n| =: C > 0,$$

hence $\left| \prod_{n=1}^{\infty} A_n \right| \geq C > 0$. This proves (i).

(ii) Let $M, N$ be positive integers such that $N > M$ and $\{\sigma(1), \ldots, \sigma(N)\}$ contains $\{1, \ldots, M\}$. Similarly to (0.2) we get

$$\left| \prod_{n=1}^{N} A_{\sigma(n)} - \prod_{n=1}^{M} A_n \right| \leq \exp \left( \sum_{n=1}^{M} a_n \right) \cdot \left( \exp \left( \sum_{n \leq N, \sigma(n) > M} a_{\sigma(n)} \right) - 1 \right).$$

If for fixed $M$ we let first $N \to \infty$ and then let $M \to \infty$, the right-hand side tends to 0. Hence $\prod_{n=1}^{\infty} A_{\sigma(n)} = \prod_{n=1}^{\infty} A_n$. \qed

### 0.3 Uniform convergence

We consider functions $f : D \to \mathbb{C}$ where $D$ can be any set. We can express each such function as $g + ih$ where $g, h$ are functions from $D$ to $\mathbb{R}$. We write $g = \Re f$ and $h = \Im f$.

We recall that if $D$ is a topological space (in this course mostly a subset of $\mathbb{R}^n$ with the usual topology), then $f$ is continuous if and only if $\Re f$ and $\Im f$ are continuous.

In case that $D \subseteq \mathbb{R}$, we say that $f$ is differentiable if and only if $\Re f$ and $\Im f$ are differentiable, and in that case we define the derivative of $f$ by $f' = (\Re f)' + i(\Im f)'$.

In what follows, let $D$ be any set and $\{F_n\} = \{F_n\}_{n=1}^{\infty}$ a sequence of functions from $D$ to $\mathbb{C}$. 9
**Definition.** We say that \( \{F_n\} \) converges pointwise on \( D \) if for every \( z \in D \) there is \( F(z) \in \mathbb{C} \) such that \( \lim_{n \to \infty} F_n(z) = F(z) \). We say that \( \{F_n\} \) converges uniformly on \( D \) if moreover,

\[
\lim_{n \to \infty} \left( \sup_{z \in D} |F_n(z) - F(z)| \right) = 0.
\]

**Facts:**

- \( \{F_n\} \) converges uniformly on \( D \) if and only if \( \lim_{M,N \to \infty} \left( \sup_{z \in D} |F_M(z) - F_N(z)| \right) = 0. \)

- Let \( D \) be a topological space, assume that all functions \( F_n \) are continuous on \( D \), and that \( \{F_n\} \) converges to a function \( F \) uniformly on \( D \). Then \( F \) is continuous on \( D \).

Let again \( D \) be any set and \( \{F_n\}_{n=1}^\infty \) a sequence of functions from \( D \) to \( \mathbb{C} \). We say that the series \( \sum_{n=1}^\infty F_n \) converges pointwise/uniformly on \( D \) if the partial sums \( \sum_{n=1}^k F_n \) converge pointwise/uniformly on \( D \). Further, we say that \( \sum_{n=1}^\infty F_n \) is pointwise absolutely convergent on \( D \) if \( \sum_{n=1}^\infty |F_n(z)| \) converges for every \( z \in D \).

**Proposition 0.5 (Weierstrass criterion for series).** Assume that there are finite real numbers \( M_n \) such that

\[
|F_n(z)| \leq M_n \text{ for } z \in D, \; n \geq 1, \; \sum_{n=1}^\infty M_n \text{ converges.}
\]

Then \( \sum_{n=1}^\infty F_n \) is both uniformly convergent, and pointwise absolutely convergent on \( D \).

We need a similar result for infinite products of functions. Let again \( D \) be any set and \( \{F_n : D \to \mathbb{C}\}_{n=1}^\infty \) a sequence of functions. We define the limit function \( \prod_{n=1}^\infty F_n \) by

\[
\prod_{n=1}^\infty F_n(z) := \lim_{N \to \infty} \prod_{n=1}^N F_n(z) \; (z \in D),
\]

provided that for every \( z \in D \) the limit exists.

We say that \( \prod_{n=1}^\infty F_n \) converges uniformly on \( D \) if the limit function \( F := \prod_{n=1}^\infty F_n \) exists for every \( z \in D \), and

\[
\lim_{N \to \infty} \left( \sup_{z \in D} \left| F(z) - \prod_{n=1}^N F_n(z) \right| \right) = 0.
\]
Proposition 0.6 (Weierstrass criterion for infinite products). Assume that there are finite real numbers $M_n$ such that

$$|F_n(z) - 1| \leq M_n \text{ for } z \in D, \ n \geq 1, \ \sum_{n=1}^{\infty} M_n \text{ converges.}$$

Then $F := \prod_{n=1}^{\infty} F_n$ is uniformly convergent on $D$ and moreover, if $z \in D$ is such that $F_n(z) \neq 0$ for all $n$, then also $F(z) \neq 0$.

Proof. Applying (0.2) with $A_n = F_n(z)$ and using $|F_n(z) - 1| \leq M_n$ for $z \in D$, we obtain that for any two integers $M, N$ with $N > M > 0$, and all $z \in D$,

$$\left| \prod_{n=1}^{N} F_n(z) - \prod_{n=1}^{M} F_n(z) \right| \leq \exp \left( \sum_{n=1}^{M} M_n \right) \cdot \left( \exp \left( \sum_{n=M+1}^{N} M_n \right) - 1 \right).$$

Since the right-hand side is independent of $z$ and tends to 0 as $M, N \to \infty$, the uniform convergence follows. Further, if $F_n(z) \neq 0$ for all $n$ then $\prod_{n=1}^{\infty} F_n(z) \neq 0$ by Proposition 0.4.

0.4 Integration

In this course, all integrals will be Lebesgue integrals of real or complex measurable functions on $\mathbb{R}^n$ (always with respect to the Lebesgue measure on $\mathbb{R}^n$). Lebesgue integrals coincide with Riemann integrals from first year calculus whenever the latter are defined, but Riemann integrals can be defined only for a much smaller class of functions. It is not really necessary to know the precise definitions of Lebesgue measure, measurable functions and Lebesgue integrals, and you will be perfectly able to follow this course without any knowledge of Lebesgue theory. But we will often have to deal with infinite integrals of infinite series of functions, and to handle these, Lebesgue theory is much more convenient than the theory of Riemann integrals. In particular, in Lebesgue theory there are some very powerful convergence theorems for sequences of functions, theorems on interchanging multiple integrals, etc., which we will frequently apply. If you are willing to take for granted that all functions appearing in this course are measurable, there will be no problem to understand or apply these theorems.

We have collected a few useful facts, which are amply sufficient for our course.
0.4.1 Measurable sets

The length of a bounded interval $I = [a, b]$, $[a, b)$, $(a, b]$ or $(a, b)$, where $a, b \in \mathbb{R}$, $a < b$, is given by $l(I) := b - a$. Let $n \in \mathbb{Z}_{\geq 1}$. An interval in $\mathbb{R}^n$ is a cartesian product of bounded intervals $I = \prod_{i=1}^{n} I_i$. We define the volume of $I$ by $l(I) := \prod_{i=1}^{n} l(I_i)$.

Let $A$ be an arbitrary subset of $\mathbb{R}^n$. We define the outer measure of $A$ by

$$\lambda^*(A) := \inf \sum_{i=1}^{\infty} l(I_i),$$

where the infimum is taken over all countable unions of intervals $\bigcup_{i=1}^{\infty} I_i \supset A$. We say that a set $A$ is measurable if

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \cap A^c) \text{ for every } S \subseteq \mathbb{R}^n,$$

where $A^c = \mathbb{R}^n \setminus A$ is the complement of $A$. In this case we define the (Lebesgue) measure of $A$ by $\lambda(A) := \lambda^*(A)$. This measure may be finite or infinite. It can be shown that intervals are measurable, and that $\lambda(I) = l(I)$ for any interval $I$ in $\mathbb{R}^n$.

**Facts:**

- A countable union $\bigcup_{i=1}^{\infty} A_i$ of measurable sets $A_i$ is measurable. Further, the complement of a measurable set is measurable. Hence a countable intersection of measurable sets is measurable.

- All open and closed subsets of $\mathbb{R}^n$ are measurable.

- Let $A = \cup_{i=1}^{\infty} A_i$ be a countable union of pairwise disjoint measurable sets. Then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_i)$, where we agree that $\lambda(A) = 0$ if $\lambda(A_i) = 0$ for all $i$.

- Under the assumption of the axiom of choice, one can construct non-measurable subsets of $\mathbb{R}^n$.

Let $A$ be a measurable subset of $\mathbb{R}^n$. We say that a particular condition holds for almost all $x \in A$, if it holds for all $x \in A$ with the exception of a subset of Lebesgue measure 0. If the condition holds for almost all $x \in \mathbb{R}^n$, we say that it holds almost everywhere.

All sets occurring in this course will be measurable; we will never bother about the verification in individual cases.
0.4.2 Measurable functions

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called measurable if for every \( a \in \mathbb{R} \), the set \( \{ x \in \mathbb{R}^n : f(x) > a \} \) is measurable.

A function \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) is measurable if both \( \text{Re} \, f \) and \( \text{Im} \, f \) are measurable.

Facts:

- If \( A \subset \mathbb{R}^n \) is measurable then its characteristic function, given by \( I_A(x) = 1 \) if \( x \in A \), \( I_A(x) = 0 \) otherwise is measurable.
- Every continuous function \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) is measurable. More generally, \( f \) is measurable if its set of discontinuities has Lebesgue measure 0.
- If \( f, g : \mathbb{R}^n \rightarrow \mathbb{C} \) are measurable then \( f + g \) and \( fg \) are measurable. Further, the function given by \( x \mapsto f(x)/g(x) \) if \( g(x) \neq 0 \) and \( x \mapsto 0 \) if \( g(x) = 0 \) is measurable.
- If \( f, g : \mathbb{R}^n \rightarrow \mathbb{R} \) are measurable, then so are \( \max(f, g) \) and \( \min(f, g) \).
- If \( \{f_k : \mathbb{R}^n \rightarrow \mathbb{C}\} \) is a sequence of measurable functions and \( f_k \rightarrow f \) pointwise on \( \mathbb{R}^n \), then \( f \) is measurable.

All functions occurring in our course can be proved to be measurable by combining the above facts. We will always omit such nasty verifications, and take the measurability of the functions for granted.

0.4.3 Lebesgue integrals

The Lebesgue integral is defined in various steps.

1) An elementary function on \( \mathbb{R}^n \) is a function of the type \( f = \sum_{i=1}^{r} c_i I_{D_i} \), where \( D_1, \ldots, D_r \) are pairwise disjoint measurable subsets of \( \mathbb{R}^n \), and \( c_1, \ldots, c_r \) positive reals. Then we define \( \int fdx := \sum_{i=1}^{r} c_i \lambda(D_i) \).

2) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be measurable and \( f \geq 0 \) on \( \mathbb{R}^n \). Then we define \( \int fdx := \sup \int gdx \) where the supremum is taken over all elementary functions \( g \leq f \). Thus, \( \int fdx \) is defined and \( \geq 0 \) but it may be infinite.
3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary measurable function. Then we define

$$\int f \, dx := \int \max(f, 0) \, dx - \int \max(-f, 0) \, dx,$$

provided that at least one of the integrals is finite. If both integrals are finite, we say that $f$ is integrable or summable.

4) Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that $f$ is integrable if both $\text{Re} \, f$ and $\text{Im} \, f$ are integrable, and in that case we define

$$\int f \, dx := \int (\text{Re} \, f) \, dx + i \int (\text{Im} \, f) \, dx.$$

5) Let $D$ be a measurable subset of $\mathbb{R}^n$. Let $f$ be a complex function defined on a set containing $D$. We define $f \cdot I_D$ by defining it to be equal to $f$ on $D$ and equal to 0 outside $D$. We say that $f$ is measurable on $D$ if $f \cdot I_D$ is measurable. Further, we say that $f$ is integrable over $D$ if $f \cdot I_D$ is integrable, and in that case we define

$$\int_D f \, dx := \int f \cdot I_D \, dx.$$

Facts:

- Let $D$ be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a measurable function. Then $f$ is integrable over $D$ if and only if $\int_D |f| \, dx < \infty$ and in that case, $|\int_D f \, dx| \leq \int_D |f| \, dx$.

- Let again $D$ be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$, $g : D \to \mathbb{R}_{\geq 0}$ measurable functions, such that $\int_D g \, dx < \infty$ and $|f| \leq g$ on $D$. Then $f$ is integrable over $D$, and $|\int_D f \, dx| \leq \int_D g \, dx$.

- Let $D$ be a closed interval in $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a bounded function which is Riemann integrable over $D$. Then $f$ is Lebesgue integrable over $D$ and the Lebesgue integral $\int_D f \, dx$ is equal to the Riemann integral $\int_D f(x) \, dx$.

- Let $f : [0, \infty) \to \mathbb{C}$ be such that the improper Riemann integral $\int_0^\infty |f(x)| \, dx$ converges. Then the improper Riemann integral $\int_0^\infty f(x) \, dx$ converges as well, and it is equal to the Lebesgue integral $\int_{[0,\infty)} f \, dx$. However, an improper Riemann integral $\int_0^{\infty} f(x) \, dx$ which is convergent, but for which $\int_0^{\infty} |f(x)| \, dx = \infty$ can not be interpreted as a Lebesgue integral. The same applies to the other types of improper Riemann integrals, e.g., $\int_a^b f(x) \, dx$ where $f$ is unbounded on $(a, b)$. 


• An absolutely convergent series of complex terms $\sum_{n=0}^{\infty} a_n$ may be interpreted as a Lebesgue integral. Define the function $A$ by $A(x) := a_n$ for $x \in \mathbb{R}$ with $n \leq x < n + 1$ and $A(x) := 0$ for $x < 0$. Then $A$ is measurable and integrable, and $\sum_{n=0}^{\infty} a_n = \int A \, dx$.

0.4.4 Important theorems

**Theorem 0.7** (Dominated Convergence Theorem). Let $D \subseteq \mathbb{R}^n$ be a measurable set and $\{f_k : D \to \mathbb{C}\}_{k \geq 0}$ a sequence of functions that are all integrable over $D$, and such that $f_k \to f$ pointwise on $D$. Assume that there is an integrable function $g : D \to \mathbb{R}_{\geq 0}$ such that $|f_k(x)| \leq g(x)$ for all $x \in D$, $k \geq 0$. Then $f$ is integrable over $D$, and $\int_D f_k \, dx \to \int_D f \, dx$.

**Corollary 0.8.** let $D \subseteq \mathbb{R}^n$ be a measurable set of finite measure and $\{f_k : D \to \mathbb{C}\}_{k \geq 0}$ a sequence of functions that are all integrable over $D$, and such that $f_k \to f$ uniformly on $D$. Then $f$ is integrable over $D$, and $\int_D f_k \, dx \to \int_D f \, dx$.

**Proof.** Let $\varepsilon > 0$. There is $k_0$ such that $|f(x) - f_k(x)| < \varepsilon$ for all $x \in D$, $k > k_0$. The constant function $x \mapsto \varepsilon$ is integrable over $D$ since $D$ has finite measure. Hence for $k > k_0$, $f - f_k$ is integrable over $D$, and so $f$ is integrable over $D$. Consequently, $|f|$ is integrable over $D$. Now $|f_k| < \varepsilon + |f|$ for $k > k_0$. So by the Dominated Convergence Theorem, $\int_D f_k \, dx \to \int_D f \, dx$. \qed

In the theorem below, we write points of $\mathbb{R}^{m+n}$ as $(x, y)$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Further, $dx$, $dy$, $d(x,y)$ denote the Lebesgue measures on $\mathbb{R}^m$, $\mathbb{R}^n$, $\mathbb{R}^{m+n}$, respectively.

**Theorem 0.9** (Fubini-Tonelli). Let $D_1, D_2$ be measurable subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively, and $f : D_1 \times D_2 \to \mathbb{C}$ a measurable function. Assume that at least one of the integrals

$$\int_{D_1 \times D_2} |f(x,y)| d(x,y), \quad \int_{D_1} \left( \int_{D_2} |f(x,y)| \, dy \right) \, dx, \quad \int_{D_2} \left( \int_{D_1} |f(x,y)| \, dx \right) \, dy$$

is finite. Then they are all finite and equal. Further, $f$ is integrable over $D_1 \times D_2$, $x \mapsto f(x,y)$ is integrable over $D_1$ for almost all $y \in D_2$, $y \mapsto f(x,y)$ is integrable over $D_2$ for almost all $x \in D_1$, and

$$\int_{D_1 \times D_2} f(x,y) \, d(x,y) = \int_{D_1} \left( \int_{D_2} f(x,y) \, dy \right) \, dx = \int_{D_2} \left( \int_{D_1} f(x,y) \, dx \right) \, dy.$$
Corollary 0.10. Let $D$ be a measurable subset of $\mathbb{R}^m$ and \{${f_k : D \to \mathbb{C}}_{k\geq 0}$ a sequence of functions that are all integrable over $D$ and such that $\sum_{k=0}^{\infty} |f_k|$ converges pointwise on $D$. Assume that at least one of the quantities

$$\sum_{k=0}^{\infty} \int_D |f_k(x)| \, dx, \quad \int_D \left( \sum_{k=0}^{\infty} |f_k(x)| \right) \, dx$$

is finite. Then $\sum_{k=0}^{\infty} f_k$ is integrable over $D$ and

$$\sum_{k=0}^{\infty} \int_D f_k(x) \, dx = \int_D \left( \sum_{k=0}^{\infty} f_k(x) \right) \, dx.$$

Proof. Apply the Theorem of Fubini-Tonelli with $n = 1$, $D_1 = D$, $D_2 = [0, \infty)$, $F(x, y) = f_k(x)$ where $k$ is the integer with $k \leq y < k + 1$. \hfill \Box

Corollary 0.11. Let \{${a_{kl}}_{k,l=0}^{\infty}$ be a double sequence of complex numbers such that at least one of

$$\sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} |a_{kl}| \right), \quad \sum_{l=0}^{\infty} \left( \sum_{k=0}^{\infty} |a_{kl}| \right)$$

converges. Then both

$$\sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} a_{kl} \right), \quad \sum_{l=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{kl} \right)$$

converge and are equal.

Proof. Apply the Theorem of Fubini-Tonelli with $m = n = 1$, $D_1 = D_2 = [0, \infty)$, $F(x, y) = a_{kl}$ where $k, l$ are the integers with $k \leq x < k + 1, l \leq y < l + 1$. \hfill \Box

0.4.5 Useful inequalities

We have collected some inequalities, stated without proof, which frequently show up in analytic number theory. The proofs belong to a course in measure theory or functional analysis.

Proposition 0.12. Let $D$ be a measurable subset of $\mathbb{R}^n$ and $f, g : D \to \mathbb{C}$ measurable functions. Let $p, q$ be reals > 1 with $\frac{1}{p} + \frac{1}{q} = 1$. Then if all integrals are defined,

$$\left| \int_D f g \, dx \right| \leq \left( \int_D |f|^p \, dx \right)^{1/p} \cdot \left( \int_D |g|^q \, dx \right)^{1/q}$$

(Hölder’s Inequality).
In particular,
\[ \left| \int_D fg \, dx \right| \leq \left( \int_D |f|^2 \, dx \right)^{1/2} \cdot \left( \int_D |g|^2 \, dx \right)^{1/2} \quad \text{(Cauchy-Schwarz' Inequality)}.\]

**Corollary 0.13.** Let \( a_1, \ldots, a_r, \ b_1, \ldots, b_r \) be complex numbers and \( p, q \) reals \( > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[ \left| \sum_{n=1}^r a_n b_n \right| \leq \left( \sum_{n=1}^r |a_n|^p \right)^{1/p} \cdot \left( \sum_{n=1}^r |b_n|^q \right)^{1/q} \quad \text{(Hölder)}.\]

In particular,
\[ \left| \sum_{n=1}^r a_n b_n \right| \leq \left( \sum_{n=1}^r |a_n|^2 \right)^{1/2} \cdot \left( \sum_{n=1}^r |b_n|^2 \right)^{1/2} \quad \text{(Cauchy-Schwarz)}.\]

This follows from Proposition 0.12 by taking \( D = [0, r) \), \( f(x) = a_n \), \( g(x) = b_n \) for \( n - 1 \leq x < n, \ n = 1, \ldots, r \).

A function \( \varphi \) from an interval \( I \subseteq \mathbb{R} \) to \( \mathbb{R} \) is called convex if \( \varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y) \) holds for all \( x, y \in I \) and all \( t \in [0,1] \). In particular, \( \varphi \) is convex on \( I \) if \( \varphi \) is twice differentiable and \( \varphi'' \geq 0 \) on \( I \).

**Proposition 0.14.** Let \( D \) be a measurable subset of \( \mathbb{R}^n \) with \( 0 < \lambda(D) < \infty \), let \( f : D \to \mathbb{R}_{>0} \) be a Lebesgue integrable function and let \( \varphi : \mathbb{R}_{>0} \to \mathbb{R} \) be a convex function. Then
\[ \varphi \left( \frac{1}{\lambda(D)} \int_D f \cdot dx \right) \leq \frac{1}{\lambda(D)} \int_D (\varphi \circ f) \, dx \quad \text{(Jensen's Inequality)}.\]

**Corollary 0.15.** Let \( a_1, \ldots, a_r \) be positive reals, and let \( \varphi : \mathbb{R}_{>0} \to \mathbb{R} \) be a convex function. Then
\[ \varphi \left( \frac{1}{r} \sum_{n=1}^r a_n \right) \leq \frac{1}{r} \sum_{n=1}^r \varphi(a_n) \]

In particular,
\[ \frac{1}{r} \sum_{n=1}^r a_n \geq \sqrt[r]{a_1 \cdots a_n} \quad \text{(arithmetic mean \geq geometric mean)}.\]

The first assertion follows by applying Proposition 0.14 with \( D = [0, r) \) and \( f(x) = a_n \) for \( x \in [n-1, n) \). The second assertion follows by applying the first with \( \varphi(x) = -\log x \).
0.5 Line integrals

0.5.1 Paths in \( \mathbb{C} \)

We consider continuous functions \( g : [a, b] \to \mathbb{C} \), where \( a, b \in \mathbb{R} \) and \( a < b \). Two continuous functions \( g_1 : [a, b] \to \mathbb{C}, \ g_2 : [c, d] \to \mathbb{C} \) are called equivalent if there is a continuous monotone increasing function \( \varphi : [a, b] \to [c, d] \) such that \( g_1 = g_2 \circ \varphi \). The equivalence classes of this relation are called paths (in \( \mathbb{C} \)), and a function \( g : [a, b] \to \mathbb{C} \) representing a path is called a parametrization of the path. Roughly speaking, a path is a curve in \( \mathbb{C} \), together with a direction in which it is traversed.

A (continuously) differentiable path is a path represented by a (continuously) differentiable function \( g : [a, b] \to \mathbb{C} \).

Let \( \gamma \) be a path. Choose a parametrization \( g : [a, b] \to \mathbb{C} \) of \( \gamma \). We call \( g(a) \) the start point and \( g(b) \) the end point of \( \gamma \). Further, \( g([a, b]) \) is called the support of \( \gamma \). By saying that a function is continuous on \( \gamma \), or that \( \gamma \) is contained in a particular set, etc., we mean the support of \( \gamma \).

The path \( \gamma \) is said to be closed if its end point is equal to its start point, i.e., if \( g(a) = g(b) \). The path \( \gamma \) is called a contour if it is closed, has no self-intersections, and is traversed counterclockwise (we will not give the cumbersome formal definition of this intuitively obvious notion).

Let \( \gamma_1, \gamma_2 \) be paths, such that the end point of \( \gamma_1 \) is equal to the start point of \( \gamma_2 \). We define \( \gamma_1 + \gamma_2 \) to be the path obtained by first traversing \( \gamma_1 \) and then \( \gamma_2 \).
For instance, if $g_1 : [a, b] \to \mathbb{C}$ is a parametrization of $\gamma_1$ then we may choose a parametrization $g_2 : [b, c] \to \mathbb{C}$ of $\gamma_2$; then $g : [a, c] \to \mathbb{C}$ defined by $g(t) := g_1(t)$ if $a \leq t \leq b$, $g(t) := g_2(t)$ if $b \leq t \leq c$ is a parametrization of $\gamma_1 + \gamma_2$.

Given a path $\gamma$, we define $-\gamma$ to be the path traversed in the opposite direction, i.e., the start point of $-\gamma$ is the end point of $\gamma$ and conversely.

Let $\gamma$ be a path and $F : \gamma \to \mathbb{C}$ a continuous function on (the support of) $\gamma$. Then $F(\gamma)$ is the path such that if $g : [a, b] \to \mathbb{C}$ is a parametrization of $\gamma$ then $F \circ g : [a, b] \to \mathbb{C}$ is a parametrization of $F(\gamma)$.

0.5.2 Definition of line integrals

All paths occurring in our course will be built up from circle segments and line segments. So for our purposes, it suffices to define integrals of continuous functions along piecewise continuously differentiable paths, these are paths of the shape $\gamma_1 + \cdots + \gamma_r$, where $\gamma_1, \ldots, \gamma_r$ are continuously differentiable paths, and for $i = 1, \ldots, r-1$, the end point of $\gamma_i$ coincides with the start point of $\gamma_{i+1}$.

Let $\gamma$ be a continuously differentiable path, and $f : \gamma \to \mathbb{C}$ a continuous function. Choose a continuously differentiable parametrization $g : [a, b] \to \mathbb{C}$ of $\gamma$. Then we define

$$\int_{\gamma} f(z) \, dz := \int_{a}^{b} f(g(t)) g'(t) \, dt.$$
Further, we define the length of $\gamma$ by

$$L(\gamma) := \int_a^b |g'(t)| \, dt.$$  

These notions do not depend on the choice of $g$.

If $\gamma = \gamma_1 + \cdots + \gamma_r$ is a piecewise continuously differentiable path with continuously differentiable pieces $\gamma_1, \ldots, \gamma_r$ and $f : \gamma \to \mathbb{C}$ is continuous, we define

$$\int_{\gamma} f(z) \, dz := \sum_{i=1}^{r} \int_{\gamma_i} f(z) \, dz$$

and

$$L(\gamma) := \sum_{i=1}^{r} L(\gamma_i).$$

In case that $\gamma$ is closed, we write $\oint_{\gamma} f(z) \, dz$. It can be shown that the value of this integral is independent of the choice of the common start point and end point of $\gamma$.

We mention here that line integrals $\int_{\gamma} f(z) \, dz$ can be defined also for paths $\gamma$ that are not necessarily piecewise continuously differentiable. For piecewise continuously differentiable paths, this new definition coincides with the one given above.

Let $\gamma$ be any path and choose a parametrization $g : [a, b] \to \mathbb{C}$ of $\gamma$. A partition of $[a, b]$ is a tuple $P = (t_0, \ldots, t_s)$ where $a = t_0 < t_1 < \cdots < t_s = b$. We define the length of $\gamma$ by

$$L(\gamma) := \sup_P \sum_{i=1}^{s} |g(t_i) - g(t_{i-1})|,$$

where the supremum is taken over all partitions $P$ of $[a, b]$. This does not depend on the choice of $g$. We call $\gamma$ rectifiable if $L(\gamma) < \infty$ (in another language, this means that the function $g$ is of bounded variation).

Let $\gamma$ be a rectifiable path, and $g : [a, b] \to \mathbb{C}$ a parametrization of $\gamma$. Given a partition $P = (t_0, \ldots, t_s)$ of $[a, b]$, we define the mesh of $P$ by

$$\delta(P) := \max_{1 \leq i \leq s} |t_i - t_{i-1}|.$$

A sequence of intermediate points of $P$ is a tuple $W = (w_1, \ldots, w_s)$ such that $t_0 < w_1 < t_1 < w_2 < t_2 < \cdots < t_s$.  

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Let $f : \gamma \to \mathbb{C}$ be a continuous function. For a partition $P$ of $[a, b]$ and a tuple of intermediate points $W$ of $P$ we define

$$S(f, g, P, W) := \sum_{i=1}^{s} f(g(w_i))(g(t_i) - g(t_{i-1})).$$

One can show that there is a finite number, denoted $\int_{\gamma} f(z)dz$, such that for any choice of parametrization $g : [a, b] \to \mathbb{C}$ of $\gamma$ and any sequence $(P_n, W_n)_{n \geq 0}$ of partitions $P_n$ of $[a, b]$ and sequences of intermediate points $W_n$ of $P_n$ with $\delta(P_n) \to 0$,

$$\int_{\gamma} f(z)dz = \lim_{n \to \infty} S(f, g, P_n, W_n).$$

In another language, $\int_{\gamma} f(z)dz$ is equal to the Riemann-Stieltjes integral $\int_{a}^{b} f(g(t))dg(t)$.

### 0.5.3 Properties of line integrals

Below (and in the remainder of the course), by a path we will mean a piecewise continuously differentiable path. In fact, all properties below hold for line integrals over rectifiable paths, but in textbooks on complex analysis, these properties are never proved in this generality.

- Let $\gamma$ be a path, and $f : \gamma \to \mathbb{C}$ continuous. Then

$$\left| \int_{\gamma} f(z)dz \right| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|.$$

- Let $\gamma_1, \gamma_2$ be two paths such that the end point of $\gamma_1$ and the start point of $\gamma_2$ coincide. Let $f : \gamma_1 + \gamma_2 \to \mathbb{C}$ continuous. Then

$$\int_{\gamma_1 + \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

- Let $\gamma$ be a path and $f : \gamma \to \mathbb{C}$ continuous. Then

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$
Let $\gamma$ be a path and \( \{ f_n : \gamma \to \mathbb{C} \}_{n=1}^\infty \) a sequence of continuous functions. Suppose that $f_n \to f$ uniformly on $\gamma$, i.e., $\sup_{z \in \gamma} |f_n(z) - f(z)| \to 0$ as $n \to \infty$. Then $f$ is continuous on $\gamma$, and $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$ as $n \to \infty$.

Call a function $F : U \to \mathbb{C}$ on an open subset $U$ of $\mathbb{C}$ analytic if for every $z \in U$ the limit

$$F'(z) = \lim_{h \in \mathbb{C}, h \to 0} \frac{F(z + h) - F(z)}{h}$$

exists. Let $\gamma$ be a path with start point $z_0$ and end point $z_1$, and let $F$ be an analytic function defined on an open set $U \subset \mathbb{C}$ that contains $\gamma$. Then

$$\int_\gamma F'(z)dz = F(z_1) - F(z_0).$$

Let $\gamma$ be a path and $F$ an analytic function defined on some open set containing $\gamma$. Further, let $f : F(\gamma) \to \mathbb{C}$ be continuous. Then

$$\int_{F(\gamma)} f(w)dw = \int_\gamma f(F(z))F'(z)dz.$$

**Examples.**

1. Let $\gamma_{a,r}$ denote the circle with center $a$ and radius $r$, traversed counterclockwise. For $\gamma_{a,r}$ we may choose a parametrization $t \mapsto a + re^{2\pi it}$, $t \in [0, 1]$. Let $n \in \mathbb{Z}$. Then

$$\oint_{\gamma_{a,r}} (z-a)^n dz = \int_0^1 r^n e^{2\pi int} \cdot 2\pi i \cdot r e^{2\pi it} dt = 2\pi ir^{n+1} \int_0^1 e^{2(n+1)\pi it} dt = \begin{cases} 2\pi i & \text{if } n = -1; \\ 0 & \text{if } n \neq -1. \end{cases}$$

2. For $z_0, z_1 \in \mathbb{C}$, denote by $[z_0, z_1]$ the line segment from $z_0$ to $z_1$. For $[z_0, z_1]$ we may choose a parametrization $t \mapsto z_0 + t(z_1 - z_0)$, $t \in [0, 1]$. Let $f : [z_0, z_1] \to \mathbb{C}$ be continuous. Then

$$\int_{[z_0, z_1]} f(z)dz = \int_0^1 f(z_0 + t(z_1 - z_0))(z_1 - z_0) dt.$$

**0.6 Topology**

We recall some facts about the topology of $\mathbb{C}$.
0.6.1 Basic facts

Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$. We define the open disk and closed disk with center $a$ and radius $r$,

$$D(a, r) := \{ z \in \mathbb{C} : |z - a| < r \}, \quad \overline{D}(a, r) := \{ z \in \mathbb{C} : |z - a| \leq r \}.$$ 

Recall that a subset $U$ of $\mathbb{C}$ is called open if either $U = \emptyset$, or for every $a \in U$ there is $\delta > 0$ with $D(a, \delta) \subset U$. A subset $U$ of $\mathbb{C}$ is called closed if its complement $U^c = \mathbb{C} \setminus U$ is open. It is easy to verify that the union of any possibly infinite collection of open subsets of $\mathbb{C}$ is open. Further, the intersection of finitely many open subsets is open. Consequently, the intersection of any possibly infinite collection of closed sets is closed, and the union of finitely many closed subsets is closed.

A subset $S$ of $\mathbb{C}$ is called compact, if for every collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{C}$ with $S \subset \bigcup_{\alpha \in I} U_\alpha$ there is a finite subset $F$ of $I$ such that $S \subset \bigcup_{\alpha \in F} U_\alpha$, in other words, every open cover of $S$ has a finite subcover.

By the Heine-Borel Theorem, a subset of $\mathbb{C}$ is compact if and only if it is closed and bounded.

Let $U$ be a non-empty subset of $\mathbb{C}$. A point $z_0 \in \mathbb{C}$ is called a limit point of $U$ if there is a sequence $\{z_n\}$ in $U$ such that all $z_n$ are distinct and $z_n \to z_0$ as $n \to \infty$. Recall that a non-empty subset $U$ of $\mathbb{C}$ is closed if and only if each of its limit points belongs to $U$.

Let $U$ be a non-empty open subset of $\mathbb{C}$, and $S \subset U$. Then $S$ is called discrete in $U$ if it has no limit points in $U$. Recall that by the Bolzano-Weierstrass Theorem, every infinite subset of a compact subset $K$ of $\mathbb{C}$ has a limit point in $K$. This implies that $S$ is discrete in $U$ if and only if for every compact subset $K$ of $\mathbb{C}$ with $K \subset U$, the intersection $K \cap S$ is finite.

Let $U$ be a non-empty, open subset of $\mathbb{C}$. We say that $U$ is connected if there are no non-empty open sets $U_1, U_2$ with $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. We say that $U$ is pathwise connected if for every $z_0, z_1 \in U$ there is a path $\gamma \subset U$ with start point $z_0$ and end point $z_1$. A fact (typical for the topological space $\mathbb{C}$) is that a non-empty open subset $U$ of $\mathbb{C}$ is connected if and only if it is pathwise connected.

Let $U$ be any, non-empty open subset of $\mathbb{C}$. We can express $U$ as a disjoint union $\bigcup_{\alpha \in I} U_\alpha$, with $I$ some index set, such that two points of $U$ belong to the same set $U_\alpha$ if and only if they are connected by a path contained in $U$. The sets $U_\alpha$ are open,
connected, and pairwise disjoint. We call these sets $U_\alpha$ the connected components of $U$.

0.6.2 Homotopy

Let $U \subseteq \mathbb{C}$ and $\gamma_1, \gamma_2$ two paths in $U$ with start point $z_0$ and end point $z_1$. Then $\gamma_1, \gamma_2$ are homotopic in $U$ if one can be continuously deformed into the other within $U$. More precisely this means the following: there are parametrizations $f : [0, 1] \to \mathbb{C}$ of $\gamma_1$, $g : [0, 1] \to \mathbb{C}$ of $\gamma_2$ and a continuous map $H : [0, 1] \times [0, 1] \to U$ with the following properties:

\[
H(0, t) = f(t), \quad H(1, t) = g(t) \quad \text{for } 0 \leq t \leq 1;
\]

\[
H(s, 0) = z_0, \quad H(s, 1) = z_1 \quad \text{for } 0 \leq s \leq 1.
\]

Let $U \subseteq \mathbb{C}$ be open and non-empty. We call $U$ simply connected ('without holes') if it is connected and if every closed path in $U$ can be contracted to a point in $U$, that is, if $z_0$ is any point in $U$ and $\gamma$ is any closed path in $U$ containing $z_0$, then $\gamma$ is homotopic in $U$ to $z_0$.

A map $f : D_1 \to D_2$, where $D_1, D_2$ are subsets of $\mathbb{C}$, is called a homeomorphism if $f$ is a bijection, and both $f$ and $f^{-1}$ are continuous. Homeomorphisms preserve topological properties of sets such as openness, closedness, compactness, (simple) connectedness, etc.

**Theorem 0.16** (Schoenflies Theorem for curves). Let $\gamma$ be a contour in $\mathbb{C}$. Then
there is a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\gamma_{0,1}) = \gamma$, where $\gamma_{0,1}$ is the unit circle with center 0 and radius 1, traversed counterclockwise.

**Corollary 0.17** (Jordan Curve Theorem). Let $\gamma$ be a contour in $\mathbb{C}$. Then $\mathbb{C} \setminus \gamma$ has two connected components, $U_1$ and $U_2$. The component $U_1$ is bounded and simply connected, while $U_2$ is unbounded.

The component $U_1$ is called the **interior** of $\gamma$, notation $\text{int}(\gamma)$, and $U_2$ the **exterior** of $\gamma$, notation $\text{ext}(\gamma)$.

## 0.7 Complex analysis

We have given an overview of the complex analysis that will be used during the course. We will need only the theorems and corollaries and the like, but not the proofs. For readers who have followed a course on complex analysis, most of this will be familiar. We hope that readers who did not follow such a course will gain sufficient confidence with complex analysis from reading these notes.

### 0.7.1 Basics

In what follows, $U$ is a non-empty open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. We say that $f$ is **holomorphic** or **analytic** in $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In that case, the limit is denoted by $f'(z_0)$. We say that $f$ is analytic on $U$ if $f$ is analytic in every $z \in U$; in that case, the derivative $f'(z)$ is defined for every $z \in U$. 

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We say that $f$ is analytic around $z_0$ if it is analytic on some open disk $D(z_0, \delta)$ for some $\delta > 0$. Finally, given a not necessarily open subset $A$ of $\mathbb{C}$ and a function $f : A \rightarrow \mathbb{C}$, we say that $f$ is analytic on $A$ if there is an open set $U \supseteq A$ such that $f$ is defined on $U$ and analytic on $U$. An everywhere analytic function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called entire.

For any two analytic functions $f, g$ on some open set $U \subseteq \mathbb{C}$, we have the usual rules for differentiation $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$ and $(f/g)' = (gf' - fg')/g^2$ (the latter is defined for any $z$ with $g(z) \neq 0$). Further, given a non-empty set $U \subseteq \mathbb{C}$, and analytic functions $f : U \rightarrow \mathbb{C}$, $g : f(U) \rightarrow \mathbb{C}$, the composition $g \circ f$ is analytic on $U$ and $(g \circ f)' = (g' \circ f) \cdot f'$.

Recall that a power series around $z_0 \in \mathbb{C}$ is an infinite sum

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}_{\geq 0}$. The radius of convergence of this series is given by

$$R = R_f = \left(\limsup_{n \to \infty} \sqrt[n]{|a_n|}\right)^{-1}.$$

We state without proof the following fact.

**Theorem 0.18.** Let $z_0 \in \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ a power series around $z_0 \in \mathbb{C}$ with radius of convergence $R > 0$. Then $f$ defines a function on $D(z_0, R)$ which is analytic infinitely often. For $k \geq 0$ the $k$-th derivative $f^{(k)}$ of $f$ has a power series expansion with radius of convergence $R$ given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)a_n(z - z_0)^{n-k}.$$

In each of the examples below, $R$ denotes the radius of convergence of the given
power series.

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad R = \infty, \quad (e^z)' = e^z. \]
\[ \cos z = \frac{(e^{iz} + e^{-iz})}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad R = \infty, \quad \cos' z = -\sin z. \]
\[ \sin z = \frac{(e^{iz} - e^{-iz})}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad R = \infty, \quad \sin' z = \cos z. \]
\[ (1 + z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad R = 1, \quad ((1 + z)^\alpha)' = \alpha (1 + z)^{\alpha - 1} \]

where \( \alpha \in \mathbb{C}, \quad \binom{\alpha}{n} = \frac{\alpha (\alpha-1) \cdots (\alpha-n+1)}{n!} \).
\[ \log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot z^n, \quad R = 1, \quad \log'(1 + z) = (1 + z)^{-1}. \]

### 0.7.2 Cauchy’s Theorem and some applications

In the remainder of this course, a path will always be a piecewise continuously differentiable path. Recall that for a piecewise continuously differentiable path \( \gamma \), say \( \gamma = \gamma_1 + \cdots + \gamma_r \) where \( \gamma_1, \ldots, \gamma_r \) are paths with continuously differentiable parametrizations \( g_i : [a_i, b_i] \to \mathbb{C} \), and for a continuous function \( f : \gamma \to \mathbb{C} \) we have
\[
\int_{\gamma} f(z)dz = \sum_{i=1}^{r} \int_{a_i}^{b_i} f(g(t))g'_i(t)dt.
\]

**Theorem 0.19** (Cauchy). Let \( U \subseteq \mathbb{C} \) be a non-empty open set and \( f : U \to \mathbb{C} \) an analytic function. Further, let \( \gamma_1, \gamma_2 \) be two paths in \( U \) with the same start point and end point that are homotopic in \( U \). Then
\[
\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.
\]

**Proof.** Any textbook on complex analysis.

**Corollary 0.20.** Let \( U \subseteq \mathbb{C} \) be a non-empty, open, simply connected set, and \( f : U \to \mathbb{C} \) an analytic function. Then for any closed path \( \gamma \) in \( U \),
\[
\oint_{\gamma} f(z)dz = 0.
\]
Proof. The path $\gamma$ is homotopic in $U$ to a point, and a line integral along a point is 0. 

**Corollary 0.21.** Let $\gamma_1, \gamma_2$ be two contours (closed paths without self-intersections traversed counterclockwise), such that $\gamma_2$ is contained in the interior of $\gamma_1$. Let $U \subset \mathbb{C}$ be an open set which contains $\gamma_1, \gamma_2$ and the region between $\gamma_1$ and $\gamma_2$. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

Proof.

Let $z_0, z_1$ be points on $\gamma_1, \gamma_2$ respectively, and let $\alpha$ be a path from $z_0$ to $z_1$ lying inside the region between $\gamma_1$ and $\gamma_2$ without self-intersections.

Then $\gamma_1$ is homotopic in $U$ to the path $\alpha + \gamma_2 - \alpha$, which consists of first traversing $\alpha$, then $\gamma_2$, and then $\alpha$ in the opposite direction. Hence

$$\oint_{\gamma_1} f(z)dz = \left(\int_{\alpha} + \oint_{\gamma_2} - \int_{\alpha}\right) f(z)dz = \oint_{\gamma_2} f(z)dz.$$

**Corollary 0.22** (Cauchy’s Integral Formula). Let $\gamma$ be a contour in $\mathbb{C}$, $U \subset \mathbb{C}$ an open set containing $\gamma$ and its interior, $z_0$ a point in the interior of $\gamma$, and $f : U \to \mathbb{C}$ an analytic function. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \cdot dz = f(z_0).$$

Proof.
Let $\gamma_{z_0, \delta}$ be the circle with center $z_0$ and radius $\delta$, traversed counterclockwise. Then by Corollary 0.21 we have for any sufficiently small $\delta > 0$,

$$
\frac{1}{2\pi i} \oint_{\gamma_{z_0, \delta}} f(z) \cdot dz = \frac{1}{2\pi i} \oint_{\gamma} f(z) \cdot dz.
$$

Now, since $f(z)$ is continuous, hence uniformly continuous on any sufficiently small compact set containing $z_0$,

$$
\left| \frac{1}{2\pi i} \oint_{\gamma} f(z) \cdot dz - f(z_0) \right| = \left| \frac{1}{2\pi i} \oint_{\gamma_{z_0, \delta}} f(z) \cdot dz - f(z_0) \right|
$$

$$
= \left| \int_0^1 \frac{f(z_0 + \delta e^{2\pi it})}{\delta e^{2\pi it}} \cdot \delta e^{2\pi it}dt - f(z_0) \right|
$$

$$
= \left| \int_0^1 \{ f(z_0 + \delta e^{2\pi it}) - f(z_0) \} dt \right| \leq \sup_{0 \leq t \leq 1} |f(z_0 + \delta e^{2\pi it}) - f(z_0)|
$$

$$
\rightarrow 0 \text{ as } \delta \downarrow 0.
$$

This completes our proof. \qed

We now show that every analytic function $f$ on a simply connected set has an anti-derivative. We first prove a simple lemma.

**Lemma 0.23.** Let $U \subseteq \mathbb{C}$ be a non-empty, open, connected set, and let $f : U \rightarrow \mathbb{C}$ be an analytic function such that $f' = 0$ on $U$. Then $f$ is constant on $U$.

**Proof.** Fix a point $z_0 \in U$ and let $z \in U$ be arbitrary. Take a path $\gamma_z$ in $U$ from $z_0$ to $z$ which exists since $U$ is (pathwise) connected. Then

$$
f(z) - f(z_0) = \int_{\gamma_z} f'(w)dw = 0.
$$
Corollary 0.24. Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected set, and $f : U \to \mathbb{C}$ an analytic function. Then there exists an analytic function $F : U \to \mathbb{C}$ with $F' = f$. Further, $F$ is determined uniquely up to addition with a constant.

Proof (sketch). If $F_1, F_2$ are any two analytic functions on $U$ with $F_1' = F_2' = f$, then $F_1' - F_2'$ is constant on $U$ since $U$ is connected. This shows that an anti-derivative of $f$ is determined uniquely up to addition with a constant. It thus suffices to prove the existence of an analytic function $F$ on $U$ with $F' = f$.

Fix $z_0 \in U$. Given $z \in U$, we define $F(z)$ by

$$F(z) := \int_{\gamma_z} f(w) dw,$$

where $\gamma_z$ is any path in $U$ from $z_0$ to $z$. This does not depend on the choice of $\gamma_z$.

For let $\gamma_1, \gamma_2$ be any two paths in $U$ from $z_0$ to $z$. Then $\gamma_1 - \gamma_2$ (the path consisting of first traversing $\gamma_1$ and then $\gamma_2$ in the opposite direction) is homotopic to $z_0$ since $U$ is simply connected, hence

$$\int_{\gamma_1} f(z) dz - \int_{\gamma_2} f(z) dz = \int_{\gamma_1 - \gamma_2} f(z) dz = 0.$$

To prove that $\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$, take a path $\gamma_z$ from $z_0$ to $z$ and then the line segment $[z, z+h]$ from $z$ to $z+h$. Then since $f$ is uniformly continuous on any sufficiently small compact set around $z$,

$$F(z+h) - F(z) = (\int_{\gamma_z + [z,z+h]} - \int_{\gamma_z}) f(w) dw = \int_{[z,z+h]} f(w) dw$$

$$= \int_0^1 f(z + th) h dt = h \left( f(z) + \int_0^1 (f(z+th) - f(z)) dt \right).$$

So

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \int_0^1 (f(z+th) - f(z)) dt \right| \leq \sup_{0 \leq t \leq 1} |f(z+th) - f(z)| \to 0 \text{ as } h \to 0.$$
Example. Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected subset of $\mathbb{C}$ with $0 \not\in U$. Then $1/z$ has an anti-derivative on $U$.

For instance, if $U = \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Re } z \leq 0\}$ we may take as anti-derivative of $1/z$,

$$\log z := \log |z| + i\text{Arg } z,$$

where $\text{Arg } z$ is the argument of $z$ in the interval $(-\pi, \pi)$ (this is called the principal value of the logarithm).

On $\{z \in \mathbb{C} : |z - 1| < 1\}$ we may take as anti-derivative of $1/z$ the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} (z - 1)^n / n.$$

### 0.7.3 Taylor series

**Theorem 0.25.** Let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in U$ and $R > 0$ be such that $D(z_0, R) \subseteq U$. Then $f$ has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for } z \in D(z_0, R).$$

Further, we have for $n \in \mathbb{Z}_{\geq 0},$

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0, r}} \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz \text{ for any } r \text{ with } 0 < r < R.$$

**Proof.** We fix $z \in D(z_0, R)$ and use $w$ to indicate a complex variable. Choose $r$ with $|z - z_0| < r < R$. By Cauchy’s integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0, r}} \frac{f(w)}{w - z} \cdot dw.$$
We rewrite the integrand. We have

$$
\frac{f(w)}{w - z} = \frac{f(w)}{(w - z_0) - (z - z_0)} = f(w) \cdot \left(1 - \frac{z - z_0}{w - z_0}\right)^{-1}
$$

$$
= \frac{f(w)}{w - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n.
$$

The latter series converges uniformly on $\gamma_{z_0,r}$. For let $M := \sup_{w \in \gamma_{z_0,r}} |f(w)|$. Then

$$
\sup_{w \in \gamma_{z_0,r}} \left| \frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n \right| \leq \frac{M}{r} \left(\frac{|z - z_0|}{r}\right)^n =: M_n
$$

and $\sum_{n=0}^{\infty} M_n$ converges since $|z - z_0| < r$. Consequently,

$$
f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \cdot dw
$$

$$
= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \sum_{n=0}^{\infty} \left(\frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n\right) dw
$$

$$
= \sum_{n=0}^{\infty} (z - z_0)^n \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{(w - z_0)^{n+1}} \cdot dw \right\}.
$$

Now Theorem 0.25 follows since by Corollary 0.21 the integral in (0.3) is independent of $r$. 

\[ \Box \]

**Corollary 0.26.** Let $U \subseteq \mathbb{C}$ be a non-empty, open set, and $f : U \to \mathbb{C}$ an analytic function. Then $f$ is analytic on $U$ infinitely often, that is, for every $k \geq 0$ the $k$-th derivative $f^{(k)}$ exists, and is analytic on $U$.

**Proof.** Pick $z \in U$. Choose $\delta > 0$ such that $D(z, \delta) \subset U$. Then for $w \in D(z, \delta)$ we have

$$
f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n \quad \text{with} \quad a_n = \frac{1}{2\pi i} \oint_{\gamma_{z,r}} \frac{f(w)}{(w - z)^n} \cdot dw \quad \text{for} \quad 0 < r < \delta.
$$

Now for every $k \geq 0$, the $k$-th derivative $f^{(k)}(z)$ exists and is equal to $k! a_k$. 

\[ \Box \]
Corollary 0.27. Let $\gamma$ be a contour in $\mathbb{C}$, and $U$ an open subset of $\mathbb{C}$ containing $\gamma$ and its interior. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for every $z$ in the interior of $\gamma$ and every $k \geq 0$ we have

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$ 

Proof. Choose $\delta > 0$ such that $\gamma_{z,\delta}$ lies in the interior of $\gamma$. By Corollary 0.21,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw = \frac{1}{2\pi i} \oint_{\gamma_{z,\delta}} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$ 

By the argument in Corollary 0.26, this is equal to $f^{(k)}(z)/k!$.

We prove a generalization of Cauchy’s integral formula.

Corollary 0.28. Let $\gamma_1, \gamma_2$ be two contours such that $\gamma_1$ is lying in the interior of $\gamma_2$. Let $U \subset \mathbb{C}$ be an open set which contains $\gamma_1, \gamma_2$ and the region between $\gamma_1, \gamma_2$. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for any $z_0$ in the region between $\gamma_1$ and $\gamma_2$ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z-z_0} \, dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z-z_0} \, dz.$$ 

Proof. We have seen that around $z_0$ the function $f$ has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Define the function on $U$,

$$g(z) := \frac{f(z) - a_0}{z-z_0} \quad (z \neq z_0); \quad g(z_0) := a_1.$$ 

The function $g$ is clearly analytic on $U \setminus \{z_0\}$. Further,

$$\frac{g(z) - g(z_0)}{z-z_0} = \sum_{n=2}^{\infty} a_n(z - z_0)^{n-2} \to a_2 \quad \text{as} \quad z \to z_0.$$ 

Hence $g$ is also analytic at $z = z_0$. In particular, $g$ is analytic in the region between $\gamma_1$ and $\gamma_2$. So by Corollary 0.21,

$$\oint_{\gamma_1} g(z) \, dz = \oint_{\gamma_2} g(z) \, dz.$$
Together with Corollaries 0.22, 0.21 this implies
\[
f(z_0) = a_0 = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{a_0}{z - z_0} \cdot dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{a_0}{z - z_0} \cdot dz
\]
\[
= \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} \cdot dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} \cdot dz.
\]

\[\square\]

0.7.4 Isolated singularities, Laurent series, meromorphic functions

We define the punctured disk with center \(z_0 \in \mathbb{C}\) and radius \(r > 0\) by
\[
D^0(z_0, r) := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.
\]

If \(f\) is an analytic function defined on \(D^0(z_0, r)\) for some \(r > 0\), we call \(z_0\) an isolated singularity of \(f\). In case that there exists an analytic function \(g\) on the non-punctured disk \(D(z_0, r)\) such that \(g(z) = f(z)\) for \(z \in D^0(z_0, r)\), we call \(z_0\) a removable singularity of \(f\).

**Theorem 0.29.** Let \(U \subseteq \mathbb{C}\) be a non-empty, open set and \(f : U \to \mathbb{C}\) an analytic function. Further, let \(z_0 \in U\), and let \(R > 0\) be such that \(D^0(z_0, R) \subseteq U\). Then \(f\) has a Laurent series expansion
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \text{ converging for } z \in D^0(z_0, R).
\]

Further, we have for \(n \in \mathbb{Z}\),
\[
a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz \text{ for any } r \text{ with } 0 < r < R.
\]

**Proof.** We fix \(z \in D^0(z_0, R)\) and use \(w\) to denote a complex variable. Choose \(r_1, r_2\) with \(0 < r_1 < |z - z_0| < r_2 < R\).

By Corollary 0.28 we have
\[
f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r_2}} \frac{f(w)}{w - z} \cdot dw - \frac{1}{2\pi i} \oint_{\gamma_{z_0,r_1}} \frac{f(w)}{w - z} \cdot dw =: I_1 - I_2,
\]

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say. Completely similarly to Theorem 0.25, one shows that

\[ I_1 = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]  

with \( a_n = \frac{1}{2 \pi i} \oint_{\gamma_{z_0,r_2}} \frac{f(w)}{(w - z_0)^{n+1}} \cdot dw \).

Notice that for \( w \) on the inner circle \( \gamma_{z_0,r_1} \) we have

\[ f(w) \frac{w - z}{w - z_0} = \frac{f(w)}{z - z_0} \left( \frac{1 - w - z_0}{z - z_0} \right)^{-1} \]

Further, one easily shows that the latter series converges uniformly to \( f(w)/(w - z) \) on \( \gamma_{z_0,r_1} \). After a substitution \( n = -m - 1 \), it follows that

\[ I_2 = - \frac{1}{2 \pi i} \oint_{\gamma_{z_0,r_2}} \left( \sum_{m=0}^{\infty} f(w)(w - w_0)^m(z - z_0)^{-m-1} \right) \cdot dw \]

By substituting the expressions for \( I_1, I_2 \) obtained above into (0.5), we obtain

\[ f(z) = I_1 - I_2 = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \]

Further, by Corollary 0.21 we have

\[ a_n = \frac{1}{2 \pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{(w - z_0)^{n+1}} \cdot dw \]

for any \( n \in \mathbb{Z} \) and any \( r \) with \( 0 < r < R \). This completes our proof. \( \square \)

We say that a function \( f \) has Laurent expansion \( \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \) (or Taylor expansion if \( a_n = 0 \) for \( n < 0 \)) around \( z_0 \) if there is \( r > 0 \) such that \( f(z) \) is equal to this Laurent series on \( D^0(z_0, r) \).

Let \( z_0 \in \mathbb{C} \) and suppose \( f \) has a Laurent series expansion

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \]
around \( z_0 \). Notice that \( z_0 \) is a removable singularity of \( f \) if \( a_n = 0 \) for all \( n < 0 \). We define the order of \( f \) at \( z_0 \) by

\[
\text{ord}_{z_0}(f) := \text{infimum of all } k \in \mathbb{Z} \text{ such that } a_k \neq 0,
\]

so that in particular \( \text{ord}_{z_0}(f) = \infty \) if \( f \equiv 0 \).

Clearly, \( f \) is analytic at \( z_0 \) if and only if \( \text{ord}_{z_0}(f) \geq 0 \). In case that \( \text{ord}_{z_0}(f) \) is finite, it is precisely the integer \( k \) such that \( g(z) := (z - z_0)^{-k}f(z) \) defines a function that is analytic and non-zero in \( z_0 \).

The point \( z_0 \) is called - an essential singularity of \( f \) if \( \text{ord}_{z_0}(f) = -\infty \); - a pole of order \( k \) of \( f \) if \( k > 0 \) and \( \text{ord}_{z_0}(f) = -k \); a simple pole is one of order 1; - a zero of order \( k \) of \( f \) if \( k > 0 \) and \( \text{ord}_{z_0}(f) = k \); a simple zero is one of order 1.

Notice that \( z_0 \) is a zero of order \( k \) of \( f \) if and only if \( f^{(j)}(z_0) = 0 \) for \( j = 0, \ldots, k - 1 \), and \( f^{(k)}(z_0) \neq 0 \).

We say that a complex function \( f \) is meromorphic around \( z_0 \) if \( f \) is analytic on \( D^0(z_0, r) \) for some \( r > 0 \), and \( z_0 \) is a pole or a removable singularity of \( f \). The meromorphic functions around \( z_0 \) contain as a subclass the functions analytic around \( z_0 \), i.e., those that are analytic in \( z_0 \) or for which \( z_0 \) is a removable singularity.

If \( f \) is meromorphic around \( z_0 \) and not identically 0, then so is \( 1/f \). Indeed, there is \( r > 0 \) such that \( f(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^n \) on \( D^0(z_0, r) \) with \( a_k \neq 0 \). We can write \( f(z) = (z - z_0)^k h(z) \) with \( h \) analytic on \( D(z_0, r) \) and \( h(z_0) = a_k \neq 0 \). By making \( r \) smaller we can achieve that \( h(z) \neq 0 \) on \( D(z_0, r) \). We thus get \( f(z) = (z - z_0)^{-k} h(z) \) with \( 1/h(\zeta) \) analytic and non-zero on \( D(z_0, r) \), and so \( 1/f \) is meromorphic around \( z_0 \) and moreover \( \text{ord}_{z_0}(1/f) = -\text{ord}_{z_0}(f) \).

It is obvious that if \( f, g \) are functions that are meromorphic around \( z_0 \) then so are \( f + g \) and \( fg \). Hence the functions meromorphic around \( z_0 \) form a field.

**Lemma 0.30.** Let \( z_0 \in \mathbb{C} \) and let \( f, g \) be two functions meromorphic around \( z_0 \). Then

\[
\text{ord}_{z_0}(f + g) \geq \min \left( \text{ord}_{z_0}(f), \text{ord}_{z_0}(g) \right);
\]

\[
\text{ord}_{z_0}(fg) = \text{ord}_{z_0}(f) + \text{ord}_{z_0}(g);
\]

\[
\text{ord}_{z_0}(f/g) = \text{ord}_{z_0}(f) - \text{ord}_{z_0}(g) \quad \text{if } g \neq 0.
\]

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Proof. Exercise.

For instance, if \( f, g \) are meromorphic functions around \( z_0 \), \( f \) has a pole of order \( k \) at \( z_0 \), \( g \) has a zero of order \( l \) at \( z_0 \) and \( l > k \), then \( fg \) is analytic around \( z_0 \), and \( fg \) has a zero of order \( l - k \) at \( z_0 \).

Lemma 0.30 shows that the function \( \text{ord}_{z_0} \) defines a discrete valuation on the field of functions meromorphic around \( z_0 \). In general, a discrete valuation on a field \( K \) is a surjective map \( v : K \to \mathbb{Z} \cup \{\infty\} \) such that \( v(0) = \infty \); \( v(x) \in \mathbb{Z} \) for \( x \in K \), \( x \neq 0 \); \( v(xy) = v(x) + v(y) \) for \( x, y \in K \); and \( v(x + y) \geq \min(v(x), v(y)) \) for \( x, y \in K \).

Other examples of discrete valuations are \( \text{ord}_p \) (\( p \) prime number) on \( \mathbb{Q} \), given by \( \text{ord}_p(0) := \infty \) and \( \text{ord}_p(\alpha) := k \) if \( \alpha = p^k a/b \), where \( k \) is an integer and \( a, b \) are integers not divisible by \( p \).

Let \( U \) be a non-empty, open subset of \( \mathbb{C} \). A meromorphic function on \( U \) is a complex function \( f \) with the following properties:
(i) there is a set \( S \) discrete in \( U \) such that \( f \) is defined and analytic on \( U \setminus S \);
(ii) all elements of \( S \) are poles of \( f \).

It is easy to verify that if \( f, g \) are meromorphic functions on \( U \) then so are \( f + g \) and \( f \cdot g \). It can be shown as well (less trivial) that if \( U \) is connected and \( g \) is a non-zero meromorphic function on \( U \), then the set of zeros of \( g \) is discrete in \( U \). The zeros of \( g \) are poles of \( 1/g \), and the poles of \( g \) are zeros of \( 1/g \). Hence \( 1/g \) is meromorphic on \( U \). Consequently, if \( U \) is an open, connected subset of \( \mathbb{C} \), then the functions meromorphic on \( U \) form a field.

0.7.5 Residues, logarithmic derivatives

Let \( z_0 \in \mathbb{C} \), \( R > 0 \) and let \( f : D^0(z_0, R) \to \mathbb{C} \) be an analytic function. Then \( f \) has a Laurent series expansion converging on \( D^0(z_0, R) \):

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.
\]

We define the residue of \( f \) at \( z_0 \) by

\[
\text{res}(z_0, f) := a_{-1}.
\]
In particular, if \( f \) is analytic at \( z_0 \) then \( \text{res}(z_0, f) = 0 \). By Theorem 0.29 we have

\[
\text{res}(z_0, f) = \frac{1}{2\pi i} \oint_{\gamma_{z_0, r}} f(z) \, dz
\]

for any \( r \) with \( 0 < r < R \).

**Theorem 0.31 (Residue Theorem).** Let \( \gamma \) be a contour in \( \mathbb{C} \) and let \( z_1, \ldots, z_q \) be points in the interior of \( \gamma \). Further, let \( f \) be a complex function that is analytic on an open set containing \( \gamma \) and the interior of \( \gamma \) minus \( \{z_1, \ldots, z_q\} \). Then

\[
\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \sum_{i=1}^{q} \text{res}(z_i, f).
\]

**Proof.** We proceed by induction on \( q \). First let \( q = 1 \). Choose \( r > 0 \) such that \( \gamma_{z_1, r} \) lies in the interior of \( \gamma \). Then by Corollary 0.21,

\[
\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \frac{1}{2\pi i} \oint_{\gamma_{z_1, r}} f(z) \, dz = \text{res}(z_1, f).
\]

Now let \( q > 1 \) and assume the Residue Theorem is true for fewer than \( q \) points. We cut \( \gamma \) into two pieces, the piece \( \gamma_1 \) from a point \( w_0 \) to \( w_1 \) and the piece \( \gamma_2 \) from \( w_1 \) to \( w_0 \) so that \( \gamma = \gamma_1 + \gamma_2 \). Then we take a path \( \gamma_3 \) from \( w_1 \) to \( w_0 \) inside the interior of \( \gamma \) without self-intersections; this gives two contours \( \gamma_1 + \gamma_3 \) and \( -\gamma_3 + \gamma_2 \).

We choose \( \gamma_3 \) in such a way that it does not hit any of the points \( z_1, \ldots, z_q \) and both the interiors of these contours contain points from \( z_1, \ldots, z_q \). Without loss of generality, we assume that the interior of \( \gamma_1 + \gamma_3 \) contains \( z_1, \ldots, z_m \) with \( 0 < m < q \), while the interior of \( -\gamma_3 + \gamma_2 \) contains \( z_{m+1}, \ldots, z_q \). Then by the induction hypothesis,

\[
\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \frac{1}{2\pi i} \oint_{\gamma_1} f(z) \, dz + \frac{1}{2\pi i} \oint_{\gamma_2} f(z) \, dz = \frac{1}{2\pi i} \oint_{\gamma_1 + \gamma_3} f(z) \, dz + \frac{1}{2\pi i} \oint_{-\gamma_3 + \gamma_2} f(z) \, dz = \sum_{i=1}^{m} \text{res}(z_i, f) + \sum_{i=m+1}^{q} \text{res}(z_i, f) = \sum_{i=1}^{q} \text{res}(z_i, f),
\]

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completing our proof. □

The next lemma gives some useful facts about residues. Both \( f, g \) are analytic functions on \( D^0(z_0, r) \) for some \( r > 0 \).

**Lemma 0.32.** (i) \( f \) has a simple pole or removable singularity at \( z_0 \) with residue \( \alpha \)

\[
\lim_{z \to z_0} (z - z_0)f(z) = \alpha \iff f(z) - \frac{\alpha}{z - z_0} \text{ is analytic around } z_0.
\]

(ii) Suppose that \( f \) is analytic at \( z_0 \). Let \( k \) be a positive integer. Then \( f/(z - z_0)^k \) has a pole of order at most \( k \) at \( z = z_0 \), and

\[
\text{res}(z_0, f/(z - z_0)^k) = \frac{f^{(k-1)}(z_0)}{(k-1)!}.
\]

(iii) Suppose \( f \) has a pole of order 1 at \( z_0 \) and \( g \) is analytic and non-zero at \( z_0 \). Then \( fg \) has a pole of order 1 at \( z_0 \) and

\[
\text{res}(z_0, fg) = g(z_0)\text{res}(z_0, f).
\]

(iv) Suppose that \( f \) is analytic and non-zero at \( z_0 \) and \( g \) has a zero of order 1 at \( z_0 \). Then \( f/g \) has a pole of order 1 at \( z_0 \), and

\[
\text{res}(z_0, f/g) = \frac{f(z_0)}{g'(z_0)}.
\]

**Proof.** (i) Assume that \( f \) has a simple pole or removable singularity at \( z = z_0 \) with residue \( \alpha \). Then \( f(z) = \frac{\alpha}{z - z_0} + \sum_{n=0}^{\infty} a_k (z - z_0)^k \) on \( D^0(z_0, r) \) and the two other assertions easily follow. Conversely, Suppose that \( f(z) - \frac{\alpha}{z - z_0} \) is analytic around \( z_0 \). Then \( f(z) = \frac{\alpha}{z - z_0} + \sum_{n=0}^{\infty} a_k (z - z_0)^k \) on \( D^0(z_0, r) \) and the first assertion easily follows.

(ii) \( f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \) around \( z_0 \). Divide by \( (z - z_0)^k \).

(iii) We have \( \lim_{z \to z_0} (z - z_0)f(z)g(z) = g(z_0) \lim_{z \to z_0} (z - z_0)f(z) \). Apply (i).

(iv) We have

\[
\lim_{z \to z_0} \frac{(z - z_0)f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}
\]

and by (i) this implies that \( f(z)/g(z) \) has a simple pole at \( z = z_0 \) and \( \text{res}(z_0, f/g) = f(z_0)/g'(z_0) \). □
Example. We compute the residues of \( f(z) = \frac{e^z}{(z-1)^3(z-2)^2} \) at \( z = 1, z = 2 \).

Let \( g_1(z) = \frac{e^z}{(z-2)^2} \) and \( g_2(z) = \frac{e^z}{(z-1)^3} \). Then

\[
g_1''(z) = \frac{(z^2 - 8z + 18)e^z}{(z-2)^4}, \quad g_2'(z) = \frac{(z-4)e^z}{(z-1)^4},
\]

and thus, by (ii), \( \text{res}(1, f) = g_1''(1)/2! = \frac{11}{2}e \), \( \text{res}(2, f) = g_2'(2) = -2e^2 \).

We deduce a useful consequence for integrals of rational functions.

Theorem 0.33. Let \( p, q \) be two polynomials in \( \mathbb{C}[X] \) such that \( \deg q \geq \deg p + 2 \) and \( q \) has no zeros on the real line. Let \( z_1, \ldots, z_m \) be the distinct zeros of \( q \) in the upper half plane. Then

\[
\int_{\infty}^{-\infty} p(x) \cdot \frac{dx}{q(x)} = 2\pi i \sum_{j=1}^{m} \text{res}(z_j, p/q).
\]

Remark. We say that \( \int_{-\infty}^{\infty} \ldots \) converges and define \( \int_{-\infty}^{\infty} \ldots := \lim_{R_1,R_2 \to \infty} \int_{-R_1}^{R_2} \ldots \) provided the limit exists and is finite, where we let \( R_1, R_2 \) tend to \( \infty \) independently of each other. If \( \int_{-\infty}^{\infty} \ldots \) converges then it is equal to \( \lim_{R \to \infty} \int_{-R}^{R} \ldots \). But conversely it may be that \( \lim_{R \to \infty} \int_{-R}^{R} \ldots \) exists and is finite while \( \int_{-\infty}^{\infty} \ldots \) diverges, e.g., \( \lim_{R \to \infty} \int_{-R}^{R} xdx = 0 \), while \( \int_{-\infty}^{\infty} xdx \) is clearly divergent.

Proof of Theorem 0.33. Let \( f(z) := \frac{p(z)}{q(z)} \). We first estimate \(|f(z)|\) from above, for \( z \in \mathbb{C} \). If \(|z|\) is large, in \( p(z) \) and \( q(z) \) the highest powers of \( z \) dominate, which implies that there are \( c_1, c_2 > 0 \) such that

\[
|f(z)| \leq c_1|z|^{|\deg p - \deg q|} \leq c_1|z|^{-2} \quad \text{for } z \in \mathbb{C} \text{ with } |z| \geq c_2.
\]

This estimate implies that the integral under consideration converges absolutely, hence converges, and so it is equal to

\[
\lim_{R \to \infty} \int_{-R}^{R} f(x)dx.
\]

We compute the limit. For \( R > 0 \), let \( \Gamma_R \) be the contour defined by first traversing from \(-R\) to \( R \) along the real line, and then traversing from \( R \) to \(-R\) along the upper
semicircle with center 0 and radius \( R \). For \( R \) sufficiently large, the poles of \( f \) in the interior of \( \Gamma_R \) are precisely \( z_1, \ldots, z_m \), so by the Residue Theorem,

\[
\oint_{\Gamma_R} f(z) \, dz = 2\pi i \sum_{j=1}^{m} \text{res}(z_j, f).
\]

On the other hand, letting \( C_R \) denote the upper semicircle with center 0 and radius \( R \),

\[
\oint_{\Gamma_R} f(z) \, dz = \int_{-R}^{R} f(x) \, dx + \int_{C_R} f(z) \, dz
\]
and, for \( R > c_2 \),

\[
\left| \int_{C_R} f(z) \, dz \right| \leq L(C_R) \cdot \sup_{z \in C_R} |f(z)| \leq \pi R \cdot c_1 R^{-2} \to 0 \quad \text{as } R \to \infty.
\]
This implies our theorem.

\[\square\]

**Example.** We compute \( \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^n} \) for any integer \( n \geq 1 \).

Notice that \( f(z) = (z^2 + 1)^{-n} \) has only one pole in the upper half plane, namely at \( z = i \). By the above theorem, the integral is equal to \( 2\pi i \cdot \text{res}(i, f) \). To compute the residue, observe that \( f(z) = g(z)/(z - i)^n \), where \( g(z) = (z + i)^{-n} \). Hence by Lemma 0.32 (ii),

\[
\text{res}(i, (z^2 + 1)^{-n}) = \frac{g^{(n-1)}(i)}{(n-1)!} = \frac{(-n)(-n-1) \cdots (-n-n+2)}{(n-1)!} (z+i)^{-(n+n+1)}|_{z=i} = \binom{2n-2}{n-1} \cdot (-1)^{n+1}(2i)^{2n+1} = \binom{2n-2}{n-1} 2^{2n+1}i^{-1}.
\]

The value of the integral is \( 2\pi i \) times this quantity, that is,

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^n} = \left( \frac{2n-2}{n-1} \right) 2^{-2n+2} \pi.
\]

Let \( U \) be a non-empty, open subset of \( \mathbb{C} \) and \( f \) a meromorphic function on \( U \) which is not identically zero. We define the *logarithmic derivative* of \( f \) by

\[
\frac{f'}{f}.
\]
Suppose that $U$ is simply connected and $f$ is analytic and has no zeros on $U$. Then $f'/f$ has an anti-derivative $h : U \to \mathbb{C}$. One easily verifies that $(e^h/f)' = 0$. Hence $e^h/f$ is constant on $U$. By adding a suitable constant to $h$ we can achieve that $e^h = f$. That is, we may view $h$ as the logarithm of $f$, and $f'/f$ as the derivative of this logarithm. But we will refer to $f'/f$ as the logarithmic derivative of $f$ also if $U$ is not simply connected and/or $f$ does have zeros or poles on $U$, although in that case it need not be the derivative of some function.

The following facts are easy to prove: if $f, g$ are two meromorphic functions on $U$ that are not identically zero, then

$$
\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}, \quad \frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}.
$$

Further, if $U$ is connected, then

$$
\frac{f'}{f} = \frac{g'}{g} \iff f = cg \text{ for some constant } c.
$$

**Lemma 0.34.** Let $z_0 \in \mathbb{C}$, $r > 0$ and let $f : D^0(z_0, r) \to \mathbb{C}$ be analytic. Assume that $z_0$ is either a removable singularity or a pole of $f$. Then $z_0$ is a simple pole or (if $z_0$ is neither a zero nor a pole of $f$) a removable singularity of $f'/f$, and

$$
\text{res}(z_0, f'/f) = \text{ord}_{z_0}(f).
$$

**Proof.** Let $\text{ord}_{z_0}(f) = k$. This means that $f(z) = (z - z_0)^k g(z)$ with $g$ analytic around $z_0$ and $g(z_0) \neq 0$. Consequently,

$$
\frac{f'}{f} = \frac{k(z - z_0)'}{z - z_0} + \frac{g'}{g} = \frac{k}{z - z_0} + \frac{g'}{g}.
$$

The function $g'/g$ is analytic around $z_0$ since $g(z_0) \neq 0$. So by Lemma 0.32, $\text{res}(z_0, f'/f) = k$. \hfill \Box

**Corollary 0.35.** Let $\gamma$ be a contour in $\mathbb{C}$, $U$ an open subset of $\mathbb{C}$ containing $\gamma$ and its interior, and $f$ a meromorphic function on $U$. Assume that $f$ has no zeros or poles on $\gamma$ and let $z_1, \ldots, z_q$ be the zeros and poles of $f$ inside $\gamma$. Then

$$
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \text{ord}_{z_i}(f) = Z - P,
$$

where $Z, P$ denote the number of zeros and poles of $f$ inside $\gamma$, counted with their multiplicities.
Proof. By Theorem 0.31 and Lemma 0.34 we have
\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \text{res}(z_i, f'/f) = \sum_{i=1}^{q} \text{ord}_{z_i}(f) = Z - P.
\]

0.7.6 Unicity of analytic functions

In this section we show that two analytic functions \(f, g\) defined on a connected open set \(U\) are equal on the whole set \(U\), if they are equal on a sufficiently large subset of \(U\).

We start with the following result.

**Theorem 0.36.** Let \(U\) be a non-empty, open, connected subset of \(\mathbb{C}\), and \(f : U \to \mathbb{C}\) an analytic function. Assume that \(f = 0\) on an infinite subset of \(U\) having a limit point in \(U\). Then \(f = 0\) on \(U\).

*Proof.* Our assumption that \(U\) is connected means, that any non-empty subset \(S\) of \(U\) that is both open and closed in \(U\), must be equal to \(U\).

Let \(Z\) be the set of \(z \in U\) with \(f(z) = 0\). Let \(S\) be the set of \(z \in U\) such that \(z\) is a limit point of \(Z\). By assumption, \(S\) is non-empty. Since \(f\) is continuous, we have \(S \subseteq Z\). Any limit point in \(U\) of \(S\) is therefore a limit point of \(Z\) and so it belongs to \(S\). Hence \(S\) is closed in \(U\). We show that \(S\) is also open; then it follows that \(S = U\) and we are done.

Pick \(z_0 \in S\). We have to show that there is \(\delta > 0\) such that \(D(z_0, \delta) \subset S\). There is \(\delta > 0\) such that \(f\) has a Taylor expansion
\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]
converging on \(D(z_0, \delta)\). Assume that \(f\) is not identically 0 on \(D(z_0, \delta)\). Then not all coefficients \(a_n\) are 0. Assume that \(a_m \neq 0\) and \(a_n = 0\) for \(n < m\), say. Then \(f(z) = (z - z_0)^m h(z)\) with \(h(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m}\). Since \(h(z_0) = a_m \neq 0\) and \(h\) is continuous, there is \(\delta_1 > 0\) such that \(h(z) \neq 0\) for all \(z \in D(z_0, \delta_1)\). But then \(f(z) \neq 0\) for all \(z\) with \(0 < |z - z_0| < \delta_1\), contradicting that \(z_0 \in S\).
Hence $f$ is identically 0 on $D(z_0, \delta)$. Clearly, every point of $D(z_0, \delta)$ is a limit point of $D(z_0, \delta)$, hence of $Z$. So $D(z_0, \delta) \subset S$. This shows that indeed, $S$ is open.

**Corollary 0.37.** Let $U$ be a non-empty, open, connected subset of $\mathbb{C}$, and let $f : U \rightarrow \mathbb{C}$ be an analytic function that is not identically 0 on $U$. Then the set of zeros of $f$ in $U$ is discrete in $U$, i.e., every compact subset of $U$ contains only finitely many zeros of $f$.

**Proof.** Suppose that some compact subset of $U$ contains infinitely many zeros of $f$. Then by the Bolzano-Weierstrass Theorem, the set of these zeros would have a limit point in this compact set, implying that $f = 0$ on $U$. \qed

**Corollary 0.38.** Let $U$ be a non-empty, open, connected subset of $\mathbb{C}$, and $f, g : U \rightarrow \mathbb{C}$ two analytic functions. Assume that $f = g$ on an infinite subset of $U$ having a limit point in $U$. Then $f = g$ on $U$.

**Proof.** Apply Theorem 0.36 to $f - g$. \qed

Let $U, V$ be open subsets of $\mathbb{C}$ with $U \subset V$. Let $f : U \rightarrow \mathbb{C}$ be an analytic function. An analytic continuation of $f$ to $V$ is an analytic function $g : V \rightarrow \mathbb{C}$ such that $g(z) = f(z)$ for $z \in U$. For instance, the function $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic on $\{z \in \mathbb{C} : |z| < 1\}$. It has an analytic continuation $\frac{1}{1-z}$ to $\mathbb{C} \setminus \{1\}$.

It is often a difficult problem to figure out whether such an analytic continuation exists. The next corollary shows that if it exists, it must be unique.

**Corollary 0.39.** Let $U, V$ be non-empty, open subsets of $\mathbb{C}$, such that $U \subset V$ and $V$ is connected. Let $f : U \rightarrow \mathbb{C}$ be an analytic function. Then $f$ has at most one analytic continuation to $V$.

**Proof.** Let $g_1, g_2$ be two analytic continuations of $f$ to $U'$. Then $g_1(z) = g_2(z) = f(z)$ for $z \in U$. Since $U$ is open, every point in $U$ is a limit point of $U$, hence of $U'$. Therefore, $g_1(z) = g_2(z)$ for $z \in U'$. \qed

The next corollary states that analytic continuations of a function $f$ to different sets can be glued together to a single continuation to the union of these sets.
**Corollary 0.40.** Let $U$ be a non-empty open subset of $\mathbb{C}$, and $\{V_i\}_{i \in I}$ with $I$ any index set a collection of connected open subsets of $\mathbb{C}$ each of which contains $U$, and such that $V_i \cap V_j$ is connected for any two $i, j \in I$. Let $f$ be an analytic function on $U$, and $g_i$ an analytic continuation of $f$ to $V_i$, for $i \in I$. Then $g_i = g_j$ holds on $V_i \cap V_j$ for any $i, j \in I$, and $f$ has a unique analytic continuation to $\bigcup_{i \in I} V_i$, which coincides with $g_i$ on $V_i$, for $i \in I$.

**Proof.** If $i, j$ are any two indices from $I$, then both $g_i, g_j$ are analytic continuations of $f$ to $V_i \cap V_j$, hence must be equal on $V_i \cap V_j$, since $V_i \cap V_j$ is assumed to be connected. Now define a function $g$ on $V := \bigcup_{i \in I} V_i$ by $g(z) := g_i(z)$ if $z \in V_i$. If $i, j$ are any two indices such that $z \in V_i$ and $z \in V_j$, then $g_i(z) = g_j(z)$, so this is well-defined. Further, $g$ clearly coincides with $f$ on $U$, and is analytic on $V$. 

Another consequence of Theorem 0.36 is the so-called Schwarz’ reflection principle, which implies that analytic functions assuming real values on the real line have nice symmetric properties.

**Corollary 0.41 (Schwarz’ reflection principle).**

Let $U$ be an open, connected subset of $\mathbb{C}$, such that $U \cap \mathbb{R} \neq \emptyset$ and such that $U$ is symmetric about $\mathbb{R}$, i.e., $\overline{z} \in U$ for every $z \in U$. Further, let $f : U \to \mathbb{C}$ be a non-identically zero analytic function with the property that

$$\{z \in U \cap \mathbb{R} : f(z) \in \mathbb{R}\}$$

has a limit point in $U$.

Then $f$ has the following properties:

(i) $f(z) \in \mathbb{R}$ for $z \in U \cap \mathbb{R}$;
(ii) $f(\overline{z}) = f(z)$ for $z \in U$;
(iii) If $z_0$ and $r > 0$ are such that $D^0(z_0, r) \subset U$, then $\operatorname{ord}_{z_0}(f) = \operatorname{ord}_{\overline{z}_0}(f)$.

**Proof.** We first show that the function $z \mapsto \overline{f(z)}$ is analytic on $U$. Indeed, for $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \left( \frac{f(z) - f(z_0)}{\overline{z} - \overline{z_0}} \right) = f'(\overline{z_0})$$
exists.

Notice that for every $z \in U \cap \mathbb{R}$ with $f(z) \in \mathbb{R}$, we have $\overline{f(z)} = f(z)$. So by our assumption on $f$, the set of $z \in U$ with $\overline{f(z)} = f(z)$ has a limit point in $U$. Now Corollary 0.38 implies that $\overline{f(z)} = f(z)$ for $z \in U$. This implies (i) and (ii).

We finish with proving (iii). Our assumption implies that $f$ has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converging on $D^0(z_0, r)$. Then for $z \in D^0(z_0, r)$ we have $\overline{z} \in D^0(z_0, r)$ and

$$f(z) = \overline{f(\overline{z})} = \left( \sum_{n=-\infty}^{\infty} a_n (\overline{z} - z_0)^n \right) = \sum_{n=-\infty}^{\infty} \overline{a_n} (z - \overline{z_0})^n,$$

which clearly implies (iii).

\[\square\]

### 0.7.7 Analytic functions defined by integrals

In analytic number theory, one often has to deal with complex functions that are defined by infinite series, infinite products, infinite integrals, or even worse, infinite integrals of infinite series. In this section we have collected some useful results that allow us to verify in a not too difficult manner that such complicated functions are analytic. Although all results we mention are well-known, we could not find a convenient reference for them, therefore we have included their not too exciting proofs.

We start with a general theorem on analytic functions defined by an integral, which will be frequently used. In practical applications, condition (i) will always be taken for granted and only (ii) and (iii) will be verified.

**Theorem 0.42.** Let $D$ be a measurable subset of $\mathbb{R}^m$, $U$ an open subset of $\mathbb{C}$ and $f : D \times U \to \mathbb{C}$ a function with the following properties:

(i) $f$ is measurable on $D \times U$ (with $U$ viewed as subset of $\mathbb{R}^2$);

(ii) for every fixed $x \in D$, the function $z \mapsto f(x, z)$ is analytic on $U$;

(iii) for every compact subset $K$ of $U$ there is a measurable function $M_K : D \to \mathbb{R}$ such that

$$|f(x, z)| \leq M_K(x) \text{ for } x \in D, z \in K, \quad \int_D M_K(x) dx < \infty.$$
Then the function $F$ given by

$$F(z) := \int_D f(x, z)dx$$

is analytic on $U$, and for every $k \geq 1$,

$$F^{(k)}(z) = \int_D f^{(k)}(x, z)dx,$$

where $f^{(k)}(x, z)$ denotes the $k$-th derivative with respect to $z$ of the analytic function $z \mapsto f(x, z)$.

Proof. Fix $z \in U$. Choose $r > 0$ such that $D(z, r) \subset U$, and let $0 < \delta < \frac{1}{2}r$. We show that $F$ can be expanded into a Taylor series around $z$ on $D(z, \delta)$; then it follows that $F$ is analytic on $D(z, \delta)$ and so in particular in $z$. By assumption, there is a measurable function $M : D \to \mathbb{R}$ such that $|f(x, w)| \leq M(x)$ for $x \in D$, $w \in D(z, r)$ and $\int_D M(x)dx < \infty$.

Let $w \in D(z, \delta)$. Then by Cauchy’s integral formula (i.e., Corollary 0.22),

$$F(w) = \int_D f(x, w)dx = \int_D \left\{ \frac{1}{2\pi i} \int_{\gamma_{z, 2\delta}} \frac{f(x, \zeta)}{\zeta - w} \cdot d\zeta \right\} dx.$$ 

By inserting

$$\frac{f(x, \zeta)}{\zeta - w} = \frac{f(x, \zeta)}{(\zeta - z) - (w - z)} = \frac{f(x, \zeta)}{\zeta - z} \left(1 - \frac{w - z}{\zeta - z}\right)^{-1} = \sum_{n=0}^{\infty} f(x, \zeta) (\zeta - z)^n \cdot (w - z)^n$$

we obtain

$$F(w) = \int_D \left\{ \frac{1}{2\pi i} \int_{\gamma_{z, 2\delta}} \left(\sum_{n=0}^{\infty} f(x, \zeta) (\zeta - z)^n\right) d\zeta \right\} dx$$

$$= \int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} \frac{f(x, z + 2\delta e^{2\pi it})}{(2\delta e^{2\pi it})^n} (w - z)^n\right) dt \right\} dx.$$ 

We want to interchange the summation with the two integrations, and require the Fubini-Tonelli Theorem to show that this is possible. We have to check that the
conditions of that theorem are satisfied, i.e., that in the above expression for \( F(w) \) we have absolute convergence. Note that since \( |w - z| < \delta \) we have

\[
\int_D \left\{ \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f(x, z + 2\delta e^{2\pi it})}{(2\delta e^{2\pi it})^n} (w - z)^n \right) dt \right\} dx
\]

\[
\leq \int_D \left\{ \int_0^1 \left( \sum_{n=0}^{\infty} M(x)2^{-n} \right) dt \right\} dx \leq \int_D 2M(x)dx < \infty,
\]

which shows that indeed, the conditions of the Fubini-Tonelli Theorem are satisfied. So in the expression for \( F(w) \) derived above we can indeed interchange the summation and the two integrations and thus obtain

\[
F(w) = \sum_{n=0}^{\infty} (w - z)^n \left( \int_D \left\{ \int_0^1 \frac{f(x, z + 2\delta e^{2\pi it})}{(2\delta e^{2\pi it})^n} dt \right\} dx \right)
\]

\[
= \sum_{n=0}^{\infty} (w - z)^n \left( \int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma, z} \frac{f(x, \zeta)}{(\zeta - z)^{n+1}} d\zeta \right\} dx \right)
\]

\[
= \sum_{n=0}^{\infty} (w - z)^n \left( \int_D \frac{f^{(n)}(x, z)}{n!} \cdot dx \right),
\]

where in the last step we have applied Corollary 0.27. This shows that indeed, \( F \) has a Taylor expansion around \( z \) converging on \( D(z, \delta) \). So in particular, \( F \) is analytic in \( z \). Further, \( F^{(k)}(z) \) is equal to \( k! \) times the coefficient of \( (w - z)^k \), that is, \( \int_D f^{(k)}(x, z)dx \). This proves our Theorem.

We deduce a result, which states that under certain conditions, the pointwise limit of a sequence of analytic functions is again analytic.

**Theorem 0.43.** Let \( U \subset \mathbb{C} \) be a non-empty open set, and \( \{f_n : U \to \mathbb{C}\}_{n=0}^{\infty} \) a sequence of analytic functions, converging pointwise to a function \( f \) on \( U \). Assume that for every compact subset \( K \) of \( U \) there is a constant \( C_K < \infty \) such that

\[
|f_n(z)| \leq C_K \quad \text{for all } z \in K, \ n \geq 0.
\]

Then \( f \) is analytic on \( U \), and \( f_n^{(k)} \to f^{(k)} \) pointwise on \( U \) for all \( k \geq 1 \).
Proof. The set $U$ can be covered by disks $D(z_0, \delta)$ with $z_0 \in U$, $\delta > 0$, such that the closed disk with center $z_0$ and radius $2\delta$, $\overline{D}(z_0, 2\delta)$ is contained in $U$. We fix such a disk $D(z_0, \delta)$ and prove that $f$ is analytic on $D(z_0, \delta)$ and $f_n^{(k)} \to f^{(k)}$ pointwise on $D(z_0, \delta)$ for $k \geq 1$. This clearly suffices.

Let $z \in D(z_0, \delta)$, $k \geq 0$. Then by Corollary 0.27, we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma_{z_0,2\delta}} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}} \cdot d\zeta$$

$$= \int_0^1 k! \cdot \frac{f_n(z_0 + 2\delta e^{2\pi it})2\delta e^{2\pi it}}{(z_0 + 2\delta e^{2\pi it} - z)^{k+1}} \cdot dt = \int_0^1 g_{n,k}(t,z)dt,$$

say. By assumption, there is $C < \infty$ such that $|f_n(w)| \leq C$ for $w \in \overline{D}(z_0, 2\delta)$, $n \geq 0$. Further, for $t \in [0, 1]$ we have $|z_0 + 2\delta e^{2\pi it} - z| > \delta$. Hence

$$(0.6) \quad |g_{n,k}(t,z)| \leq C \cdot k! \cdot 2\delta/\delta^{k+1} = 2C \cdot k!\delta^{-k} \quad \text{for } n, k \geq 0.$$ 

Notice that for $k \geq 0$, $t \in [0, 1]$, $z \in D(z_0, \delta)$ we have

$$\lim_{n \to \infty} g_{n,k}(t,z) = k! \cdot \frac{f(z_0 + 2\delta e^{2\pi it})2\delta e^{2\pi it}}{(z_0 + 2\delta e^{2\pi it} - z)^{k+1}} = g^{(k)}(t,z),$$

where

$$g(t,z) := \frac{f(z_0 + 2\delta e^{2\pi it})2\delta e^{2\pi it}}{z_0 + 2\delta e^{2\pi it} - z}$$

and $g^{(k)}(t,z)$ is the $k$-th derivative of the analytic function in $z$, $z \mapsto g(t,z)$.

Thanks to (0.6) we can apply the dominated convergence theorem, and obtain

$$\lim_{n \to \infty} f_n^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt \quad \text{for } z \in D(z_0, \delta), k \geq 0.$$ 

Applying this with $k = 0$ and using $f_n \to f$ pointwise, we obtain

$$f(z) = \int_0^1 g(t,z)dt \quad \text{for } z \in D(z_0, \delta).$$

It follows from Theorem 0.42 that the right-hand side, and hence $f$, is analytic on $D(z_0, \delta)$, and moreover,

$$f^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt \quad \text{for } z \in D(z_0, \delta), k \geq 1.$$
Indeed, $g(t, z)$ is measurable on $[0, 1] \times D(z_0, \delta)$ and for every fixed $t$, the function $z \mapsto g(t, z)$ is analytic on $D(z_0, \delta)$. Further, by (0.6) and since $g_{n,0}(t, z) \rightarrow g(t, z)$, we have $|g(t, z)| \leq 2C$ for $t \in [0, 1], z \in D(z_0, \delta)$. So all conditions of Theorem 0.42 are satisfied.

Now it follows that

$$\lim_{n \to \infty} f_n^{(k)}(z) = \int_0^1 g^{(k)}(t, z)dt = f^{(k)}(z) \quad \text{for} \quad z \in D(z_0, \delta), \quad k \geq 1,$$

which is what we wanted to prove.

**Corollary 0.44.** Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^\infty$ a sequence of analytic functions, converging to a function $f$ pointwise on $U$, and uniformly on every compact subset of $U$. Then $f$ is analytic on $U$ and $f_n^{(k)} \to f^{(k)}$ pointwise on $U$ for every $k \geq 1$.

**Proof.** Take a compact subset $K$ of $U$. Let $\varepsilon > 0$. Then there is $N$ such that $|f_n(z) - f_m(z)| < \varepsilon$ for all $z \in K$, $m, n \geq N$. Choose $m \geq N$. Then there is $C > 0$ such that $|f_m(z)| \leq C$ for $z \in K$ since $f_m$ is continuous. Hence $|f_n(z)| \leq C + \varepsilon$ for $z \in K, n \geq N$. Now our Corollary follows at once from Theorem 0.43.

**Corollary 0.45.** Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^\infty$ a sequence of analytic functions, converging to a function $f$ pointwise on $U$ and uniformly on every compact subset of $U$. Then

$$\lim_{n \to \infty} \frac{f_n'(z)}{f_n(z)} = \frac{f'(z)}{f(z)}$$

for all $z \in U$ with $f(z) \neq 0$, where the limit is taken over those $n$ for which $f_n(z) \neq 0$.

**Proof.** Obvious.

**Corollary 0.46.** Let $U \subset \mathbb{C}$ be a non-empty open set and $\{f_n : U \to \mathbb{C}\}_{n=0}^\infty$ a sequence of analytic functions. Assume that for every compact subset $K$ of $U$ there are reals $M_{n,K}$ such that $|f_n(z)| \leq M_{n,K}$ for $z \in K$ and $\sum_{n=0}^{\infty} M_{n,K}$ converges. Then

(i) $\sum_{n=0}^{\infty} f_n$ is analytic on $U$, and $(\sum_{n=0}^{\infty} f_n)^{(k)} = \sum_{n=0}^{\infty} f_n^{(k)}$ for $k \geq 0$,

(ii) $\prod_{n=0}^{\infty}(1 + f_n)$ is analytic on $U$.
Proof. Our assumption on the functions $f_n$ implies that both the series $\sum_{n=0}^{\infty} f_n$ and the infinite product $\prod_{n=0}^{\infty} (1 + f_n)$ converge uniformly on every compact subset of $U$ (see Propositions 0.5 and 0.6). Now apply Corollary 0.44.

**Corollary 0.47.** Let $U$, $\{f_n\}_{n=0}^{\infty}$ be as in Corollary 0.46 and assume in addition that $f_n \neq -1$ on $U$ for every $n \geq 0$. Then for the function $F = \prod_{n=0}^{\infty} (1 + f_n)$ we have

$$\frac{F'}{F} = \sum_{n=0}^{\infty} \frac{f'_n}{1 + f_n}.$$ 

Proof. Let $F_m := \prod_{n=0}^{m} (1 + f_n)$. Then $F_m \to F$ uniformly on every compact subset of $U$. Hence by Corollary 0.45,

$$\frac{F'}{F} = \lim_{m \to \infty} \frac{F'_m}{F_m} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{f'_n}{1 + f_n}$$

which clearly implies Corollary 0.47. 

\[ \Box \]