ANALYTIC NUMBER THEORY (MASTERMATH)

PART I: PRIME NUMBER THEORY

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Literature

Below is a list of recommended additional literature. Much of the material of part I of this course has been taken from the books of Jameson and that of Davenport on multiplicative number theory.

A. Cojocaru and M. R. Murty, *An Introduction to Sieve Methods and Their Applications*, Cambridge University Press; 1st edition, January 30, 2006.

H. Davenport, *Multiplicative Number Theory (2nd ed.)*, Springer Verlag, Graduate Texts in Mathematics 74, 1980.

H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Cambridge University Press, 1963, reissued in 2005 in the Cambridge Mathematical Library series.

J. Friedlander, H. Iwaniec, *Opera de Cribo,* American Mathematical Society Colloquium Publications 57, American Mathematical Society, 2010.

A. Granville, *What is the best approach to counting primes*, arXiv:1406.3754 [math.NT].

A.E. Ingham, *The distribution of prime numbers,* Cambridge University Press, 1932 (reissued in 1990).

H. Iwaniec, E. Kowalski, *Analytic Number Theory,* American Mathematical Society Colloquium Publications 53, American Mathematical Society, 2004.

G.J.O. Jameson, *The Prime Number Theorem,* London Mathematical Society, Student Texts 53, Cambridde University Press, 2003.

S. Lang, *Algebraic Number Theory,* Addison-Wesley, 1970. S. Lang, *Complex Analysis (4th. ed.),* Springer Verlag, Graduate Texts in Mathematics 103, 1999.

H.L. Montgomery, R.C. Vaughan, *Multiplicative Number Theory I. Classical Theory,* Cambridge studies in advanced mathematics 97, Cambridge University Press 2007.

D.J. Newman, *Analytic Number Theory,* Springer Verlag, Graduate Texts in Mathematics 177, 1998.

P. Pollack, *Not Always Buried Deep,* American Mathematical Society; New ed. edition (October 14, 2009)

E.C. Titchmarsh, *The theory of the Riemann zeta function (2nd. ed., revised by D.R. Heath-Brown),* Oxford Science Publications, Clarendon Press Oxford, 1986.

R.C. Vaughan, *The Hardy-Littlewood method (2nd ed.)*, Cambridge University Press, 1997.

Notation

- $\limsup_{n\to\infty} x_n$ or $\overline{\lim}_{n\to\infty} x_n$

  For a sequence of reals $\{x_n\}$ we define $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \big(\sup_{m\geqslant n} x_m\big)$. We have $\limsup_{n\to\infty} x_n = \infty$ if and only if the sequence $\{x_n\}$ is not bounded from above, i.e., if for every $A > 0$ there is n with $x_n > A$.
  In case that the sequence $\{x_n\}$ is bounded from above, we have $\limsup_{n\to\infty} x_n = \alpha$ where α is the largest limit point ('limes superior') of the sequence $\{x_n\}$, in other words, for every $\varepsilon > 0$ there are infinitely many n such that $x_n \geqslant \alpha - \varepsilon$, while there are only finitely many n such that $x_n \geqslant \alpha + \varepsilon$.

- $\liminf_{n\to\infty} x_n$ or $\underline{\lim}_{n\to\infty} x_n$

  For a sequence of reals $\{x_n\}$ we define $\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \big(\inf_{m\geqslant n} x_m\big)$. We have $\liminf_{n\to\infty} x_n = -\infty$ if the sequence $\{x_n\}$ is not bounded from below, and the smallest limit point ('limes inferior') of the sequence $\{x_n\}$ otherwise.

- $f(x) = g(x) + o(e(x))$ as $x \to \infty$ (for functions $f, g : \mathcal{S} \to \mathbb{C}$ with $\mathcal{S}$ any subset of $\mathbb{R}$ containing arbitrary large reals and $e : \mathcal{S} \to \mathbb{R}_{\geqslant 0}$)

  $\lim_{x\to\infty} \dfrac{f(x) - g(x)}{e(x)} = 0$, i.e., $f(x) - g(x)$ is of smaller order of magnitude than $e(x)$.

  Examples: $f(x) = g(x) + o(1)$ as $x \to \infty$ means that $\lim_{x\to\infty}(f(x) - g(x)) = 0$; $\log x = o(x^{\varepsilon})$ as $x \to \infty$ for every $\varepsilon > 0$ since $\lim_{x\to\infty}(\log x)/x^{\varepsilon} = 0$ for every $\varepsilon > 0$.

- $f(x) = g(x) + O(e(x))$ as $x \to \infty$ (with f, g, e as above)

  There are constants $x_0 > 0, C > 0$ such that $|f(x) - g(x)| \leqslant Ce(x)$ for all $x \geqslant x_0$, i.e., $f(x) - g(x)$ is of order of magnitude at most $e(x)$.

  We call $g(x) + O(e(x))$ as $x \to \infty$ an asymptotic formula for $f(x)$, with main term $g(x)$ and error term $O(e(x))$. Of course, such an asymptotic formula is

interesting only if the error term is of smaller order of magnitude than the main term, i.e., $e(x) = o(|g(x)|)$ as $x \to \infty$. If $g(x)$ is of order of magnitude at most $e(x)$, i.e., $g(x) = O(e(x))$ as $x \to \infty$, we can just as well write $f(x) = O(e(x))$ as $x \to \infty$.
Likewise, if $f(x) = g(x) + o(e(x))$ as $x \to \infty$, we call $g(x)$ the main term and $o(e(x))$ the error term.

Examples:
$f(x) = g(x) + O(1)$ as $x \to \infty$ means that $|f(x) - g(x)|$ is bounded;

$\log(1 + x^{-1}) = x^{-1} + O(x^{-2})$ as $x \to \infty$ (from the expansion $\log(1 + x^{-1}) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{-n}/n$ for $|x| > 1$);

$(1+x^{-1})^{\alpha} = 1+\alpha x^{-1}+O(x^{-2})$ as $x \to \infty$ for every $\alpha \in \mathbb{R}$ (from the expansion $(1 + x^{-1})^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^{-n}$ for $|x| > 1$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$);

$e^{1/x} = 1 + x^{-1} + O(x^{-2})$ as $x \to \infty$ (from the expansion $e^{1/x} = \sum_{n=0}^{\infty} x^{-n}/n!$).

- $f(x) \sim g(x)$ as $x \to \infty$ (with f, g as above)

  $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

- $f(x) \ll g(x)$, $g(x) \gg f(x)$ as $x \to \infty$ (with f, g as above)
  (Vinogradov symbols; used only if $g(x) > 0$ for all sufficiently large x, i.e., there is x_0 such that $g(x) > 0$ for all $x \geqslant x_0$).

  $f(x) = O(g(x))$ as $x \to \infty$, that is, there are constants $x_0 > 0, C > 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \geqslant x_0$.

- $f(x) \asymp g(x)$ as $x \to \infty$ (with f, g as above, used only if $f(x) > 0$, $g(x) > 0$ for all sufficiently large x)

  there are constants $x_0, C_1, C_2 > 0$ such that $C_1 f(x) \leqslant g(x) \leqslant C_2 f(x)$ for all $x \geqslant x_0$.

- $f(x) = \Omega(g(x))$ as $x \to \infty$ (with f, g as above, defined only if $g(x) > 0$ for $x \geqslant x_0$ for some $x_0 > 0$)

  $\limsup_{x\to\infty} \dfrac{|f(x)|}{g(x)} > 0$, that is, there is a sequence $\{x_n\}$ with $x_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \dfrac{|f(x_n)|}{g(x_n)} > 0$ (possibly ∞).

- $f(x) = \Omega^{\pm}(g(x))$ as $x \to \infty$ (with f, g as above, defined only if $g(x) > 0$ for $x \geqslant x_0$ for some $x_0 > 0$)

  $\limsup_{x\to\infty} \dfrac{f(x)}{g(x)} > 0$, $\liminf_{x\to\infty} \dfrac{f(x)}{g(x)} < 0$, that is, there are sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to \infty$, $y_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \dfrac{f(x_n)}{g(x_n)} > 0$ (possibly ∞) and $\lim_{n\to\infty} \dfrac{f(y_n)}{g(y_n)} < 0$ (possibly $-\infty$)

- $f(x) = g(x) + O(e(x))$ for functions $f, g : \mathcal{S} \to \mathbb{C}$ (with $\mathcal{S}$ any infinite set, not necessarily contained in the reals and $e : \mathcal{S} \to \mathbb{R}_{\geqslant 0}$; we drop here $x \to \infty$)

  There is $C > 0$ such that $|f(x) - g(x)| \leqslant Ce(x)$ for all $x \in \mathcal{S}$.

- γ (Euler-Mascheroni constant)

  $\lim_{N\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N\right) = 0.5772156649...$

- $|\mathcal{A}|$

  Cardinality of a set $\mathcal{A}$.

- $\sum_{n\leqslant x} \cdots, \quad \sum_{p\leqslant x} \cdots, \quad \sum_{d|n} \cdots, \quad \sum_{p|n} \cdots$

  Summations over all positive integers $\leqslant x$, all primes $\leqslant x$, all positive divisors of n (including n itself), all primes dividing n; there is a similar notation for products $\prod_{\cdots}$. In general, in summations or products, n will be used to denote a positive integer, p to denote a prime, and d to denote a positive divisor of a given integer.

- $\sum_p \cdots, \quad \prod_p \cdots$

  Infinite sum, infinite product over all primes.

- $\pi(x)$

  Number of primes $\leqslant x$.

- $\theta(x), \quad \psi(x)$

  $\sum_{p \leqslant x} \log p$, $\sum_{p^k \leqslant x} \log p$, where the summations are over all primes $\leqslant x$, respectively all prime powers $\leqslant x$.

- $\pi(x; q, a)$

  Number of primes p with $p \equiv a \pmod q$ and $p \leqslant x$; here q is any integer $\geqslant 2$ and a is any integer coprime with q.

- $\theta(x; q, a), \quad \psi(x; q, a)$

  $\sum_{p \leqslant x, p \equiv a \,(\mathrm{mod}\, q)} \log p$, $\sum_{p^k \leqslant x, p^k \equiv a \,(\mathrm{mod}\, q)} \log p$, where the summations are over all primes $\leqslant x$ that are congruent to a modulo q, respectively all prime powers $\leqslant x$ that are congruent to a modulo q.

- $\mathrm{Li}(x)$

  $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$; this is a good approximation for $\pi(x)$.

- $\Lambda(n)$

  Von Mangoldt function; it is given by $\Lambda(n) = \log p$ if $n = p^k$ for some prime p and exponent $k \geqslant 1$, and $\Lambda(n) = 0$ if $n = 1$ or n is not a prime power; it should be verified that $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$, where the summation is over all positive integers $n \leqslant x$.

- $\varphi(n)$

  Euler's totient function, given by
  $\varphi(n) := |\{a \in \mathbb{Z} :\ 1 \leqslant a < n,\ \gcd(a, n) = 1\}|$.

- $\mu(n)$

  Möbius function, given by $\mu(1) = 1$, $\mu(n) = (-1)^t$ if n is a product $p_1 \cdots p_t$ of distinct primes, and $\mu(n) = 0$ if n is not square-free, i.e., divisible by p^2 for some prime number p.

- $\omega(n), \quad \Omega(n)$

  number of primes dividing n, number of prime powers dividing n, i.e., if $n = p_1^{k_1} \cdots p_t^{k_t}$ with $p_1, \ldots, p_t$ distinct primes and $k_1, \ldots, k_t$ positive integers, then $\omega(n) = t$ and $\Omega(n) = k_1 + \cdots + k_t$; in particular, $\omega(1) = \Omega(1) = 0$.

- $E(n)$

  $E(n) = 1$ for every positive integer n.

- $e(n)$

  $e(1) = 1$ and $e(n) = 0$ for all integers $n > 1$.

- $\tau(n)$ (or $\sigma_0(n)$)

  number of positive divisors of n, including n itself, i.e., $\sum_{d|n} 1$, for instance $\tau(6) = 4$, since $1, 2, 3, 6$ are the divisors of 6.

- $\sigma(n)$ (or $\sigma_1(n)$)

  sum of the positive divisors of n including n itself, i.e., $\sum_{d|n} d$, for instance $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

- $\sigma_\alpha(n)$

  $\sum_{d|n} d^\alpha$

Chapter 0

Prerequisites

We have collected some facts from algebra and analysis which we will not discuss during our course, which will not be a subject of the examination, but which will be used frequently in the course and the exercises. Students are expected to be familiar with the definitions and results in these prerequisites so that we can use them in our course without much explanation.

We need only a little bit of algebra, basically elementary group theory. As for analysis, most of the facts we mention are covered by standard courses on analysis, Lebesgue integration and complex analysis, with the exception maybe of subsections 0.2.1, 0.2.2, 0.6.6, 0.6.7.

In some cases we have provided proofs, either since they may help to gain some confidence with the material, or since we couldn't find a good reference for them. These proofs will not be used in our course, nor will they be examined.

Apart from what is mentioned in these prerequisites, nothing else from Lebesgue integration theory or complex analysis is used, so also students who did not follow courses on these topics should be able to follow our course after having read these prerequisites.

0.1 Groups

Literature:
P. Stevenhagen: *Collegedictaat Algebra 1 (Dutch)*, Universiteit Leiden.
S. Lang: *Algebra, 2nd ed.*, Addison-Wesley, 1984.

0.1.1 Definition

A group is a set G, together with an operation $\cdot : G \times G \to G$ satisfying the following axioms:

- $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$;
- there is $e_G \in G$ such that $g \cdot e_G = e_G \cdot g = g$ for all $g \in G$;
- for all $g \in G$ there is $h \in G$ with $g \cdot h = h \cdot g = e_G$.

From these axioms it follows that the unit element e_G is uniquely determined, and that the inverse h defined by the last axiom is uniquely determined; henceforth we write g^{-1} for this h.

If moreover, $g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g_1, g_2 \in G$, we say that the group G is *abelian* or *commutative*.

Remark. For $n \in \mathbb{Z}_{>0}$, $g \in G$ we write g^n for g multiplied with itself n times. Further, $g^0 := e_G$ and $g^n := (g^{-1})^{|n|}$ for $n \in \mathbb{Z}_{<0}$. This is well-defined by the associative axiom, and we have $(g^m)(g^n) = g^{m+n}$, $(g^m)^n = g^{mn}$ for $m, n \in \mathbb{Z}$.

0.1.2 Subgroups

Let G be a group with group operation $\cdot$. A subgroup of G is a subset H of G that is a group with the group operation of G. This means that $g_1 \cdot g_2 \in H$ for all $g_1, g_2 \in H$; $e_G \in H$; and $g^{-1} \in H$ for all $g \in H$. It is easy to see that H is a subgroup of G if and only if $g_1 \cdot g_2^{-1} \in H$ for all $g_1, g_2 \in H$. We write $H \leqslant G$ if H is a subgroup of G.

0.1.3 Cosets, order, index

Let G be a group and H a subgroup of G. The left cosets of G with respect to H are the sets $gH = \{g \cdot h : h \in H\}$. Two left cosets g_1H, g_2H are equal if and only if $g_1^{-1}g_2 \in H$ and otherwise disjoint.

The right cosets of G with respect to H are the sets $Hg = \{h \cdot g : h \in H\}$. Two right cosets Hg_1, Hg_2 are equal if and only if $g_2g_1^{-1} \in H$ and otherwise disjoint.

There is a one-to-one correspondence between the left cosets and right cosets of G with respect to H, given by $gH \leftrightarrow Hg^{-1}$. Thus, the collection of left cosets has the same cardinality as the collection of right cosets. This cardinality is called the *index* of H in G, notation $(G : H)$.

The order of a group G is its cardinality, notation $|G|$. Assume that $|G|$ is finite. Let again H be a subgroup of G. Since the left cosets w.r.t. H are pairwise disjoint and have the same number of elements as H, and likewise for right cosets, we have

$$(G : H) = \frac{|G|}{|H|}.$$

An important consequence of this is, that $|H|$ divides $|G|$.

0.1.4 Normal subgroup, factor group

Let G be a group, and H a subgroup of G. We call H a *normal subgroup* of G if $gH = Hg$, that is, if $gHg^{-1} = H$ for every $g \in G$.

Let H be a normal subgroup of G. Then the cosets of G with respect to H form a group with group operation $(g_1H) \cdot (g_2H) = (g_1g_2) \cdot H$. This operation is well-defined. We denote this group by G/H; it is called the *factor group* of G with respect to H. Notice that the unit element of G/H is $e_GH = H$. If G is finite, we have $|G/H| = (G : H) = |G|/|H|$.

0.1.5 Order of an element

Let G be a group, and $g \in G$. The *order* of g, notation $\mathrm{ord}(g)$, is the smallest positive integer n such that $g^n = e_G$; if such an integer n does not exist we say that g has infinite order.

We recall some properties of orders of group elements. Suppose that $g \in G$ has finite order n.

- $g^a = g^b \iff a \equiv b \pmod{n}$.
- Let $k \in \mathbb{Z}$. Then $\mathrm{ord}(g^k) = n/\gcd(k, n)$.
- $\{e_G, g, g^2, \ldots, g^{n-1}\}$ is a subgroup of G of cardinality $n = \mathrm{ord}(g)$. Hence if G is finite, then $\mathrm{ord}(g)$ divides $|G|$. Consequently, $g^{|G|} = e_G$.

Example. Let q be a positive integer. A prime residue class modulo q is a residue class of the type $a \bmod q$, where $\gcd(a, q) = 1$. The prime residue classes form a group under multiplication, which is denoted by $(\mathbb{Z}/q\mathbb{Z})^*$. The unit element of this group is $1 \bmod q$, and the order of this group is $\varphi(q)$, that is the number of positive integers $\leqslant q$ that are coprime with q. It follows that if $\gcd(a, q) = 1$, then $a^{\varphi(q)} \equiv 1 \pmod{q}$.

0.1.6 Cyclic groups

The cyclic group generated by g, denoted by $\langle g \rangle$, is given by $\{g^k : k \in \mathbb{Z}\}$. In case that $G = \langle g \rangle$ is finite, say of order $n \geqslant 2$, we have

$$\langle g \rangle = \{e_G = g^0, g, g^2, \ldots, g^{n-1}\}, \quad g^n = e_G.$$

So g has order n.

Example 1. $\mu_n = \{\rho \in \mathbb{C}^* : \rho^n = 1\}$, that is the group of roots of unity of order n is a cyclic group of order n. For a generator of μ_n one may take any primitive root of unity of order n, i.e., $e^{2\pi i k/n}$ with $k \in \mathbb{Z}$, $\gcd(k, n) = 1$.

Example 2. Let p be a prime number, and $(\mathbb{Z}/p\mathbb{Z})^* = \{a \bmod p, \gcd(a, p) = 1\}$ the group of prime residue classes modulo p with multiplication. This is a cyclic group of order $p - 1$.

Let $G = \langle g \rangle$ be a cyclic group and H a subgroup of G. Let k be the smallest positive integer such that $g^k \in H$. Using, e.g., division with remainder, one shows that $g^r \in H$ if and only if $r \equiv 0 \pmod{k}$. Hence $H = \langle g^k \rangle$ and $(G : H) = k$.

0.1.7 Homomorphisms and isomorphisms

Let G_1, G_2 be two groups. A *homomorphism* from G_1 to G_2 is a map $f : G_1 \to G_2$ such that $f(g_1 g_2) = f(g_1) f(g_2)$ for all $g_1, g_2 \in G$ and $f(e_{G_1}) = e_{G_2}$. This implies that $f(g^{-1}) = f(g)^{-1}$ for $g \in G_1$.

Let $f : G_1 \to G_2$ be a homomorphism. The kernel and image of f are given by

$$\mathrm{Ker}(f) := \{g \in G_1 : f(g) = e_{G_2}\}, \quad f(G_1) = \{f(g) : g \in G_1\},$$

respectively. Notice that $\mathrm{Ker}(f)$ is a normal subgroup of G_1. It is easy to check that f is injective if and only if $\mathrm{Ker}(f) = \{e_{G_1}\}$.

Let G be a group and H a normal subgroup of G. Then

$$f : G \to G/H : \ g \mapsto gH$$

is a surjective homomorphism from G to G/H, the *canonical homomorphism* from G to G/H. Notice that the kernel of this homomorphism is H. Thus, every normal subgroup of G occurs as the kernel of some homomorphism.

A homomorphism $f : G_1 \to G_2$ which is bijective is called an *isomorphism* from G_1 to G_2. In case that there is an isomorphism from G_1 to G_2 we say that G_1, G_2 are isomorphic, notation $G_1 \cong G_2$. Notice that a homomorphism $f : G_1 \to G_2$ is an isomorphism if and only if $\mathrm{Ker}(f) = \{e_{G_1}\}$ and $f(G_1) = G_2$. Further, in this case the inverse map $f^{-1} : G_2 \to G_1$ is also an isomorphism.

Let $f : G_1 \to G_2$ be a homomorphism of groups and $H = \mathrm{Ker}(f)$. This yields an isomorphism

$$\overline{f} : G_1/H \to f(G_1) : \ \overline{f}(gH) = f(g).$$

Proposition 0.1.1. *Let C be a cyclic group. If C is infinite, then it is isomorphic to $\mathbb{Z}^+$ (the additive group of $\mathbb{Z}$). If C has finite order n, then it is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^+$ (the additive group of residue classes modulo n).*

Proof. Let $C = \langle g \rangle$. Define $f : \mathbb{Z}^+ \to C$ by $x \mapsto g^x$. This is a surjective homomorphism; let H denote its kernel. Thus, $\mathbb{Z}^+/H \cong C$. We have $H = \{0\}$ if C is infinite, and $H = n\mathbb{Z}^+$ if C has order n. This implies the proposition. $\square$

0.1.8 Direct products

Let $G_1, \ldots, G_r$ be groups. Denote by e_{G_i} the unit element of G_i. The *(external) direct product* $G_1 \times \cdots \times G_r$ is the set of tuples $(g_1, \ldots, g_r)$ with $g_i \in G_i$ for $i = 1, \ldots, r$, endowed with the group operation

$$(g_1, \ldots, g_r) \cdot (h_1, \ldots, h_r) = (g_1 h_1, \ldots, g_r h_r).$$

This is obviously a group, with unit element $(e_{G_1}, \ldots, e_{G_r})$ and inverse $(g_1, \ldots, g_r)^{-1} = (g_1^{-1}, \ldots, g_r^{-1})$.

Let G be a group and $G_1, \ldots, G_r$ subgroups of G. We say that G is the *internal direct product* of $G_1, \ldots, G_r$ if:

(a) $G = G_1 \cdots G_r$, i.e., every element of G can be expressed as $g_1 \cdots g_r$ with $g_i \in G_i$ for $i = 1, \ldots, r$;

(b) $G_1, \ldots, G_r$ commute, that is, for all $i, j = 1, \ldots, r$ and all $g_i \in G_i$, $g_j \in G_j$ we have $g_i g_j = g_j g_i$;

(c) $G_1, \ldots, G_r$ are independent, i.e., if $g_i \in G_i$ $(i = 1, \ldots, r)$ are any elements such that $g_1 \cdots g_r = e_G$, then $g_i = e_G$ for $i = 1, \ldots, r$.

A consequence of (a), (b), (c) is that every element of G can be expressed uniquely as a product $g_1 \cdots g_r$ with $g_i \in G_i$ for $i = 1, \ldots, r$.

Proposition 0.1.2. *Let G, $G_1, \ldots, G_r$ be groups.*

(i) Suppose G is the internal direct product of $G_1, \ldots, G_r$. Then $G \cong G_1 \times \cdots \times G_r$.
(ii) Suppose $G \cong G_1 \times \cdots \times G_r$. Then there are subgroups $H_1, \ldots, H_r$ of G such that $H_i \cong G_i$ for $i = 1, \ldots, r$ and G is the internal direct product of $H_1, \ldots, H_r$.

Proof. (i) The map $G_1 \times \cdots \times G_r \to G : (g_1, \ldots, g_r) \mapsto g_1 \cdots g_r$ is easily seen to be an isomorphism.

(ii) Let $G' := G_1 \times \cdots \times G_r$ and for $i = 1, \ldots, r$, define the group

$$G'_i := \{(e_{G_1}, \ldots, g_i, \ldots, e_{G_r}) : g_i \in G_i\}$$

where the i-th coordinate is g_i and the other components are the unit elements of the respective groups. Clearly, G' is the internal direct product of $G'_1, \dots, G'_r$, and $G'_i \cong G_i$ for $i = 1, \dots, r$. Let $f : G \to G_1 \times \cdots \times G_r$ be an isomorphism. Then G is the internal direct product of $H_i := f^{-1}(G'_i)$ $(i = 1, \dots, r)$, and $H_i \cong G'_i \cong G_i$ for $i = 1, \dots, r$. □

We will sometimes be sloppy and write $G = G_1 \times \cdots \times G_r$ if G is the internal direct product of subgroups $G_1, \dots, G_r$.

0.1.9 Abelian groups

The group operation of an abelian group is often denoted by $+$, but in this course we stick to the multiplicative notation. The unit element of an abelian group A is denoted by 1 or 1_A. It is obvious that every subgroup of an abelian group is a normal subgroup. In Proposition 0.1.2, the condition that $H_1, \dots, H_r$ commute holds automatically so it can be dropped.

The following important theorem, which we state without proof, implies that the finite cyclic groups are the building blocks of the finite abelian groups.

Theorem 0.1.3. *Every finite abelian group is isomorphic to a direct product of finite cyclic groups.*

Proof. See S. Lang, Algebra, 2nd ed. Addison-Wesley, 1984, Ch.1, §10. □

Let A be a finite, multiplicatively written abelian group of order $\geqslant 2$ with unit element 1. Theorem 0.1.3 implies that A is the internal direct product of cyclic subgroups, say $C_1, \dots, C_r$. Assume that C_i has order $n_i \geqslant 2$; then $C_i = \langle h_i \rangle$, where $h_i \in A$ is an element of order n_i. We call $\{h_1, \dots, h_r\}$ a *basis* for A.

Every element of A can be expressed uniquely as $g_1 \cdots g_r$, where $g_i \in C_i$ for $i = 1, \dots, r$. Further, every element of C_i can be expressed as a power h_i^k, and $h_i^k = 1$ if and only if $k \equiv 0 \pmod{n_i}$. Together with Proposition 0.1.2 this implies the following characterization of a basis for A:

$$\left\{ \begin{array}{l} A = \{h_1^{k_1} \cdots h_r^{k_r} : \quad k_i \in \mathbb{Z} \text{ for } i = 1, \dots, r\}, \\ \text{there are integers } n_1, \dots, n_r \geqslant 2 \text{ such that} \\ h_1^{k_1} \cdots h_r^{k_r} = 1 \Longleftrightarrow k_i \equiv 0 \pmod{n_i} \text{ for } i = 1, \dots, r. \end{array} \right. \tag{0.1.1}$$

0.2 Basic concepts from analysis

0.2.1 Asymptotic formulas

In analytic number theory texts there is a frequent occurrence of asymptotic formulas, in which a complicated, not well understood function is approximated by a simple, well understood function, and an estimate for the order of magnitude for the error is given. In this section we recall some notation and some basic facts. Most of this is first year calculus, formulated in a somewhat different manner.

Let $\mathcal{S}$ be an unbounded subset of $\mathbb{R}$ (for instance, the positive reals, the positive integers or the primes), let f (the complicated function) and g (the simple function) be functions from $\mathcal{S}$ to $\mathbb{C}$ and e (the estimate for the error) a function from $\mathcal{S}$ to $\mathbb{R}_{\geqslant 0}$. We write

$$f(x) = g(x) + O(e(x)) \quad \text{as } |x| \to \infty \tag{0.2.1}$$

if there are $C, x_0 > 0$ such that $|f(x) - g(x)| \leqslant C \cdot e(x)$ for all $x \in \mathcal{S}$ with $|x| \geqslant x_0$. We call C a constant *implied by the O-symbol*, or a constant implicit in the O-symbol. Further, we write

$$f(x) = g(x) + o(e(x)) \quad \text{as } |x| \to \infty \tag{0.2.2}$$

if $\lim_{x \in \mathcal{S}, |x| \to \infty} (f(x) - g(x))/e(x) = 0$.

The interpretation of (0.2.1) is that $f(x)$ can be approximated by $g(x)$ with error of order of magnitude at most $e(x)$, and the interpretation of (0.2.2) is that $f(x)$ can be approximated by $g(x)$ with error of order of magnitude smaller than $e(x)$. We call (0.2.1) and (0.2.2) *asymptotic formulas*, with *main term* $g(x)$ and *error term* $O(e(x))$, respectively $o(e(x))$.

In addition to the above, the notation $f(x) = g(x) + O(e(x))$ (without $x \to \infty$) is used. This is defined for functions $f, g : \mathcal{S} \to \mathbb{C}$ for any infinite set $\mathcal{S}$, not necessarily contained in the reals, and $e : \mathcal{S} \to \mathbb{R}_{\geqslant 0}$. It means that there is $C > 0$ such that $|f(x) - g(x)| \leqslant C \cdot e(x)$ for all $x \in \mathcal{S}$.

We should mention here that in case f, g, e are defined on a subset $\mathcal{S}$ of $\mathbb{R}$ and $f, g, 1/e$ are bounded on bounded subsets of $\mathcal{S}$, then $f(x) = g(x) + O(e(x))$ as $x \to \infty$ and $f(x) = g(x) + O(e(x))$ (without $x \to \infty$) have the same meaning. Indeed, suppose that $f(x) = g(x) + O(e(x))$ as $x \to \infty$. Then there are $x_0 > 0, C > 0$ such that $|f(x) - g(x)| \leqslant C \cdot e(x)$ for all $x \in \mathcal{S}$ with $|x| \geqslant x_0$. However, by assumption on f, g, e, there is $C' > 0$ such that $|(f(x) - g(x))/e(x)| \leqslant C'$ for all $x \in \mathcal{S}$ with $|x| \leqslant x_0$. Consequently, $|f(x) - g(x)| \leqslant \max(C, C')e(x)$ for all $x \in \mathcal{S}$, i.e., $f(x) = g(x) + O(e(x))$.

We introduce some further notation:

• $f(x) \ll e(x)$ or $e(x) \gg f(x)$ as $|x| \to \infty$ has the same meaning as $f(x) = O(e(x))$ as $|x| \to \infty$, i.e., there are $C, x_0 > 0$ such that $|f(x)| \leqslant C \cdot e(x)$ for all $x \in \mathcal{S}$ with $|x| \geqslant x_0$; we call C a constant implied by $\ll$ or $\gg$.

• $f(x) \asymp g(x)$ as $|x| \to \infty$ (defined for functions $f, g : \mathcal{S} \to \mathbb{R}_{\geqslant 0}$) means that there are $C_1, C_2, x_0 > 0$ such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$ for all $x \in \mathcal{S}$ with $|x| \geqslant x_0$. In other words, $f(x) \asymp g(x)$ as $|x| \to \infty$ means that both $f(x) \ll g(x)$ as $|x| \to \infty$ and $g(x) \ll f(x)$ as $|x| \to \infty$.

• $f(x) \sim g(x)$ as $|x| \to \infty$ (defined for functions $f, g : \mathcal{S} \to \mathbb{R}$) means that $\lim_{x \in \mathcal{S}, |x| \to \infty} f(x)/g(x) = 1$.

Of course, asymptotic formulas such as (0.2.1) or (0.2.2) are of interest only if the error term is of smaller order of magnitude than the main term. Thus, in (0.2.1) we require that $\lim_{x \in \mathcal{S}, |x| \to \infty} e(x)/|g(x)| = 0$, i.e., $e(x) = o(|g(x)|)$ as $|x| \to \infty$, while in (0.2.2) we require that there are x_0 and C such that $e(x) \leqslant C|g(x)|$ for $x \in \mathcal{S}$ with $|x| \geqslant x_0$, that is, $e(x) = O(|g(x)|)$ as $|x| \to \infty$.

We mention some basic facts.

Lemma 0.2.1. *(i) Let f_i, g_i $(i = 1, 2)$ be functions from $\mathcal{S}$ to $\mathbb{R}$ and e a function from $\mathcal{S}$ to $\mathbb{R}_{\geqslant 0}$ such that $f_1(x) = g_1(x) + O(e(x))$, $f_2(x) = g_2(x) + O(e(x))$ as $|x| \to \infty$ and let a, b be reals. Then*

$$af_1(x) + bf_2(x) = ag_1(x) + bg_2(x) + O(e(x)) \ \textit{as } |x| \to \infty. \tag{0.2.3}$$

(ii) Let f_i, g_i $(i = 1, 2)$ be functions from $\mathcal{S}$ to $\mathbb{R}$ and e a function from $\mathcal{S}$ to $\mathbb{R}_{\geqslant 0}$ such that $e(x) = o(1)$ as $|x| \to \infty$, that is, $\lim_{x \in \mathcal{S}, |x| \to \infty} e(x) = 0$. Further, let a_1, a_2 be reals such that $f_1(x) = a_1 + O(e(x))$, $f_2(x) = a_2 + O(e(x))$ as $|x| \to \infty$. Then

$$f_1(x) f_2(x) = a_1 a_2 + O(e(x)) \quad \textit{as } |x| \to \infty. \tag{0.2.4}$$

(iii) Let g be a function from $\mathcal{S}$ to $\mathbb{R}$ with $g(x) = o(1)$ as $|x| \to \infty$ and a a real. Further, let φ be a function defined on a neighbourhood of a that is $n + 1$ times

continuously differentiable. Then

(0.2.5)

$$\varphi(a+g(x)) = \varphi(a) + \varphi'(a)g(x) + \cdots + \frac{\varphi^{(n)}(a)}{n!}\cdot g(x)^n + O(|g(x)|^{n+1}) \text{ as } |x| \to \infty.$$

Proof. (i) and (ii) are obvious, while (iii) follows from the Taylor-Lagrange formula

$$\varphi(a+t) = \varphi(a) + \varphi'(a)t + \cdots + \frac{\varphi^{(n)}(a)}{n!}\cdot t^n + \frac{\varphi^{(n+1)}(a+\theta)}{(n+1)!}\cdot t^{n+1}$$

where $|t|$ is small enough such that $a+t$ falls within the domain of definition of φ, and θ lies between 0 and t. Suppose φ is defined on $(a-\epsilon, a+\epsilon)$ and let x_0 be such that $|g(x)| < \frac{1}{2}\epsilon$ for all $x \in \mathcal{S}$ with $|x| \geqslant x_0$. Since $\varphi^{(n+1)}$ is continuous, there is C such that $|\varphi^{(n+1)}(a+t)| \leqslant C$ for all t with $|t| \leqslant \frac{1}{2}\epsilon$. Now by substituting $t = g(x)$, formula (0.2.5) follows. $\square$

Examples.

$$\begin{aligned}
\frac{1}{a+g(x)} &= a - a^{-2}g(x) + \tfrac{1}{2}a^{-3}g(x)^2 + O(|g(x)|^3) \text{ as } |x| \to \infty, \\
\log(1+g(x)) &= g(x) - \tfrac{1}{2}g(x)^2 + \tfrac{1}{3}g(x)^3 + O(|g(x)|^4) \text{ as } |x| \to \infty, \\
e^{g(x)} &= 1 + g(x) + \tfrac{1}{2}g(x)^2 + \tfrac{1}{3!}g(x)^3 + O(|g(x)|^4) \text{ as } |x| \to \infty.
\end{aligned}$$

Next, we derive asymptotic formulas for sums $\sum_{a\leqslant n\leqslant x} f(n)$, where the sum is taken over all positive integers n with $a \leqslant n \leqslant x$ (with a an integer and x a real), and where f is a continuous, monotone decreasing function on $[a, \infty)$ with $\lim_{x\to\infty} f(x) = 0$. We start with a lemma.

Lemma 0.2.2. *Let a be an integer and let $f : [a, \infty) \to \mathbb{R}$ be a continuous, monotone decreasing function with $\lim_{x\to\infty} f(x) = 0$. Then there is $\gamma_f \geqslant 0$ such that for every integer $N \geqslant a$,*

$$\sum_{n=a}^{N} f(n) = \int_a^N f(t)dt + \gamma_f + r_f(N), \quad \textit{where } 0 \leqslant r_f(N) \leqslant f(N). \tag{0.2.6}$$

Remark. This formula is valid irrespective of whether $\sum_{n=a}^{\infty} f(n)$ converges or not.

Proof. Since f is monotone decreasing, we have $f(n+1) \leqslant \int_n^{n+1} f(t)dt \leqslant f(n)$, hence

$$0 \leqslant b_n := f(n) - \int_n^{n+1} f(t)dt \leqslant f(n) - f(n+1) \quad \text{for } n \geqslant a. \tag{0.2.7}$$

The series $\sum_{n=a}^{\infty}(f(n) - f(n+1)) = f(a)$ converges, so

$$\gamma_f := \sum_{n=a}^{\infty} b_n = \lim_{N\to\infty} \sum_{n=a}^{N-1} b_n = \lim_{N\to\infty} \Big(\sum_{n=a}^{N-1} f(n) - \int_a^N f(t)dt \Big)$$

converges as well and is $\geqslant 0$. Further

$$\sum_{n=a}^{N} f(n) - \int_a^N f(t)dt = f(N) + \sum_{n=a}^{N-1} b_n = \gamma_f + f(N) - \sum_{n=N}^{\infty} b_n = \gamma_f + r_f(N),$$

where by (0.2.7) we have

$$f(N) \geqslant r_f(N) \geqslant f(N) - \sum_{n=N}^{\infty}(f(n) - f(n+1)) = 0.$$

$\square$

Corollary 0.2.3. *Let a be an integer and let $f : [a, \infty) \to \mathbb{R}$ be a continuous, monotone decreasing function with $\lim_{x\to\infty} f(x) = 0$. Assume that $\sum_{n=a}^{\infty} f(n)$ converges. Then for every integer $N \geqslant a$,*

$$\sum_{n=a}^{N} f(n) = \sum_{n=a}^{\infty} f(n) - \int_a^{\infty} f(t)dt + r_f(N) \quad \textit{where } 0 \leqslant r_f(N) \leqslant f(N). \tag{0.2.8}$$

Proof. Letting $N \to \infty$ in (0.2.6), we get $\gamma_f = \sum_{n=a}^{\infty} f(n) - \int_a^{\infty} f(t)dt$. Substituting this into (0.2.6) we immediately get (0.2.8). $\square$

Corollary 0.2.4. *Let a be an integer and let $f : [a, \infty) \to \mathbb{R}$ be a continuous, monotone decreasing function with $\lim_{x\to\infty} f(x) = 0$. Assume in addition that the quotient $f(x-1)/f(x)$ is bounded as $x \to \infty$. Then for every real $x \geqslant a$,*

$$\sum_{a\leqslant n\leqslant x} f(n) = \int_a^x f(t)dt + \gamma_f + O(f(x)) \quad \textit{as } x \to \infty.$$

Further, if $\sum_{n=a}^{\infty} f(n)$ converges, we have

$$\sum_{a\leqslant n\leqslant x} f(n) = \sum_{n=a}^{\infty} f(n) - \int_x^{\infty} f(t)dt + O(f(x)) \quad \textit{as } x \to \infty.$$

Proof. We prove only the first asymptotic formula. The proof of the second is very similar. Let $N = [x]$ be the largest integer $\leqslant x$. Then

$$\begin{aligned}
\sum_{a \leqslant n \leqslant x} f(n) &= \sum_{n=a}^{N} f(n) = \int_a^N f(t)dt + \gamma_f + r_f(N) \\
&= \int_a^x f(t)dt + \gamma_f - \int_N^x f(t)dt + r_f(N).
\end{aligned}$$

Note that $f(N)/f(x) \leqslant f(x-1)/f(x)$ is bounded as $x \to \infty$. So

$$0 \leqslant \int_N^x f(t)dt \leqslant f(N) = O(f(x)), \quad 0 \leqslant r_f(N) \leqslant f(N) = O(f(x)) \ \text{ as } x \to \infty,$$

implying $\sum_{a \leqslant n \leqslant x} f(n) = \int_a^x f(t)dt + \gamma_f + O(f(x))$ as $x \to \infty$. □

Examples.
a) By applying Corollary 0.2.4 with $f(x) = x^{-1}$ we get

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + \gamma + O(\tfrac{1}{x}) \ \text{ as } x \to \infty,$$

where $\gamma = \gamma_{x^{-1}}$ is the *Euler-Mascheroni constant.*
b) By applying Corollary 0.2.4 with $f(x) = x^{-2}$ and using Euler's formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ we get

$$\sum_{n \leqslant x} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{1}{x} + O(\tfrac{1}{x^2}) \ \text{ as } x \to \infty.$$

0.2.2 Infinite products

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers converges if there is $\ell \in \mathbb{C}$ such that $\lim_{n\to\infty} a_n = \ell$, i.e., $\lim_{n\to\infty} |a_n - \ell| = 0$. By the completeness of $\mathbb{C}$, this is equivalent to $\lim_{m,n\to\infty} |a_m - a_n| = 0$. For a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ we say that $\lim_{n\to\infty} a_n$ exists if either the sequence converges or the limit is $\pm\infty$. A limit can be $\pm\infty$ only if $a_n \in \mathbb{R}$ for all sufficiently large n. So for instance $\lim_{n\to\infty}(-1)^n$ does not exist.

We define a series of complex numbers $\sum_{n=1}^{\infty} A_n$ by $\lim_{N\to\infty} \sum_{n=1}^{N} A_n$, provided the limit exists; if the limit exists and is not $\pm\infty$, we say that the series converges.

If $\sum_{n=1}^\infty |A_n|$ converges, we say that $\sum_{n=1}^\infty A_n$ converges absolutely. Absolute convergence of a series implies convergence. Just as for series of real numbers, a series of complex numbers $\sum_{n=1}^\infty A_n$ is absolutely convergent if and only if it is unconditionally convergent, i.e., after any rearrangement of its terms, the series remains convergent and its value remains the same.

In what follows, we consider infinite products. Let $\{A_n\}_{n=1}^\infty$ be a sequence of complex numbers. We define

$$\prod_{n=1}^\infty A_n := \lim_{N\to\infty} \prod_{n=1}^N A_n$$

provided the limit exists (so if it is finite or $\pm\infty$).

Clearly, $\prod_{n=1}^\infty A_n = 0$ if $A_n = 0$ for some n. But if $A_n \neq 0$ for all n then it may still happen that $\prod_{n=1}^\infty A_n = 0$, for instance $\prod_{n=1}^\infty \left(1 - \frac{1}{n+1}\right) = 0$. (It is common practice to say that $\prod_{n=1}^\infty A_n$ converges if there is *non-zero* $\ell \in \mathbb{C}$ such that $\lim_{N\to\infty} \prod_{n=1}^N A_n = \ell$. We will not use this notion of convergence and say instead that $\prod_{n=1}^\infty A_n$ exists and is $\neq 0, \pm\infty$).

Define the principal complex logarithm of $z \in \mathbb{C}\setminus\{0\}$ by $\operatorname{Log} z := \log|z| + i\operatorname{Arg} z$, where $\operatorname{Arg} z$ is the principal argument of z, i.e., the argument in $(-\pi, \pi]$. Then we have

$$\prod_{n=1}^\infty A_n \text{ exists and is} \neq 0, \pm\infty \iff A_n \neq 0 \text{ for all } n \text{ and } \sum_{n=1}^\infty \operatorname{Log} A_n \text{ converges.}$$

The following criterion is more useful for our purposes.

Proposition 0.2.5. *Assume that $\sum_{n=1}^\infty |A_n - 1| < \infty$. Then the following hold:*
(i) $\prod_{n=1}^\infty A_n$ exists and is $\neq \pm\infty$, and $\prod_{n=1}^\infty A_n \neq 0$ if $A_n \neq 0$ for all n.
(ii) $\prod_{n=1}^\infty A_n$ is invariant under rearrangements of the A_n, i.e., if σ is any bijection of $\mathbb{Z}_{>0}$, then $\prod_{n=1}^\infty A_{\sigma(n)}$ exists and is equal to $\prod_{n=1}^\infty A_n$.

Proof. (i) Let $a_n := |A_n - 1|$ for $n = 1, 2, \ldots$. Let M, N be integers with $N > M > 0$. Then, using $|1 + z| \leqslant e^{|z|}$ for $z \in \mathbb{C}$ and

$$\Big|\prod_{i=1}^r (1 + z_i) - 1\Big| \leqslant \prod_{i=1}^r (1 + |z_i|) - 1 \leqslant \exp\Big(\sum_{i=1}^r |z_i|\Big) - 1 \quad \text{for } z_1, \ldots, z_r \in \mathbb{C},$$

we get

$$\left|\prod_{n=1}^{N} A_n - \prod_{n=1}^{M} A_n\right| = \prod_{n=1}^{M} |A_n| \cdot \left|\prod_{n=M+1}^{N} A_n - 1\right| \leqslant \exp\Big(\sum_{n=1}^{M} a_n\Big) \cdot \left(\exp\Big(\sum_{n=M+1}^{N} a_n\Big) - 1\right) \tag{0.2.9}$$

which tends to 0 as $M, N \to \infty$. Hence $\prod_{n=1}^{\infty} A_n = \lim_{N\to\infty} \prod_{n=1}^{N} A_n$ exists and is finite.

Assume that $A_n \neq 0$ for all n. Since $\sum_{n=1}^{\infty} a_n$ converges, there exists M such that $\sum_{n=M}^{\infty} a_n < \frac{1}{2}$. Then noting that $|A_n| \geqslant 1 - a_n \geqslant e^{-a_n}$ we get for all $N > M$,

$$\begin{aligned}\left|\prod_{n=1}^{N} A_n\right| &= \prod_{n=1}^{M} |A_n| \cdot \prod_{n=M+1}^{N} |A_n| \\ &\geqslant \Big(\prod_{n=1}^{M} |A_n|\Big) \cdot e^{-\sum_{n=M+1}^{N} a_n} \geqslant e^{-1/2} \prod_{n=1}^{M} |A_n| =: C > 0,\end{aligned}$$

and then, letting $N \to \infty$, $\left|\prod_{n=1}^{\infty} A_n\right| \geqslant C > 0$. This proves (i).

(ii) Let M, N be positive integers such that $N > M$ and $\{\sigma(1), \ldots, \sigma(N)\}$ contains $\{1, \ldots, M\}$. Similarly to (0.2.9) we get

$$\left|\prod_{n=1}^{N} A_{\sigma(n)} - \prod_{n=1}^{M} A_n\right| \leqslant \exp\Big(\sum_{n=1}^{M} a_n\Big) \cdot \left(\exp\Big(\sum_{n\leqslant N, \sigma(n)>M} a_{\sigma(n)}\Big) - 1\right).$$

If for fixed M we let first $N \to \infty$ and then let $M \to \infty$, the right-hand side tends to 0. Hence $\prod_{n=1}^{\infty} A_{\sigma(n)} = \prod_{n=1}^{\infty} A_n$. □

0.2.3 Uniform convergence

We consider functions $f : D \to \mathbb{C}$ where D can be any set. We can express each such function as $g + ih$ where g, h are functions from D to $\mathbb{R}$. We write $g = \mathrm{Re}\, f$ and $h = \mathrm{Im}\, f$.

We recall that if D is a topological space (in this course mostly a subset of $\mathbb{R}^n$ with the usual topology, i.e., the open subsets of D are the unions of open balls in $\mathbb{R}^n$ intersected with D) then f is continuous if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous.

In case that $D \subseteq \mathbb{R}$, we say that f is differentiable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are differentiable; then we define the derivative of f by $f' := (\operatorname{Re} f)' + i(\operatorname{Im} f)'$.

In what follows, let D be any set and $\{F_n\} = \{F_n\}_{n=1}^{\infty}$ a sequence of functions from D to $\mathbb{C}$.

Definition. We say that $\{F_n\}$ converges pointwise on D if for every $z \in D$ there is $F(z) \in \mathbb{C}$ such that $\lim_{n\to\infty} F_n(z) = F(z)$. In this case, we write $F_n \to F$ pointwise. We say that $\{F_n\}$ converges uniformly on D if moreover,

$$\lim_{n\to\infty} \left(\sup_{z\in D} |F_n(z) - F(z)| \right) = 0.$$

In this case, we write $F_n \to F$ uniformly.

Facts:

- $\{F_n\}$ converges uniformly on D if and only if $\lim_{M,N\to\infty} \left(\sup_{z\in D} |F_M(z) - F_N(z)| \right) = 0.$
- Let D be a topological space, assume that all functions F_n are continuous on D, and that $\{F_n\}$ converges to a function F uniformly on D. Then F is continuous on D.

Let again D be any set and $\{F_n\}_{n=1}^{\infty}$ a sequence of functions from D to $\mathbb{C}$. We say that the series $\sum_{n=1}^{\infty} F_n$ converges pointwise/uniformly on D if the partial sums $\sum_{n=1}^{N} F_n$ converge pointwise/uniformly on D. Further, we say that $\sum_{n=1}^{\infty} F_n$ is pointwise absolutely convergent on D if $\sum_{n=1}^{\infty} |F_n(z)|$ converges for every $z \in D$.

Proposition 0.2.6 (Weierstrass criterion for series)**.** *Assume that there are finite real numbers M_n such that*

$$|F_n(z)| \leqslant M_n \text{ for } z \in D,\ n \geqslant 1, \quad \sum_{n=1}^{\infty} M_n \text{ converges.}$$

Then $\sum_{n=1}^{\infty} F_n$ is both uniformly convergent, and pointwise absolutely convergent on D.

Proof. We have for $N > M \geqslant 1$,

$$\begin{aligned}\sup_{z\in D}\Big|\sum_{n=1}^{N}F_n(z)-\sum_{n=1}^{M}F_n(z)\Big| &= \sup_{z\in D}\Big|\sum_{n=M+1}^{N}F_n(z)\Big| \\ &\leqslant \sup_{z\in D}\sum_{n=M+1}^{N}|F_n(z)| \leqslant \sum_{n=M+1}^{N}M_n \to 0 \ \text{ as } M,N\to\infty.\end{aligned}$$

□

We need a similar result for infinite products of functions. Let again D be any set and $\{F_n : D \to \mathbb{C}\}_{n=1}^{\infty}$ a sequence of functions. We define the limit function $\prod_{n=1}^{\infty} F_n$ by

$$\prod_{n=1}^{\infty}F_n(z) := \lim_{N\to\infty}\prod_{n=1}^{N}F_n(z) \quad (z\in D),$$

provided that for every $z \in D$ the limit exists.

We say that $\prod_{n=1}^{\infty} F_n$ converges uniformly on D if the limit function $F := \prod_{n=1}^{\infty} F_n$ exists and is $\neq \pm\infty$ on D, and

$$\lim_{N\to\infty}\left(\sup_{z\in D}\left|F(z)-\prod_{n=1}^{N}F_n(z)\right|\right)=0.$$

Proposition 0.2.7 (Weierstrass criterion for infinite products)**.** *Assume that there are finite real numbers M_n such that*

$$|F_n(z)-1| \leqslant M_n \text{ for } z\in D,\ n\geqslant 1, \qquad \sum_{n=1}^{\infty}M_n \text{ converges.}$$

Then $F := \prod_{n=1}^{\infty} F_n$ is uniformly convergent on D and moreover, if $z \in D$ is such that $F_n(z) \neq 0$ for all n, then also $F(z) \neq 0$.

Proof. Applying (0.2.9) with $A_n = F_n(z)$ and using $|F_n(z) - 1| \leqslant M_n$ for $z \in D$, we obtain that for any two integers M, N with $N > M > 0$, and all $z \in D$,

$$\left|\prod_{n=1}^{N}F_n(z)-\prod_{n=1}^{M}F_n(z)\right| \leqslant \exp\Big(\sum_{n=1}^{M}M_n\Big)\cdot\left(\exp\Big(\sum_{n=M+1}^{N}M_n\Big)-1\right).$$

Since the right-hand side is independent of z and tends to 0 as $M, N \to \infty$, the uniform convergence follows. Further, if $F_n(z) \neq 0$ for all n then $\prod_{n=1}^{\infty} F_n(z) \neq 0$ by Proposition 0.2.5. □

0.3 Integration

In this course, all integrals will be Lebesgue integrals of real or complex measurable functions on $\mathbb{R}^n$ (always with respect to the Lebesgue measure on $\mathbb{R}^n$). Lebesgue integrals coincide with the Riemann integrals from first year calculus whenever the latter are defined, but Riemann integrals can be defined only for a much smaller class of functions. It is not really necessary to know the precise definitions of Lebesgue measure, measurable functions and Lebesgue integrals, and you will be perfectly able to follow this course without any knowledge of Lebesgue theory. But we will frequently have to deal with infinite integrals of infinite series of functions, and to handle these, Lebesgue theory is much more convenient than the theory of Riemann integrals. In particular, in Lebesgue theory there are some very powerful convergence theorems for sequences of functions, theorems on interchanging multiple integrals, etc., which we will frequently apply. If you are willing to take for granted that all functions appearing in this course are measurable, there will be no problem to understand or apply these theorems.

We have collected a few useful facts, which are amply sufficient for our course.

0.3.1 Measurable sets

The length of a bounded interval $I = [a, b], [a, b), (a, b]$ or (a, b), where $a, b \in \mathbb{R}$, $a < b$, is given by $l(I) := b - a$. Let $n \in \mathbb{Z}_{\geqslant 1}$. An *interval* in $\mathbb{R}^n$ is a cartesian product of bounded intervals $I = \prod_{i=1}^{n} I_i$. We define the volume of I by $l(I) := \prod_{i=1}^{n} l(I_i)$.

Let A be an arbitrary subset of $\mathbb{R}^n$. We define the *outer measure* of A by

$$\lambda^*(A) := \inf \sum_{i=1}^{\infty} l(I_i),$$

where the infimum is taken over all countable unions of intervals $\bigcup_{i=1}^{\infty} I_i \supset A$. We say that a set A is *measurable* if

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \cap A^c) \text{ for every } S \subseteq \mathbb{R}^n,$$

where $A^c = \mathbb{R}^n \setminus A$ is the complement of A. In this case we define the (Lebesgue) measure of A by $\lambda(A) := \lambda^*(A)$. This measure may be finite or infinite. It can be shown that intervals are measurable, and that $\lambda(I) = l(I)$ for any interval I in $\mathbb{R}^n$.

Facts:

- A countable union $\bigcup_{i=1}^{\infty} A_i$ of measurable sets A_i is measurable. Further, the complement of a measurable set is measurable. Hence a countable intersection of measurable sets is measurable.
- All open and closed subsets of $\mathbb{R}^n$ are measurable.
- Let $A = \cup_{i=1}^{\infty} A_i$ be a countable union of pairwise disjoint measurable sets. Then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_i)$, where we agree that $\lambda(A) = 0$ if $\lambda(A_i) = 0$ for all i.
- Under the assumption of the axiom of choice, one can construct non-measurable subsets of $\mathbb{R}^n$.

Let A be a measurable subset of $\mathbb{R}^n$. We say that a particular condition holds for *almost all* $x \in A$, it if holds for all $x \in A$ with the exception of a subset of Lebesgue measure 0. If the condition holds for almost all $x \in \mathbb{R}^n$, we say that it holds *almost everywhere.*

An important subcollection of the collection of measurable subsets of $\mathbb{R}^n$ is the collection of *Borel sets:* it is the smallest collection of subsets of $\mathbb{R}^n$ which contains all open sets, and which is closed under taking complements and under taking countable unions.

All sets occurring in this course will be Borel sets, hence measurable; we will never bother about the verification in individual cases.

0.3.2 Measurable functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called measurable if for every $a \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : f(x) > a\}$ is measurable.

A function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable.

Facts:

- If $A \subset \mathbb{R}^n$ is measurable then its characteristic function, given by $I_A(x) = 1$ if $x \in A$, $I_A(x) = 0$ otherwise is measurable.

- Every continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is measurable. More generally, f is measurable if its set of discontinuities has Lebesgue measure 0.

- If $f, g : \mathbb{R}^n \to \mathbb{C}$ are measurable then $f+g$ and fg are measurable. Further, the function given by $x \mapsto f(x)/g(x)$ if $g(x) \neq 0$ and $x \mapsto 0$ if $g(x) = 0$ is measurable.

- If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable, then so are $\max(f, g)$ and $\min(f, g)$.

- If $\{f_k : \mathbb{R}^n \to \mathbb{C}\}$ is a sequence of measurable functions and $f_k \to f$ pointwise on $\mathbb{R}^n$, then f is measurable.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *Borel function* if $\{x \in \mathbb{R}^n : f(x) > a\}$ is a Borel set for every $a \in \mathbb{R}$. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a Borel function if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both Borel functions. All functions occurring in our course can be proved to be Borel, hence measurable. We will always omit the nasty verifications in individual cases.

0.3.3 Lebesgue integrals

The Lebesgue integral is defined in various steps.

1) An elementary function on $\mathbb{R}^n$ is a function of the type $f = \sum_{i=1}^{r} c_i I_{D_i}$, where $D_1, \dots, D_r$ are pairwise disjoint measurable subsets of $\mathbb{R}^n$, and $c_1, \dots, c_r$ positive reals. Then we define $\int f dx := \sum_{i=1}^{r} c_i \lambda(D_i)$.

2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and $f \geqslant 0$ on $\mathbb{R}^n$. Then we define $\int f dx := \sup \int g dx$ where the supremum is taken over all elementary functions $g \leqslant f$. Thus, $\int f dx$ is defined and $\geqslant 0$ but it may be infinite.

3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary measurable function. Then we define

$$\int f dx := \int \max(f, 0) dx - \int \max(-f, 0) dx,$$

provided that at least one of the integrals is finite. If both integrals are finite, we say that f is *integrable* or *summable*.

4) Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that f is integrable or summable if both

$\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable, and in that case we define

$$\int f dx := \int (\operatorname{Re} f) dx + i \int (\operatorname{Im} f) dx.$$

5) Let D be a measurable subset of $\mathbb{R}^n$. Let f be a complex function defined on a set containing D. We define $f \cdot I_D$ by defining it to be equal to f on D and equal to 0 outside D. We say that f is measurable on D if $f \cdot I_D$ is measurable. Further, we say that f is integrable over D if $f \cdot I_D$ is integrable, and in that case we define $\int_D f dx := \int f \cdot I_D dx$.

Facts:

- Let D be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a measurable function. Then f is integrable over D if and only if $\int_D |f| dx < \infty$ and in that case, $|\int_D f dx| \leqslant \int_D |f| dx$.

- Let again D be a measurable subset of $\mathbb{R}^n$ and $f : D \to \mathbb{C}$, $g : D \to \mathbb{R}_{\geqslant 0}$ measurable functions, such that $\int_D g dx < \infty$ and $|f| \leqslant g$ on D. Then f is integrable over D, and $|\int_D f dx| \leqslant \int_D g dx$.

- Let D be a closed interval in $\mathbb{R}^n$ and $f : D \to \mathbb{C}$ a bounded function which is Riemann integrable over D. Then f is Lebesgue integrable over D and the Lebesgue integral $\int_D f dx$ is equal to the Riemann integral $\int_D f(x) dx$.

- Let $f : [0, \infty) \to \mathbb{C}$ be such that the improper Riemann integral $\int_0^\infty |f(x)| dx := \lim_{T \to \infty} \int_0^T |f(x)| dx$ converges. Then the improper Riemann integral $\int_0^\infty f(x) dx := \lim_{T \to \infty} \int_0^T f(x) dx$ converges as well, and it is equal to the Lebesgue integral $\int_{[0,\infty)} f dx$. However, an improper Riemann integral $\int_0^\infty f(x) dx$ which itself is convergent, but for which $\int_0^\infty |f(x)| dx = \infty$ can not be interpreted as a Lebesgue integral. The same applies to the other types of improper Riemann integrals, e.g., $\int_a^b f(x) dx$ where f is unbounded on (a, b).

- An absolutely convergent series of complex terms $\sum_{n=0}^\infty a_n$ may be interpreted as a Lebesgue integral. Define the function A by $A(x) := a_n$ for $x \in \mathbb{R}$ with $n \leqslant x < n + 1$ and $A(x) := 0$ for $x < 0$. Then A is measurable and integrable, and $\sum_{n=0}^\infty a_n = \int A dx$.

0.3.4 Important theorems

Theorem 0.3.1 (Dominated Convergence Theorem). *Let $D \subseteq \mathbb{R}^n$ be a measurable set and $\{f_k : D \to \mathbb{C}\}_{k\geqslant 0}$ a sequence of functions that are all integrable over D, and such that $f_k \to f$ pointwise on D. Assume that there is an integrable function $g : D \to \mathbb{R}_{\geqslant 0}$ such that $|f_k(x)| \leqslant g(x)$ for all $x \in D$, $k \geqslant 0$. Then f is integrable over D, and $\int_D f_k dx \to \int_D f dx$.*

Corollary 0.3.2. *let $D \subset \mathbb{R}^n$ be a measurable set of finite measure and $\{f_k : D \to \mathbb{C}\}_{k\geqslant 0}$ a sequence of functions that are all integrable over D, and such that $f_k \to f$ uniformly on D. Then f is integrable over D, and $\int_D f_k dx \to \int_D f dx$.*

Proof. Let $\varepsilon > 0$. There is k_0 such that $|f(x) - f_k(x)| < \varepsilon$ for all $x \in D$, $k > k_0$. The constant function $x \mapsto \varepsilon$ is integrable over D since D has finite measure. Hence for $k > k_0$, $f - f_k$ is integrable over D, and so f is integrable over D. Consequently, $|f|$ is integrable over D. Now $|f_k| < \varepsilon + |f|$ for $k > k_0$. So by the Dominated Convergence Theorem, $\int_D f_k dx \to \int_D f dx$. □

In the theorem below, we write points of $\mathbb{R}^{m+n}$ as (x, y) with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Further, dx, dy, $d(x, y)$ denote the Lebesgue measures on $\mathbb{R}^m$, $\mathbb{R}^n$, $\mathbb{R}^{m+n}$, respectively.

Theorem 0.3.3 (Fubini-Tonelli). *Let D_1, D_2 be measurable subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively, and $f : D_1 \times D_2 \to \mathbb{C}$ a measurable function. Assume that at least one of the integrals*

$$\int_{D_1\times D_2} |f(x,y)|d(x,y), \quad \int_{D_1}\left(\int_{D_2} |f(x,y)|dy\right)dx, \quad \int_{D_2}\left(\int_{D_1} |f(x,y)|dx\right)dy$$

is finite. Then they are all finite and equal.
Further, f is integrable over $D_1 \times D_2$, $x \mapsto f(x, y)$ is integrable over D_1 for almost all $y \in D_2$, $y \mapsto f(x, y)$ is integrable over D_2 for almost all $x \in D_1$, and

$$\int_{D_1\times D_2} f(x,y)d(x,y) = \int_{D_1}\left(\int_{D_2} f(x,y)dy\right)dx = \int_{D_2}\left(\int_{D_1} f(x,y)dx\right)dy.$$

Corollary 0.3.4. *Let D be a measurable subset of $\mathbb{R}^m$ and $\{f_k : D \to \mathbb{C}\}_{k\geqslant 0}$ a sequence of functions that are all integrable over D and such that $\sum_{k=0}^{\infty} |f_k|$ converges* *pointwise on D. Assume that at least one of the quantities*

$$\sum_{k=0}^{\infty} \int_D |f_k(x)| dx, \quad \int_D \Big(\sum_{k=0}^{\infty} |f_k(x)| \Big) dx$$

is finite. Then $\sum_{k=0}^{\infty} f_k$ is integrable over D and

$$\sum_{k=0}^{\infty} \int_D f_k(x) dx = \int_D \Big(\sum_{k=0}^{\infty} f_k(x) \Big) dx.$$

Proof. Apply the Theorem of Fubini-Tonelli with $n = 1$, $D_1 = D$, $D_2 = [0, \infty)$, $F(x, y) = f_k(x)$ where k is the integer with $k \leqslant y < k + 1$. □

Corollary 0.3.5. *Let $\{a_{kl}\}_{k,l=0}^{\infty}$ be a double sequence of complex numbers such that at least one of*

$$\sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} |a_{kl}| \Big), \quad \sum_{l=0}^{\infty} \Big(\sum_{k=0}^{\infty} |a_{kl}| \Big)$$

converges. Then both

$$\sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} a_{kl} \Big), \quad \sum_{l=0}^{\infty} \Big(\sum_{k=0}^{\infty} a_{kl} \Big)$$

converge and are equal.

Proof. Apply the Theorem of Fubini-Tonelli with $m = n = 1$, $D_1 = D_2 = [0, \infty)$, $F(x, y) = a_{kl}$ where k, l are the integers with $k \leqslant x < k + 1$, $l \leqslant y < l + 1$. □

0.3.5 Useful inequalities

We have collected some inequalities, stated without proof, which frequently show up in analytic number theory. The proofs belong to a course in measure theory or functional analysis.

Proposition 0.3.6. *Let D be a measurable subset of $\mathbb{R}^n$ and $f, g : D \to \mathbb{C}$ measurable functions. Let p, q be reals > 1 with $\frac{1}{p} + \frac{1}{q} = 1$. Then if all integrals are defined,*

$$\Big| \int_D fg \cdot dx \Big| \leqslant \Big(\int_D |f|^p dx \Big)^{1/p} \cdot \Big(\int_D |g|^q dx \Big)^{1/q} \quad \textit{(Hölder's Inequality).}$$

In particular,

$$\Big|\int_D fg dx\Big| \leqslant \Big(\int_D |f|^2 dx\Big)^{1/2}\cdot\Big(\int_D |g|^2 dx\Big)^{1/2} \quad \textit{(Cauchy-Schwarz' Inequality).}$$

Corollary 0.3.7. *Let $a_1,\dots,a_r$, $b_1,\dots,b_r$ be complex numbers and p,q reals > 1 with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Big|\sum_{n=1}^{r} a_n b_n\Big| \leqslant \Big(\sum_{n=1}^{r}|a_n|^p\Big)^{1/p}\cdot\Big(\sum_{n=1}^{r}|b_n|^q\Big)^{1/q} \quad \textit{(Hölder).}$$

In particular,

$$\Big|\sum_{n=1}^{r} a_n b_n\Big| \leqslant \Big(\sum_{n=1}^{r}|a_n|^2\Big)^{1/2}\cdot\Big(\sum_{n=1}^{r}|b_n|^2\Big)^{1/2} \quad \textit{(Cauchy-Schwarz).}$$

This follows from Proposition 0.3.6 by taking $D=[0,r)$, $f(x)=a_n$, $g(x)=b_n$ for $n-1\leqslant x<n$, $n=1,\dots,r$.

A function φ from an interval $I\subseteq\mathbb{R}$ to $\mathbb{R}$ is called *convex* if $\varphi((1-t)x+ty)\leqslant(1-t)\varphi(x)+t\varphi(y)$ holds for all $x,y\in I$ and all $t\in[0,1]$. In particular, φ is convex on I if φ is differentiable twice and $\varphi''\geqslant 0$ on I.

Proposition 0.3.8. *Let D be a measurable subset of $\mathbb{R}^n$ with $0<\lambda(D)<\infty$, let $f: D\to\mathbb{R}_{>0}$ be a Lebesgue integrable function and let $\varphi:\mathbb{R}_{>0}\to\mathbb{R}$ be a convex function. Then*

$$\varphi\Big(\frac{1}{\lambda(D)}\int_D f\cdot dx\Big)\leqslant\frac{1}{\lambda(D)}\int_D(\varphi\circ f)dx \quad \textit{(Jensen's Inequality).}$$

Corollary 0.3.9. *Let $a_1,\dots,a_r$ be positive reals, and let $\varphi:\mathbb{R}_{>0}\to\mathbb{R}$ be a convex function. Then*

$$\varphi\Big(\frac{1}{r}\sum_{n=1}^{r}a_n\Big)\leqslant\frac{1}{r}\sum_{n=1}^{r}\varphi(a_n).$$

In particular,

$$\frac{1}{r}\sum_{n=1}^{r}a_n\geqslant\sqrt[r]{a_1\cdots a_n} \quad \textit{(arithmetic mean} \geqslant \textit{geometric mean).}$$

The first assertion follows by applying Proposition 0.3.8 with $D=[0,r)$ and $f(x)=a_n$ for $x\in[n-1,n)$. The second assertion follows by applying the first with $\varphi(x)=-\log x$.

0.4 Contour integrals

0.4.1 Paths in $\mathbb{C}$

We consider continuous functions $g : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$ and $a < b$. Two continuous functions $g_1 : [a, b] \to \mathbb{C}$, $g_2 : [c, d] \to \mathbb{C}$ are called equivalent if there is a continuous monotone increasing function $\varphi : [a, b] \to [c, d]$ such that $g_1 = g_2 \circ \varphi$. The equivalence classes of this relation are called *paths* (in $\mathbb{C}$), and a function $g : [a, b] \to \mathbb{C}$ representing a path is called a *parametrization* of the path. Roughly speaking, a path is a curve in $\mathbb{C}$, together with a direction in which it is traversed.

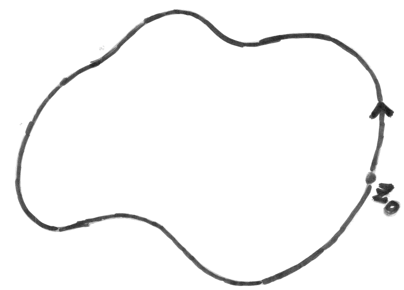
A *smooth path* is a path represented by a function $g : [a, b] \to \mathbb{C}$ such that g is continuously differentiable on $[a, b]$ (here 'differentiable' means differentiable on (a, b), right differentiable in a and left differentiable in b).

Let γ be a path. Choose a parametrization $g : [a, b] \to \mathbb{C}$ of γ. We call $g(a)$ the *start point* and $g(b)$ the *end point* of γ. Further, $g([a, b])$ is called the *support* of γ. By saying that a function is continuous on γ, or that γ is contained in a particular set, etc., we mean the support of γ.

Let γ be a path and $F : \gamma \to \mathbb{C}$ a continuous function on (the support of) γ. Then $F(\gamma)$ is the path such that if $g : [a, b] \to \mathbb{C}$ is a parametrization of γ then $F \circ g : [a, b] \to \mathbb{C}$ is a parametrization of $F(\gamma)$.

The path γ is said to be *closed* if its end point is equal to its start point, i.e., if $g(a) = g(b)$. The path γ is called *simple* if it has no self-intersections, other than its start point and end point if γ is closed. Finally, a closed, simple path is said to be *positively oriented* if it is traversed counterclockwise (we will not give the cumbersome formal definition of this intuitively obvious notion).

closed, not simple contour

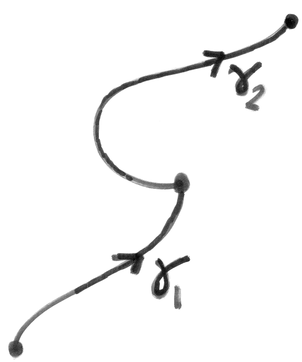


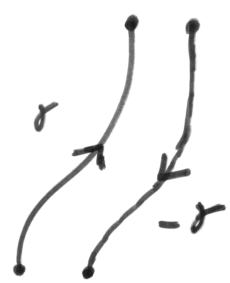
closed, simple contour

Let γ_1, γ_2 be paths, such that the end point of γ_1 is equal to the start point of γ_2. We define $\gamma_1 + \gamma_2$ to be the path obtained by first traversing γ_1 and then γ_2. For instance, if $g_1 : [a, b] \to \mathbb{C}$ is a parametrization of γ_1 then we may choose a parametrization $g_2 : [b, c] \to \mathbb{C}$ of γ_2; then $g : [a, c] \to \mathbb{C}$ defined by $g(t) := g_1(t)$ if $a \leqslant t \leqslant b$, $g(t) := g_2(t)$ if $b \leqslant t \leqslant c$ is a parametrization of $\gamma_1 + \gamma_2$.

This is easily extended to $\gamma_1 + \cdots + \gamma_r$, where first γ_1 is traversed, then γ_2, etc., and the end point of γ_i coincides with the start point of γ_{i+1}, for $i = 1, \ldots, r-1$.

Given a path γ, we define $-\gamma$ to be the path traversed in the opposite direction, i.e., the start point of $-\gamma$ is the end point of γ and conversely.





0.4.2 Definition of the contour integral

A *contour* is a piecewise smooth path, i.e., a path of the shape $\gamma_1 + \cdots + \gamma_r$ where $\gamma_1, \ldots, \gamma_r$ are smooth paths, such that the end point of γ_i coincides with the start point of γ_{i+1}, for $i = 1, \ldots, r-1$. We define integrals along contours.

All paths occurring in our course will be built up from circle segments and line segments, hence are contours.

First, let γ be a smooth path, and $f : \gamma \to \mathbb{C}$ a continuous function. Choose a continuously differentiable parametrization $g : [a, b] \to \mathbb{C}$ of γ. Then we define

$$\int_\gamma f(z)dz := \int_a^b f(g(t))g'(t)dt.$$

Further, we define the *length* of γ by

$$L(\gamma) := \int_a^b |g'(t)|dt.$$

These notions do not depend on the choice of g.

If $\gamma = \gamma_1 + \cdots + \gamma_r$ is a contour with smooth pieces $\gamma_1, \ldots, \gamma_r$, and $f : \gamma \to \mathbb{C}$ is continuous, then we define

$$\int_\gamma f(z)dz := \sum_{i=1}^r \int_{\gamma_i} f(z)dz$$

and

$$L(\gamma) := \sum_{i=1}^r L(\gamma_i).$$

In case that γ is closed, we write $\oint_\gamma f(z)dz$. It can be shown that the value of this integral is independent of the choice of the common start point and end point of γ.

We mention here that we can define more generally line integrals $\int_\gamma f(z)dz$ for paths γ that are not necessarily contours, i.e., not piecewise continuously differentiable. For contours, this new definition coincides with the one given above.

Let γ be any path and choose a parametrization $g : [a, b] \to \mathbb{C}$ of γ. A partition of $[a, b]$ is a tuple $P = (t_0, \ldots, t_s)$ where $a = t_0 < t_1 < \cdots < t_s = b$. We define the length of γ by

$$L(\gamma) := \sup_P \sum_{i=1}^s |g(t_i) - g(t_{i-1})|,$$

where the supremum is taken over all partitions P of $[a, b]$. This does not depend on the choice of g. We call γ *rectifiable* if $L(\gamma) < \infty$ (in another language, this means that the function g is of *bounded variation*).

Let γ be a rectifiable path, and $g : [a, b] \to \mathbb{C}$ a parametrization of γ. Given a partition $P = (t_0, \dots, t_s)$ of $[a, b]$, we define the mesh of P by

$$\delta(P) := \max_{1 \leqslant i \leqslant s} |t_i - t_{i-1}|.$$

A sequence of intermediate points of P is a tuple $W = (w_1, \dots, w_s)$ such that $t_0 < w_1 < t_1 < w_2 < t_2 < \dots < t_s$.

Let $f : \gamma \to \mathbb{C}$ be a continuous function. For a partition P of $[a, b]$ and a tuple of intermediate points W of P we define

$$S(f, g, P, W) := \sum_{i=1}^{s} f(g(w_i))(g(t_i) - g(t_{i-1})).$$

One can show that there is a finite number, denoted $\int_\gamma f(z)dz$, such that for any choice of parametrization $g : [a, b] \to \mathbb{C}$ of γ and any sequence $(P_n, W_n)_{n \geqslant 0}$ of partitions P_n of $[a, b]$ and sequences of intermediate points W_n of P_n with $\delta(P_n) \to 0$,

$$\int_\gamma f(z)dz = \lim_{n \to \infty} S(f, g, P_n, W_n).$$

In another language, $\int_\gamma f(z)dz$ is equal to the *Riemann-Stieltjes integral* $\int_a^b f(g(t))dg(t)$.

0.4.3 Properties of contour integrals

- Let γ be a contour, and $f : \gamma \to \mathbb{C}$ continuous. Then

$$\left| \int_\gamma f(z)dz \right| \leqslant L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|.$$

- Let γ_1, γ_2 be two contours such that the end point of γ_1 and the start point of γ_2 coincide. Let $f : \gamma_1 + \gamma_2 \to \mathbb{C}$ continuous. Then

$$\int_{\gamma_1 + \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

- Let γ be a contour and $f : \gamma \to \mathbb{C}$ be continuous. Then

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

- Let γ be a contour and $\{f_n : \gamma \to \mathbb{C}\}_{n=1}^{\infty}$ a sequence of continuous functions. Suppose that $f_n \to f$ uniformly on γ, i.e., $\sup_{z\in\gamma} |f_n(z) - f(z)| \to 0$ as $n \to \infty$. Then f is continuous on γ, and $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$ as $n \to \infty$.

- Call a function $F : U \to \mathbb{C}$ on an open subset U of $\mathbb{C}$ *analytic* if for every $z \in U$ the limit

$$F'(z) = \lim_{h\in\mathbb{C},\, h\to 0} \frac{F(z+h) - F(z)}{h}$$

  exists and is finite (much more on this later). Let γ be a contour with start point z_0 and end point z_1, and let F be an analytic function defined on an open set $U \subset \mathbb{C}$ that contains γ. Then

$$\int_{\gamma} F'(z)dz = F(z_1) - F(z_0).$$

- Let γ be a contour and F an analytic function defined on some open set containing γ. Further, let $f : F(\gamma) \to \mathbb{C}$ be continuous. Then

$$\int_{F(\gamma)} f(w)dw = \int_{\gamma} f(F(z))F'(z)dz.$$

We mention that all properties mentioned above can be generalized to line integrals along rectifiable paths, but in textbooks they are never proved in this generality.

Examples. **1.** Let $\gamma_{a,r}$ denote the circle with center a and radius r, traversed counterclockwise. For $\gamma_{a,r}$ we may choose a parametrization $t \mapsto a + re^{2\pi it}$, $t \in [0,1]$. Let $n \in \mathbb{Z}$. Then

$$\begin{aligned} \oint_{\gamma_{a,r}} (z-a)^n dz &= \int_0^1 r^n e^{2n\pi it} \cdot 2\pi i \cdot re^{2\pi it} dt \\ &= 2\pi i r^{n+1} \int_0^1 e^{2(n+1)\pi it} dt = \begin{cases} 2\pi i & \text{if } n = -1; \\ 0 & \text{if } n \neq -1. \end{cases} \end{aligned}$$

2. For $z_0, z_1 \in \mathbb{C}$, denote by $[z_0, z_1]$ the line segment from z_0 to z_1. For $[z_0, z_1]$ we may choose a parametrization $t \mapsto z_0 + t(z_1 - z_0)$, $t \in [0, 1]$. Let $f : [z_0, z_1] \to \mathbb{C}$ be continuous. Then

$$\int_{[z_0,z_1]} f(z)d(z) = \int_0^1 f(z_0 + t(z_1 - z_0))(z_1 - z_0)dt.$$

0.5 Topology

We recall some facts about the topology of $\mathbb{C}$.

0.5.1 Basic facts

Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$. We define the open disk and closed disk with center a and radius r,

$$D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}, \quad \overline{D}(a,r) := \{z \in \mathbb{C} : |z-a| \leqslant r\}.$$

Recall that a subset U of $\mathbb{C}$ is called *open* if either $U = \emptyset$, or for every $a \in U$ there is $\delta > 0$ with $D(a,\delta) \subset U$. A subset U of $\mathbb{C}$ is called *closed* if its complement $U^c = \mathbb{C}\backslash U$ is open. It is easy to verify that the union of any possibly infinite collection of open subsets of $\mathbb{C}$ is open. Further, the intersection of finitely many open subsets is open. Consequently, the intersection of any possibly infinite collection of closed sets is closed, and the union of finitely many closed subsets is closed.

A subset S of $\mathbb{C}$ is called *compact*, if for every collection $\{U_\alpha\}_{\alpha\in I}$ of open subsets of $\mathbb{C}$ with $S \subset \bigcup_{\alpha\in I} U_\alpha$ there is a finite subset F of I such that $S \subset \bigcup_{\alpha\in F} U_\alpha$, in other words, every open cover of S has a finite subcover.

By the *Heine-Borel Theorem*, a subset of $\mathbb{C}$ is compact if and only if it is closed and bounded.

Let U be a non-empty subset of $\mathbb{C}$. A point $z_0 \in \mathbb{C}$ is called a *limit point* of U if there is a sequence $\{z_n\}$ in U such that all z_n are distinct and $z_n \to z_0$ as $n \to \infty$. Recall that a non-empty subset U of $\mathbb{C}$ is closed if and only if each of its limit points belongs to U.

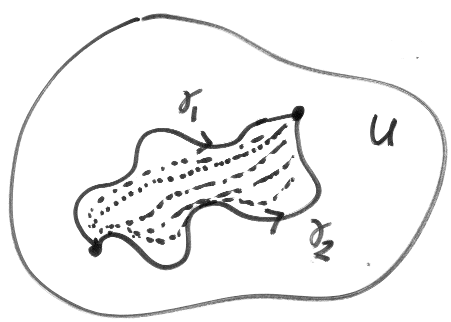
Let U be a non-empty open subset of $\mathbb{C}$, and $S \subset U$. Then S is called *discrete in* U if it has no limit points in U. Recall that by the *Bolzano-Weierstrass Theorem*, every infinite subset of a compact subset K of $\mathbb{C}$ has a limit point in K. This implies that S is discrete in U if and only if for every compact subset K of $\mathbb{C}$ with $K \subset U$, the intersection $K \cap S$ is finite.

Let U be a non-empty, open subset of $\mathbb{C}$. We say that U is *connected* if there are no non-empty open sets U_1, U_2 with $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. We say that U is *pathwise connected* if for every $z_0, z_1 \in U$ there is a path $\gamma \subset U$ with start point z_0

and end point z_1. A fact (typical for the topological space $\mathbb{C}$) is that a non-empty open subset U of $\mathbb{C}$ is connected if and only if it is pathwise connected.

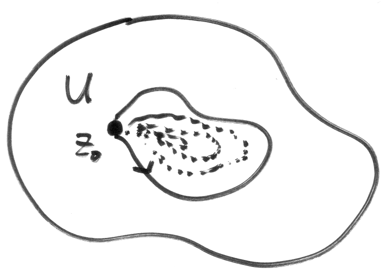
Let U be any, non-empty open subset of $\mathbb{C}$. We can express U as a disjoint union $\bigcup_{\alpha\in I} U_\alpha$, with I some index set, such that two points of U belong to the same set U_α if and only if they are connected by a path contained in U. The sets U_α are open, connected, and pairwise disjoint. We call these sets U_α the connected components of U.

0.5.2 Homotopy



Let $U \subseteq \mathbb{C}$ and γ_1, γ_2 two paths in U with start point z_0 and end point z_1. Then γ_1, γ_2 are homotopic in U if one can be continuously deformed into the other within U. More precisely this means the following: there are parametrizations $f : [0,1] \to \mathbb{C}$ of γ_1, $g : [0,1] \to \mathbb{C}$ of γ_2 and a continuous map $H : [0,1] \times [0,1] \to U$ with the following properties:

$$H(0,t) = f(t), \quad H(1,t) = g(t) \ \text{ for } 0 \leqslant t \leqslant 1;$$
$$H(s,0) = z_0, \quad H(s,1) = z_1 \ \text{ for } 0 \leqslant s \leqslant 1.$$

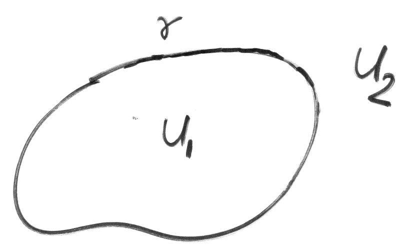


Let $U \subseteq \mathbb{C}$ be open and non-empty. We call U *simply connected* ('without holes') if it is connected and if every closed path in U can be contracted to a point in U, that is, if z_0 is any point in U and γ is any closed path in U containing z_0, then γ is homotopic in U to z_0.

A map $f : D_1 \to D_2$, where D_1, D_2 are subsets of $\mathbb{C}$, is called a *homeomorphism* if f is a bijection, and both f and f^{-1} are continuous. Homeomorphisms preserve topological properties of sets such as openness, closedness, compactness, (simple) connectedness, etc.

Theorem 0.5.1 (Schoenflies Theorem for curves). *Let γ be a closed, simple path in $\mathbb{C}$. Then there is a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\gamma_{0,1}) = \gamma$, where $\gamma_{0,1}$ is the unit circle with center 0 and radius 1, traversed counterclockwise.*

Corollary 0.5.2 (Jordan Curve Theorem). *Let γ be a closed, simple path in $\mathbb{C}$. Then $\mathbb{C} \setminus \gamma$ has two connected components, U_1 and U_2. The component U_1 is bounded and simply connected, while U_2 is unbounded.*



The component U_1 is called the *interior* of γ, notation $\mathrm{int}(\gamma)$, and U_2 the *exterior* of γ, notation $\mathrm{ext}(\gamma)$.

0.6 Complex analysis

We give an overview of the complex analysis that will be used during the course. We will need only the theorems and corollaries and the like, but not the proofs. For readers who have followed a course on complex analysis, most of this will be familiar. We hope that readers who did not follow such a course will gain sufficient confidence with complex analysis from reading these notes.

0.6.1 Basics

In what follows, U is a non-empty open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. We say that f is *holomorphic* or *analytic* in $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists and is finite.}$$

In this case, the limit is denoted by $f'(z_0)$. We say that f is analytic on U if f is analytic in every $z \in U$; in this case, the derivative $f'(z)$ is defined for every $z \in U$. We say that f is analytic around z_0 if it is analytic on some open disk $D(z_0, \delta)$ for some $\delta > 0$. Finally, given a not necessarily open subset A of $\mathbb{C}$ and a function $f : A \to \mathbb{C}$, we say that f is analytic on A if there is an open set $U \supseteq A$ such that f is defined on U and analytic on U. An everywhere analytic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.

For any two analytic functions f, g on some open set $U \subseteq \mathbb{C}$,we have the usual rules for differentiation $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$ and $(f/g)' = (gf' - fg')/g^2$ (the latter is defined for any z with $g(z) \neq 0$). Further, given a non-empty set $U \subseteq \mathbb{C}$, and analytic functions $f : U \to \mathbb{C}$, $g : f(U) \to \mathbb{C}$, the composition $g \circ f$ is analytic on U and $(g \circ f)' = (g' \circ f) \cdot f'$.

Recall that a power series around $z_0 \in \mathbb{C}$ is an infinite sum

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. The results on convergence/divergence and differentiation of power series over the complex numbers are completely similar to the corresponding results for real power series treated in a basic calculus course and the proofs are also the same. Thus, the radius of convergence of the power series $f(z)$ above is

$$R = R_f = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1},$$

and the series converges if $|z - z_0| < R_f$ and diverges if $|z - z_0| > R_f$. Further, we have the following:

Theorem 0.6.1. *Let $z_0 \in \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ a power series around $z_0 \in \mathbb{C}$ with radius of convergence $R > 0$. Then f defines a function on $D(z_0, R)$ which is analytic infinitely often. For $k \geqslant 0$ the k-th derivative $f^{(k)}$ of f has a power series expansion with radius of convergence R given by*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z - z_0)^{n-k}.$$

In particular, $a_k = f^{(k)}(z_0)/k!$.

In each of the examples below, R denotes the radius of convergence of the given power series.

$$\begin{aligned}
&e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, && R = \infty, \quad (e^z)' = e^z. \\
&\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, && R = \infty, \quad \cos' z = -\sin z. \\
&\sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, && R = \infty, \quad \sin' z = \cos z. \\
&(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, && R = 1, \quad ((1+z)^\alpha)' = \alpha(1+z)^{\alpha-1} \\
&\text{where } \alpha \in \mathbb{C}, \quad \binom{\alpha}{n} = \tfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}. && \\
&\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot z^n, && R = 1, \quad \log'(1+z) = (1+z)^{-1}.
\end{aligned}$$

0.6.2 Cauchy's Theorem and some applications

Recall that for a contour γ, say $\gamma = \gamma_1 + \cdots + \gamma_r$ where $\gamma_1, \ldots, \gamma_r$ are smooth paths with continuously differentiable parametrizations $g_i : [a_i, b_i] \to \mathbb{C}$, and for a continuous function $f : \gamma \to \mathbb{C}$ we have $\int_\gamma f(z)dz = \sum_{i=1}^{r} \int_{a_i}^{b_i} f(g_i(t))g_i'(t)dt$.

Theorem 0.6.2 (Cauchy). *Let $U \subseteq \mathbb{C}$ be a non-empty open set and $f : U \to \mathbb{C}$ an analytic function. Further, let γ_1, γ_2 be two contours in U with the same start point and end point that are homotopic in U. Then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. Any textbook on complex analysis. □

Corollary 0.6.3. *Let $U \subseteq \mathbb{C}$ be a non-empty, open, simply connected set, and $f : U \to \mathbb{C}$ an analytic function. Then for any closed contour γ in U,*

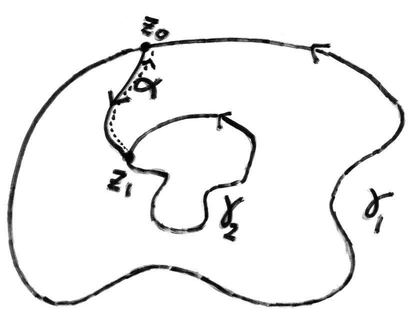
$$\oint_{\gamma} f(z)dz = 0.$$

Proof. The path γ is homotopic in U to a point, and a contour integral along a point is 0. □

Corollary 0.6.4. *Let γ_1, γ_2 be two closed, simple, positively oriented contours, such that γ_2 is contained in the interior of γ_1. Let $U \subset \mathbb{C}$ be an open set which contains γ_1, γ_2 and the region between γ_1 and γ_2. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then*

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

Proof.



Let z_0, z_1 be points on γ_1, γ_2 respectively, and let α be a path from z_0 to z_1 lying inside the region between γ_1 and γ_2 without self-intersections.

Then γ_1 is homotopic in U to the contour $\alpha + \gamma_2 - \alpha$, which consists of first traversing α, then γ_2, and then α in the opposite direction. Hence

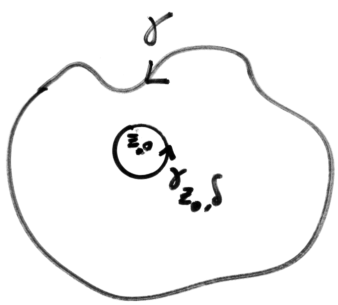
$$\oint_{\gamma_1} f(z)dz = \left(\int_\alpha + \oint_{\gamma_2} - \int_\alpha\right) f(z)dz = \oint_{\gamma_2} f(z)dz.$$

□

Corollary 0.6.5 (Cauchy's Integral Formula)**.** *Let γ be a closed, simple, positively oriented contour in $\mathbb{C}$, $U \subset \mathbb{C}$ an open set containing γ and its interior, z_0 a point in the interior of γ, and $f : U \to \mathbb{C}$ an analytic function. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0} \cdot dz = f(z_0).$$

Proof.



Let $\gamma_{z_0,\delta}$ be the circle with center z_0 and radius δ, traversed counterclockwise. Then by Corollary 0.6.4 we have for any sufficiently small $\delta > 0$,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0} \cdot dz = \frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}} \frac{f(z)}{z - z_0} \cdot dz.$$

Now, since $f(z)$ is continuous, hence uniformly continuous on any sufficiently small compact set containing z_0,

$$\left|\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0} \cdot dz \;-\; f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}} \frac{f(z)}{z - z_0} \cdot dz \;-\; f(z_0)\right|$$

$$= \left|\int_0^1 \frac{f(z_0 + \delta e^{2\pi i t})}{\delta e^{2\pi i t}} \cdot \delta e^{2\pi i t} dt \;-\; f(z_0)\right|$$

$$= \left|\int_0^1 \left\{f(z_0 + \delta e^{2\pi i t}) - f(z_0)\right\} dt\right| \leqslant \sup_{0 \leqslant t \leqslant 1} |f(z_0 + \delta e^{2\pi i t}) - f(z_0)|$$

$$\to 0 \quad \text{as } \delta \downarrow 0.$$

This completes our proof. $\square$

We now show that every analytic function f on a simply connected set has an anti-derivative. We first prove a simple lemma.

Lemma 0.6.6. *Let $U \subseteq \mathbb{C}$ be a non-empty, open, connected set, and let $f : U \to \mathbb{C}$ be an analytic function such that $f' = 0$ on U. Then f is constant on U.*

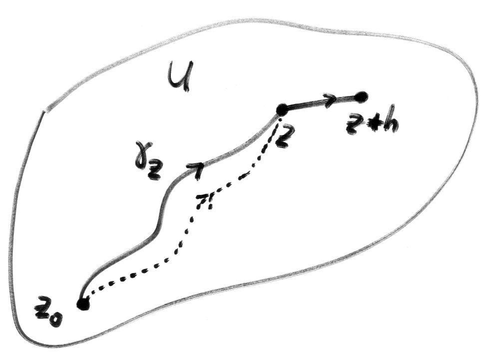
Proof. Fix a point $z_0 \in U$ and let $z \in U$ be arbitrary. Take a contour γ_z in U from z_0 to z which exists since U is (pathwise) connected. Then

$$f(z) - f(z_0) = \int_{\gamma_z} f'(w)dw = 0.$$

$\square$

Corollary 0.6.7. *Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected set, and $f : U \to \mathbb{C}$ an analytic function. Then there exists an analytic function $F : U \to \mathbb{C}$ with $F' = f$. Further, F is determined uniquely up to addition with a constant.*

Proof (sketch). If F_1, F_2 are any two analytic functions on U with $F_1' = F_2' = f$, then $F_1' - F_2'$ is constant on U since U is connected. This shows that an anti-derivative of f is determined uniquely up to addition with a constant. It thus suffices to prove the existence of an analytic function F on U with $F' = f$.



Fix $z_0 \in U$. Given $z \in U$, we define $F(z)$ by

$$F(z) := \int_{\gamma_z} f(w)dw,$$

where γ_z is any contour in U from z_0 to z. This does not depend on the choice of γ_z. For let γ_1, γ_2 be any two contours in U from z_0 to z. Then $\gamma_1 - \gamma_2$ (the contour consisting of first traversing γ_1 and then γ_2 in the opposite direction) is homotopic to z_0 since U is simply connected, hence

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \oint_{\gamma_1 - \gamma_2} f(z)dz = 0.$$

To prove that $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$, take a contour γ_z from z_0 to z and then the line segment $[z, z+h]$ from z to $z+h$. Then since f is uniformly continuous on any sufficiently small compact set around z,

$$\begin{aligned} F(z+h) - F(z) &= \Big(\int_{\gamma_z + [z,z+h]} - \int_{\gamma_z} \Big) f(w)dw = \int_{[z,z+h]} f(w)dw \\ &= \int_0^1 f(z+th)hdt = h\left(f(z) + \int_0^1 (f(z+th) - f(z))dt \right). \end{aligned}$$

So

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \int_0^1 (f(z+th) - f(z))dt \right| \\ &\leqslant \sup_{0\leqslant t\leqslant 1} |f(z+th) - f(z)| \to 0 \quad \text{as } h \to 0. \end{aligned}$$

This completes our proof. □

Example. Let $U \subset \mathbb{C}$ be a non-empty, open, simply connected subset of $\mathbb{C}$ with $0 \notin U$. Then $1/z$ has an anti-derivative on U.

For instance, if $U = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0\}$ we may take as anti-derivative of $1/z$,

$$\operatorname{Log} z := \log |z| + i \operatorname{Arg} z, \tag{0.6.1}$$

where $\operatorname{Arg} z$ is the argument of z in the interval $(-\pi, \pi)$ (this is called the *principal value* of the logarithm).

On $\{z \in \mathbb{C} : |z - 1| < 1\}$ we may take as anti-derivative of $1/z$ the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}. \tag{0.6.2}$$

On $\{z \in \mathbb{C} : |z - 1| < 1\}$ the functions given by (0.6.1) and (0.6.2) are equal since they are both anti-derivatives of $1/z$ and assume the value 0 at $z = 1$.

0.6.3 Taylor series

Theorem 0.6.8. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in U$ and $R > 0$ be such that $D(z_0, R) \subseteq U$. Then f has a* *unique Taylor series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{converging for } z \in D(z_0, R).$$

Further, we have for $n \in \mathbb{Z}_{\geqslant 0}$*,* $a_n = f^{(n)}(z_0)/n!$ *and*

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz \quad \textit{for any } r \textit{ with } 0 < r < R. \tag{0.6.3}$$

Proof. If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $z \in D(z_0, R)$, then according to Theorem 0.6.1, $a_k = f^{(k)}(z_0)/k!$ for $k \geqslant 0$. This shows that the coefficients a_k are determined by f. So if f has a Taylor expansion on $D(z_0, R)$, it is unique. We now show that such an expansion exists.

We fix $z \in D(z_0, R)$ and use w to indicate a complex variable. Choose r with $|z - z_0| < r < R$. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \cdot dw.$$

We rewrite the integrand. We have

$$\begin{aligned} \frac{f(w)}{w - z} &= \frac{f(w)}{(w - z_0) - (z - z_0)} = \frac{f(w)}{w - z_0} \cdot \left(1 - \frac{z - z_0}{w - z_0}\right)^{-1} \\ &= \frac{f(w)}{w - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n. \end{aligned}$$

The latter series converges uniformly on $\gamma_{z_0,r}$. For let $M := \sup_{w \in \gamma_{z_0,r}} |f(w)|$. Then

$$\sup_{w \in \gamma_{z_0,r}} \left| \frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n \right| \leqslant \frac{M}{r} \left(\frac{|z - z_0|}{r} \right)^n =: M_n$$

and $\sum_{n=0}^{\infty} M_n$ converges since $|z - z_0| < r$. Consequently,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \cdot dw \\ &= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \sum_{n=0}^{\infty} \left(\frac{f(w)}{(w - z_0)^{n+1}} \cdot (z - z_0)^n \right) dw \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{(w - z_0)^{n+1}} \cdot dw \right\}. \end{aligned}$$

Now Theorem 0.6.8 follows since by Corollary 0.6.4 the integral in (0.6.3) is independent of r. $\square$

Corollary 0.6.9. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set, and $f : U \to \mathbb{C}$ an analytic function. Then f is analytic on U infinitely often, that is, for every $k \geqslant 0$ the k-the derivative $f^{(k)}$ exists, and is analytic on U.*

Proof. Pick $z \in U$. Choose $\delta > 0$ such that $D(z, \delta) \subset U$. Then for $w \in D(z, \delta)$ we have

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n \quad \text{with } a_n = \frac{1}{2\pi i} \oint_{\gamma_{z,r}} \frac{f(w)}{(w-z)^{n+1}} \cdot dw \text{ for } 0 < r < \delta.$$

Now for every $k \geqslant 0$, the k-th derivative $f^{(k)}(z)$ exists and is equal to $k! a_k$. $\square$

Corollary 0.6.10. *Let γ be a closed, simple, positively oriented contour in $\mathbb{C}$, and U an open subset of $\mathbb{C}$ containing γ and its interior. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for every z in the interior of γ and every $k \geqslant 0$ we have*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$

Proof. Choose $\delta > 0$ such that $\gamma_{z,\delta}$ lies in the interior of γ. By Corollary 0.6.4,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \cdot dw = \frac{1}{2\pi i} \oint_{\gamma_{z,\delta}} \frac{f(w)}{(w-z)^{k+1}} \cdot dw.$$

By the argument in Corollary 0.6.9, this is equal to $f^{(k)}(z)/k!$. $\square$

We prove a generalization of Cauchy's integral formula.

Corollary 0.6.11. *Let γ_1, γ_2 be two closed, simple, positively oriented contours such that γ_1 is lying in the interior of γ_2. Let $U \subset \mathbb{C}$ be an open set which contains γ_1, γ_2 and the region between γ_1, γ_2. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for any z_0 in the region between γ_1 and γ_2 we have*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(z)}{z - z_0} dz.$$

Proof. We have seen that around z_0 the function f has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Define the function on U,

$$g(z) := \frac{f(z)-a_0}{z-z_0} \quad (z \neq z_0); \qquad g(z_0) := a_1.$$

The function g is clearly analytic on $U \setminus \{z_0\}$. Further, for z close to z_0 we have

$$\frac{g(z)-g(z_0)}{z-z_0} = \sum_{n=2}^{\infty} a_n(z-z_0)^{n-2} \to a_2 \text{ as } z \to z_0.$$

Hence g is also analytic at $z = z_0$. In particular, g is analytic in the region between γ_1 and γ_2. So by Corollary 0.6.4,

$$\oint_{\gamma_1} g(z)dz = \oint_{\gamma_2} g(z)dz.$$

Together with Corollaries 0.6.5, 0.6.4 this implies

$$\begin{aligned} f(z_0) = a_0 &= \frac{1}{2\pi i}\oint_{\gamma_2} \frac{a_0}{z-z_0}\cdot dz - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{a_0}{z-z_0}\cdot dz \\ &= \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(z)}{z-z_0}\cdot dz - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(z)}{z-z_0}\cdot dz. \end{aligned}$$

□

0.6.4 Isolated singularities, Laurent series, meromorphic functions

We define the *punctured disk* with center $z_0 \in \mathbb{C}$ and radius $r > 0$ by

$$D^0(z_0, r) := \{z \in \mathbb{C} : 0 < |z-z_0| < r\}.$$

If f is an analytic function defined on $D^0(z_0, r)$ for some $r > 0$, we call z_0 an *isolated singularity* of f. In case that there exists an analytic function g on the non-punctured disk $D(z_0, r)$ such that $g(z) = f(z)$ for $z \in D^0(z_0, r)$, we call z_0 a *removable* singularity of f. In this case, we also say that f is analytic at z_0.

Theorem 0.6.12. *Let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in \mathbb{C}$ and $R > 0$ be such that $D^0(z_0, R) \subseteq U$. Then f has a unique Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \quad \textit{converging for } z \in D^0(z_0, R).$$

Further, we have for $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz \quad \textit{for any } r \textit{ with } 0 < r < R. \tag{0.6.4}$$

Proof. We first show that if $f(z)$ has a Laurent series expansion as above on $D^0(z_0, R)$, then its coefficients a_n satisfy (0.6.4), and thus are uniquely determined by f. After that, we prove the existence of a Laurent series expansion.

Thus, suppose that $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ on $D^0(z_0, R)$. By definition of convergence of a doubly infinite series, this means that both $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$ converge on $D^0(z_0, R)$. Let $0 < r < R$. We show that the series converges uniformly to $f(z)$ on $\gamma_{z_0,r}$. Choose r_1, r_2 with $0 < r_1 < r < r_2 < R$. The series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges if $|z - z_0| = r_2$, so $|a_n| \cdot r_2^n \to 0$ as $n \to \infty$. Likewise, $\sum_{n=\infty}^{-1} a_n(z - z_0)^n$ converges if $|z - z_0| = r_1$, so $|a_n| \cdot r_1^n \to 0$ as $n \to -\infty$. Hence there is $M > 0$ such that $|a_n| \cdot r_2^n \leqslant M$ for $n \geqslant 0$ and $|a_n| \cdot r_1^n \leqslant M$ for $n < 0$. Now for $z \in \gamma_{z_0,r}$ we have

$$|a_n(z - z_0)^n| \leqslant M(r/r_2)^n =: M_n \text{ if } n \geqslant 0, \quad |a_n(z - z_0)^n| \leqslant M(r/r_1)^n =: M_n \text{ if } n < 0.$$

Now since $\sum_{n=-\infty}^{\infty} M_n$ converges, we know from Proposition 0.2.6 that the series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly to $f(z)$. This implies for $k \in \mathbb{Z}$,

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(z)}{(z - z_0)^{k+1}} dz &= \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \lim_{M,N \to \infty} \sum_{n=-M}^{N} a_n(z - z_0)^{n-k-1} dz \\
&= \frac{1}{2\pi i} \lim_{M,N \to \infty} \sum_{n=-M}^{N} a_n \oint_{\gamma_{z_0,r}} (z - z_0)^{n-k-1} dz = a_k,
\end{aligned}$$

where we have used that $\oint_{\gamma_{z_0,r}} (z - z_0)^{n-k-1} dz = 2\pi i$ if $n = k$ and 0 otherwise.

We now prove the existence of the Laurent series expansion. We fix $z \in D^0(z_0, R)$ and use w to denote a complex variable. Choose r_1, r_2 with $0 < r_1 < |z - z_0| < r_2 <$

R. By Corollary 0.6.11 we have

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r_2}} \frac{f(w)}{w-z}\cdot dw - \frac{1}{2\pi i}\oint_{\gamma_{z_0,r_1}} \frac{f(w)}{w-z}\cdot dw =: I_1 - I_2, \tag{0.6.5}$$

say. Completely similarly to Theorem 0.6.8, one shows that

$$I_1 = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{with } a_n = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r_2}} \frac{f(w)}{(w-z_0)^{n+1}}\cdot dw.$$

Notice that for w on the inner circle γ_{z_0,r_1} we have

$$\begin{aligned} \frac{f(w)}{w-z} &= \frac{f(w)}{(w-z_0)-(z-z_0)} = -\frac{f(w)}{z-z_0}\cdot\left(1-\frac{w-z_0}{z-z_0}\right)^{-1} \\ &= -\sum_{m=0}^{\infty} f(w)(z-z_0)^{-m-1}(w-z_0)^m. \end{aligned}$$

Similarly as above, one shows that the latter series converges uniformly to $f(w)/(w-z)$ on γ_{z_0,r_1}. After a substitution $n=-m-1$, it follows that

$$\begin{aligned} I_2 &= \frac{-1}{2\pi i}\oint_{\gamma_{z_0,r_2}} \left(\sum_{m=0}^{\infty} f(w)(w-w_0)^m(z-z_0)^{-m-1}\right)\cdot dw \\ &= -\sum_{n=-\infty}^{-1} a_n(z-z_0)^n, \quad \text{with } a_n = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r_1}} \frac{f(w)}{(w-z_0)^{n+1}}\cdot dw. \end{aligned}$$

By substituting the expressions for I_1, I_2 obtained above into (0.6.5), we obtain

$$f(z) = I_1 - I_2 = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$

This completes our proof. □

We say that a function f has Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ (or Taylor expansion if $a_n = 0$ for $n<0$) around z_0 if there is $r>0$ such that $f(z)$ is equal to this Laurent series on $D^0(z_0, r)$.

Let $z_0 \in \mathbb{C}$ and suppose f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

around z_0. Notice that z_0 is a removable singularity of f if $a_n = 0$ for all $n < 0$. We define the *order* of f at z_0 by

$$\mathrm{ord}_{z_0}(f) := \text{infimum of all } k \in \mathbb{Z} \text{ such that } a_k \neq 0,$$

so that in particular $\mathrm{ord}_{z_0}(f) = \infty$ if $f \equiv 0$.

Clearly, f is analytic at z_0 if and only if $\mathrm{ord}_{z_0}(f) \geqslant 0$. In case that $\mathrm{ord}_{z_0}(f)$ is finite, it is precisely the integer k such that $g(z) := (z - z_0)^{-k} f(z)$ defines a function that is analytic and non-zero in z_0.

The point z_0 is called
- an *essential singularity* of f if $\mathrm{ord}_{z_0}(f) = -\infty$;
- a *pole of order* k of f if $k > 0$ and $\mathrm{ord}_{z_0}(f) = -k$; a *simple pole* is one of order 1;
- a *zero of order* k of f if $k > 0$ and $\mathrm{ord}_{z_0}(f) = k$; a *simple zero* is one of order 1.

Notice that z_0 is a zero of order k of f if and only if $f^{(j)}(z_0) = 0$ for $j = 0, \ldots, k-1$, and $f^{(k)}(z_0) \neq 0$.

We say that a complex function f is *meromorphic around* z_0 if f is analytic on $D^0(z_0, r)$ for some $r > 0$, and z_0 is a pole or a removable singularity of f. The meromorphic functions around z_0 contain as a subclass the functions analytic around z_0, i.e., those that are analytic in z_0 or for which z_0 is a removable singularity.

If f is meromorphic around z_0 and not identically 0, then so is $1/f$. Indeed, there is $r > 0$ such that $f(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n$ on $D^0(z_0, r)$ with $a_k \neq 0$. We can write $f(z) = (z - z_0)^k h(z)$ with h analytic on $D(z_0, r)$ and $h(z_0) = a_k \neq 0$. By making r smaller we can achieve that $h(z) \neq 0$ on $D(z_0, r)$. We thus get $\frac{1}{f(z)} = (z - z_0)^{-k} \frac{1}{h(z)}$ with $1/h$ analytic and non-zero on $D(z_0, r)$, and so $1/f$ is meromorphic around z_0 and moreover $\mathrm{ord}_{z_0}(1/f) = -\mathrm{ord}_{z_0}(f)$.

It is obvious that if f, g are functions that are meromorphic around z_0 then so are $f + g$ and fg. Hence the functions meromorphic around z_0 form a *field*.

Lemma 0.6.13. *Let $z_0 \in \mathbb{C}$ and let f, g be two functions meromorphic around z_0. Then*

$$\begin{aligned} &\mathrm{ord}_{z_0}(f + g) \geqslant \min\left(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\right); \\ &\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g); \\ &\mathrm{ord}_{z_0}(f/g) = \mathrm{ord}_{z_0}(f) - \mathrm{ord}_{z_0}(g) \quad \textit{if } g \not\equiv 0. \end{aligned}$$

Proof. Exercise. □

For instance, if f, g are meromorphic functions around z_0, f has a pole of order k at z_0, g ha a zero of order l at z_0 and $l > k$, then fg is analytic around z_0, and fg has a zero of order $l - k$ at z_0.

Lemma 0.6.13 shows that the function ord_{z_0} defines a *discrete valuation* on the field of functions meromorphic around z_0. In general, a discrete valuation on a field K is a surjective map $v : K \to \mathbb{Z} \cup \{\infty\}$ such that $v(0) = \infty$; $v(x) \in \mathbb{Z}$ for $x \in K$, $x \neq 0$; $v(xy) = v(x)+v(y)$ for $x, y \in K$; and $v(x+y) \geqslant \min(v(x), v(y))$ for $x, y \in K$.

Other examples of discrete valuations are ord_p (p prime number) on $\mathbb{Q}$, given by $\mathrm{ord}_p(0) := \infty$ and $\mathrm{ord}_p(\alpha) := k$ if $\alpha = p^k a/b$, where k is an integer and a, b are integers not divisible by p.

Let U be a non-empty, open subset of $\mathbb{C}$. A meromorphic function on U is a complex function f with the following properties:
(i) there is a set S discrete in U such that f is defined and analytic on $U \setminus S$;
(ii) all elements of S are poles of f.

It is easy to verify that if f, g are meromorphic functions on U then so are $f + g$ and $f \cdot g$. It can be shown as well (less trivial) that if U is connected and g is a non-zero meromorphic function on U, then the set of zeros of g is discrete in U. The zeros of g are poles of $1/g$, and the poles of g are zeros of $1/g$. Hence $1/g$ is meromorphic on U. Consequently, if U is an open, connected subset of $\mathbb{C}$, then the functions meromorphic on U form a field.

0.6.5 Residues, logarithmic derivatives

Let $z_0 \in \mathbb{C}$, $R > 0$ and let $f : D^0(z_0, R) \to \mathbb{C}$ be an analytic function. Then f has a unique Laurent series expansion converging on $D^0(z_0, R)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We define the *residue of f at z_0* by

$$\operatorname{res}(z_0, f) := a_{-1}.$$

In particular, if f is analytic or has a removable singularity at z_0 then $\operatorname{res}(z_0, f) = 0$. By Theorem 0.6.12 we have

$$\operatorname{res}(z_0, f) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} f(z)dz$$

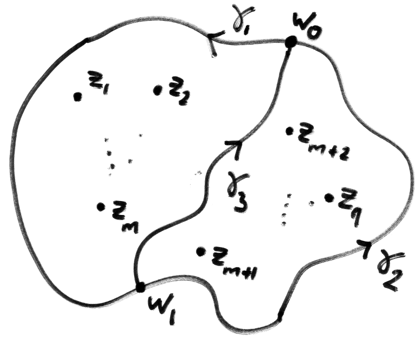
for any r with $0 < r < R$.

Theorem 0.6.14 (Residue Theorem)**.** *let γ be a closed, simple, positively oriented contour in $\mathbb{C}$ and let $z_1, \ldots, z_q$ be points in the interior of γ. Further, let f be a complex function that is analytic on an open set containing γ and the interior of γ minus $\{z_1, \ldots, z_q\}$. Then*

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)dz = \sum_{i=1}^{q} \operatorname{res}(z_i, f).$$

Proof. We proceed by induction on q. First let $q = 1$. Choose $r > 0$ such that $\gamma_{z_1,r}$ lies in the interior of γ. Then by Corollary 0.6.4,

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)dz = \frac{1}{2\pi i} \oint_{\gamma_{z_1,r}} f(z)dz = \operatorname{res}(z_1, f).$$



Now let $q > 1$ and assume the Residue Theorem is true for fewer than q points. We cut γ into two pieces, the piece γ_1 from a point w_0 to w_1 and the piece γ_2 from w_1 to w_0 so that $\gamma = \gamma_1 + \gamma_2$. Then we take a path γ_3 from w_1 to w_0 inside the interior of γ without self-intersections; this gives two contours $\gamma_1 + \gamma_3$ and $-\gamma_3 + \gamma_2$.

We choose γ_3 in such a way that it does not hit any of the points $z_1, \dots, z_q$ and both the interiors of these contours contain points from $z_1, \dots, z_q$. Without loss of generality, we assume that the interior of $\gamma_1+\gamma_3$ contains $z_1, \dots, z_m$ with $0 < m < q$, while the interior of $-\gamma_3 + \gamma_2$ contains $z_{m+1}, \dots, z_q$. Then by the induction hypothesis,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\gamma f(z)dz &= \frac{1}{2\pi i}\oint_{\gamma_1} f(z)dz + \frac{1}{2\pi i}\oint_{\gamma_2} f(z)dz \\
&= \frac{1}{2\pi i}\oint_{\gamma_1+\gamma_3} f(z)dz + \frac{1}{2\pi i}\oint_{-\gamma_3+\gamma_2} f(z)dz \\
&= \sum_{i=1}^{m}\mathrm{res}(z_i, f) + \sum_{i=m+1}^{q}\mathrm{res}(z_i, f) = \sum_{i=1}^{q}\mathrm{res}(z_i, f),
\end{aligned}$$

completing our proof. □

The next lemma gives some useful facts about residues. Both f, g are analytic functions on $D^0(z_0, r)$ for some $r > 0$.

Lemma 0.6.15. *(i) f has a pole of order 1 or removable singularity at z_0 with residue α*

$$\iff f(z) - \frac{\alpha}{z - z_0} \text{ is analytic around } z_0 \iff \lim_{z\to z_0}(z - z_0)f(z) = \alpha.$$

(ii) Suppose that f is analytic at z_0. Let k be a positive integer. Then $f/(z - z_0)^k$ has a pole of order at most k at $z = z_0$, and

$$\mathrm{res}(z_0, f/(z - z_0)^k) = \frac{f^{(k-1)}(z_0)}{(k-1)!}.$$

(iii) Suppose f has a pole of order 1 at z_0 and g is analytic and non-zero at z_0. Then fg has a pole of order 1 at z_0 and

$$\mathrm{res}(z_0, fg) = g(z_0)\mathrm{res}(z_0, f).$$

(iv) Suppose that f is analytic and non-zero at z_0 and g has a zero of order 1 at z_0. Then f/g has a pole of order 1 at z_0, and

$$\mathrm{res}(z_0, f/g) = f(z_0)/g'(z_0).$$

Proof. (i) Assume that f has a simple pole or removable singularity at $z = z_0$ with residue α. Then $f(z) = \frac{\alpha}{z-z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $D^0(z_0, r)$ and the two other assertions easily follow.

Conversely, suppose that $\lim_{z\to z_0}(z-z_0)f(z) = \alpha$. Recall that $f(z)$ has a Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ on $D^0(z_0, r)$. Let $h(z) := (z-z_0)f(z)-\alpha$; then $h(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$ on $D^0(z_0, r)$, where $b_n = a_{n-1}$ if $n \neq 0$ and $b_0 = a_{-1} - \alpha$. By Theorem 0.6.12, we can express the b_n as

$$b_n = \frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}} \frac{h(z)}{(z - z_0)^{n+1}}dz \quad \text{for } n \in \mathbb{Z},\ 0 < \delta < r.$$

Let $h(0) := 0$. Then $h(z)$ is continuous on $D(z_0, r)$, hence uniformly continuous on every compact subset of $D(z_0, r)$. Therefore, $\lim_{\delta\downarrow 0}\sup_{z\in\gamma_{z_0,\delta}} |h(z)| = 0$. Consequently, we have for $n \leqslant 0$, $0 < \delta < r$,

$$|b_n| \leqslant \frac{1}{2\pi} \cdot 2\pi\delta \cdot \sup_{z\in\gamma_{z_0,\delta}} \frac{|h(z)|}{|z - z_0|^{n+1}} \leqslant \delta^{|n|} \sup_{z\in\gamma_{z_0,\delta}} |h(z)| \to 0 \text{ as } \delta \downarrow 0.$$

This implies $b_n = 0$ for $n \leqslant 0$, hence $a_{-1} = \alpha$ and $a_n = 0$ for $n \leqslant -2$. As a consequence, $f(z) - \frac{\alpha}{z-z_0}$ is analytic around $z = z_0$, and so f either has a removable singularity (if $\alpha = 0$) or a simple pole with residue α at $z = z_0$.

(ii) $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ around z_0. Divide by $(z - z_0)^k$.

(iii) We have $\lim\limits_{z\to z_0}(z - z_0)f(z)g(z) = g(z_0)\lim\limits_{z\to z_0}(z - z_0)f(z)$. Apply (i).

(iv) We have

$$\lim_{z\to z_0} \frac{(z - z_0)f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

and by (i) this implies that $f(z)/g(z)$ has a simple pole at $z = z_0$ and $\mathrm{res}(z_0, f/g) = f(z_0)/g'(z_0)$. □

Example. We compute the residues of $f(z) = \dfrac{e^z}{(z-1)^3(z-2)^2}$ at $z = 1$, $z = 2$. Let $g_1(z) = \dfrac{e^z}{(z-2)^2}$ and $g_2(z) = \dfrac{e^z}{(z-1)^3}$. Then

$$g_1''(z) = \frac{(z^2 - 8z + 18)e^z}{(z-2)^4}, \quad g_2'(z) = \frac{(z-4)e^z}{(z-1)^4},$$

and thus, by (ii), $\mathrm{res}(1, f) = g_1''(1)/2! = \frac{11}{2}e$, $\mathrm{res}(2, f) = g_2'(2) = -2e^2$.

We deduce a useful consequence for integrals of rational functions.

Theorem 0.6.16. *Let p, q be two polynomials in $\mathbb{C}[X]$ such that $\deg q \geqslant \deg p + 2$ and q has no zeros on the real line. Let $z_1, \dots, z_m$ be the distinct zeros of q in the upper half plane. Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot dx = 2\pi i \sum_{j=1}^{m} \mathrm{res}(z_j, p/q).$$

Remark. We say that $\int_{-\infty}^{\infty} \dots$ converges and define $\int_{-\infty}^{\infty} \dots := \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} \dots$ provided the limit exists and is finite, where we let R_1, R_2 tend to ∞ independently of each other. If $\int_{-\infty}^{\infty} \dots$ converges then it is equal to $\lim_{R\to\infty} \int_{-R}^{R} \dots$. But conversely it may be that $\lim_{R\to\infty} \int_{-R}^{R} \dots$ exists and is finite while $\int_{-\infty}^{\infty} \dots$ diverges, e.g., $\lim_{R\to\infty} \int_{-R}^{R} x dx = 0$, while $\int_{-\infty}^{\infty} x dx$ is clearly divergent.

Proof of Theorem 0.6.16. Let $f(z) := p(z)/q(z)$. We first estimate $|f(z)|$ from above, for $z \in \mathbb{C}$. If $|z|$ is large, in $p(z)$ and $q(z)$ the highest powers of z dominate, which implies that there are $c_1, c_2 > 0$ such that

$$|f(z)| \leqslant c_1 |z|^{\deg p - \deg q} \leqslant c_1 |z|^{-2} \quad \text{for } z \in \mathbb{C} \text{ with } |z| \geqslant c_2.$$

This estimate implies that the integral under consideration converges absolutely, hence converges, and so it is equal to

$$\lim_{R\to\infty} \int_{-R}^{R} f(x) dx.$$

We compute the limit. For $R > 0$, let Γ_R be the closed, simple, positively orientend contour defined by first traversing from $-R$ to R along the real line, and then traversing from R to $-R$ along the upper semicircle with center 0 and radius R. For R sufficiently large, the poles of f in the interior of Γ_R are precisely $z_1, \dots, z_m$, so by the Residue Theorem,

$$\oint_{\Gamma_R} f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{res}(z_j, f).$$

On the other hand, letting C_R denote the upper semicircle with center 0 and radius R,

$$\oint_{\Gamma_R} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$$

and, for $R > c_2$,

$$\left| \int_{C_R} f(z)dz \right| \leqslant L(C_R) \cdot \sup_{z \in C_R} |f(z)| \leqslant \pi R \cdot c_1 R^{-2} \to 0 \quad \text{as } R \to \infty.$$

This implies our theorem. $\square$

Example. We compute $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^n}$ for any integer $n \geqslant 1$.
Notice that $f(z) = (z^2+1)^{-n}$ has only one pole in the upper half plane, namely at $z = i$. By the above theorem, the integral is equal to $2\pi i \cdot \mathrm{res}(i, f)$. To compute the residue, observe that $f(z) = g(z)/(z-i)^n$, where $g(z) = (z+i)^{-n}$. Hence by Lemma 0.6.15 (ii),

$$\begin{aligned}
\mathrm{res}(i, (z^2+1)^{-n}) &= \frac{g^{(n-1)}(i)}{(n-1)!} \\
&= \frac{(-n)(-n-1)\cdots(-n-n+2)}{(n-1)!}(z+i)^{-n-n+1}|_{z=i} \\
&= \binom{2n-2}{n-1}(-1)^{n-1}(2i)^{-2n+1} = \binom{2n-2}{n-1} 2^{-2n+1} i^{-1}.
\end{aligned}$$

The value of the integral is $2\pi i$ times this quantity, that is,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^n} = \binom{2n-2}{n-1} 2^{-2n+2}\pi.$$

Let U be a non-empty, open subset of $\mathbb{C}$ and f a meromorphic function on U which is not identically zero. We define the *logarithmic derivative* of f by

$$f'/f.$$

Suppose that U is simply connected and f is analytic and has no zeros on U. Then f'/f has an anti-derivative $h : U \to \mathbb{C}$. One easily verifies that $(e^h/f)' = 0$. Hence e^h/f is constant on U. By adding a suitable constant to h we can achieve that $e^h = f$. That is, we may view h as the logarithm of f, and f'/f as the derivative of this logarithm. But we will refer to f'/f as the logarithmic derivative of f also if U is not simply connected and/or f does have zeros or poles on U, although in that case it need not be the derivative of some function.

The following facts are easy to prove: if f, g are two meromorphic functions on U that are not identically zero, then

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}, \quad \frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}.$$

Further, if U is connected, then

$$\frac{f'}{f} = \frac{g'}{g} \Longleftrightarrow f = cg \text{ for some constant } c.$$

Lemma 0.6.17. *Let $z_0 \in \mathbb{C}$, $r > 0$ and let $f : D^0(z_0, r) \to \mathbb{C}$ be analytic. Assume that z_0 is either a removable singularity or a pole of f. Then z_0 is a simple pole or (if z_0 is neither a zero nor a pole of f) a removable singularity of f'/f, and*

$$\operatorname{res}(z_0, f'/f) = \operatorname{ord}_{z_0}(f).$$

Proof. Let $\operatorname{ord}_{z_0}(f) = k$. This means that $f(z) = (z - z_0)^k g(z)$ with g analytic around z_0 and $g(z_0) \neq 0$. Consequently,

$$\frac{f'}{f} = k\frac{(z - z_0)'}{z - z_0} + \frac{g'}{g} = \frac{k}{z - z_0} + \frac{g'}{g}.$$

The function g'/g is analytic around z_0 since $g(z_0) \neq 0$. So by Lemma 0.6.15, $\operatorname{res}(z_0, f'/f) = k$. □

Corollary 0.6.18. *Let γ be a closed, simple, positively oriented contour in $\mathbb{C}$, U an open subset of $\mathbb{C}$ containing γ and its interior, and f a meromorphic function on U. Assume that f has no zeros or poles on γ and let $z_1, \ldots, z_q$ be the zeros and poles of f inside γ. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \operatorname{ord}_{z_i}(f) = Z - P,$$

where Z, P denote the number of zeros and poles of f inside γ, counted with their multiplicities.

Proof. By Theorem 0.6.14 and Lemma 0.6.17 we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)} \cdot dz = \sum_{i=1}^{q} \mathrm{res}(z_i, f'/f) = \sum_{i=1}^{q} \mathrm{ord}_{z_i}(f) = Z - P.$$

□

0.6.6 Unicity of analytic functions

In this section we show that two analytic functions f, g defined on a connected open set U are equal on the whole set U, if they are equal on a sufficiently large subset of U.

We start with the following result.

Theorem 0.6.19. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ an analytic function. Assume that $f = 0$ on an infinite subset of U having a limit point in U. Then $f = 0$ on U.*

Proof. Our assumption that U is connected means, that any non-empty subset S of U that is both open and closed in U, must be equal to U.

Let Z be the set of $z \in U$ with $f(z) = 0$. Let S be the set of $z \in U$ such that z is a limit point of Z. By assumption, S is non-empty. Since f is continuous, we have $S \subseteq Z$. Any limit point in U of S is therefore a limit point of Z and so it belongs to S. Hence S is closed in U. We show that S is also open; then it follows that $S = U$ and we are done.

Pick $z_0 \in S$. We have to show that there is $\delta > 0$ such that $D(z_0, \delta) \subset S$. There is $\delta > 0$ such that f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converging on $D(z_0, \delta)$. Assume that f is not identically 0 on $D(z_0, \delta)$. Then not all coefficients a_n are 0. Assume that $a_m \neq 0$ and $a_n = 0$ for $n < m$, say. Then $f(z) = (z - z_0)^m h(z)$ with $h(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}$. Since $h(z_0) = a_m \neq 0$ and h is continuous, there is $\delta_1 > 0$ such that $h(z) \neq 0$ for all $z \in D(z_0, \delta_1)$. But then $f(z) \neq 0$ for all z with $0 < |z - z_0| < \delta_1$, contradicting that $z_0 \in S$.

Hence f is identically 0 on $D(z_0, \delta)$. Clearly, every point of $D(z_0, \delta)$ is a limit point of $D(z_0, \delta)$, hence of Z. So $D(z_0, \delta) \subset S$. This shows that indeed, S is open. $\square$

Corollary 0.6.20. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be an analytic function that is not identically 0 on U. Then the set of zeros of f in U is discrete in U, i.e., every compact subset of U contains only finitely many zeros of f.*

Proof. Suppose that some compact subset of U contains infinitely many zeros of f. Then by the Bolzano-Weierstrass Theorem, the set of these zeros would have a limit point in this compact set, implying that $f = 0$ on U. $\square$

Corollary 0.6.21. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and $f, g : U \to \mathbb{C}$ two analytic functions. Assume that $f = g$ on an infinite subset of U having a limit point in U. Then $f = g$ on U.*

Proof. Apply Theorem 0.6.19 to $f - g$. $\square$

Let U, V be open subsets of $\mathbb{C}$ with $U \subset V$. Let $f : U \to \mathbb{C}$ be an analytic function. An *analytic continuation* of f to V is an analytic function $g : V \to \mathbb{C}$ such that $g(z) = f(z)$ for $z \in U$.

Examples. 1. The function $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic on $\{z \in \mathbb{C} : |z| < 1\}$. It has an analytic continuation $\frac{1}{1-z}$ to $\mathbb{C} \setminus \{1\}$.

2. The function $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$ is analytic on $\{z \in \mathbb{C} : |z-1| < 1\}$. It has an analytic continuation $\operatorname{Log} z := \log|z| + i\operatorname{Arg} z$ to $\mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$. More generally, if V is any simply connected subset of $\mathbb{C}$ containing $\{z \in \mathbb{C} : |z-1| < 1\}$ but with $0 \notin V$ then it has an analytic continuation to V, namely the anti-derivative F of $1/z$ on V with $F(1) = 0$.

It is often a difficult problem to figure out whether an analytic continuation of U to a larger connected set V exists, and there is no general procedure to decide this. The next corollary shows that if such an analytic continuation exists, then it is unique.

Corollary 0.6.22 (Unicity of analytic continuations)**.** *Let U, V be non-empty, open subsets of $\mathbb{C}$, such that $U \subset V$ and V is connected. Let $f : U \to \mathbb{C}$ be an analytic function. Then f has at most one analytic continuation to V.*

Proof. Let g_1, g_2 be two analytic continuations of f to V. Then $g_1(z) = g_2(z) = f(z)$ for $z \in U$. Since U is open, every point in U is a limit point of U, hence of V. Therefore, $g_1(z) = g_2(z)$ for $z \in V$. $\square$

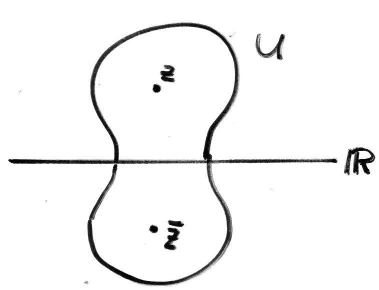
The next corollary states that under certain circumstances, analytic continuations of a function f to different sets can be glued together to a single continuation to the union of these sets.

Corollary 0.6.23. *Let U be a non-empty open subset of $\mathbb{C}$, and $\{V_i\}_{i\in I}$ with I any index set a collection of connected open subsets of $\mathbb{C}$ each of which contains U, and such that $V_i \cap V_j$ is connected for any two $i, j \in I$. Let f be an analytic function on U, and g_i an analytic continuation of f to V_i, for $i \in I$. Then $g_i = g_j$ holds on $V_i \cap V_j$ for any $i, j \in I$, and f has a unique analytic continuation to $\bigcup_{i\in I} V_i$, which coincides with g_i on V_i, for $i \in I$.*

Proof. If i, j are any two indices from I, then both g_i, g_j are analytic continuations of f to $V_i \cap V_j$, hence must be equal on $V_i \cap V_j$, since $V_i \cap V_j$ is assumed to be connected. Now define a function g on $V := \bigcup_{i\in I} V_i$ by $g(z) := g_i(z)$ if $z \in V_i$. If i, j are any two indices such that $z \in V_i$ and $z \in V_j$, then $g_i(z) = g_j(z)$, so this is well-defined. Further, g clearly coincides with f on U, and is analytic on V. $\square$

Another consequence of Theorem 0.6.19 is the so-called Schwarz' reflection principle, which implies that analytic functions assuming real values on the real line have nice symmetric properties.

Corollary 0.6.24 (Schwarz' reflection principle)**.**



Let U be an open, connected subset of $\mathbb{C}$, such that $U \cap \mathbb{R} \neq \emptyset$ and such that U is symmetric about $\mathbb{R}$, i.e., $\overline{z} \in U$ for every $z \in U$. Further, let $f : U \to \mathbb{C}$ be a non-identically zero analytic function with the property that

$$\{z \in U \cap \mathbb{R} : f(z) \in \mathbb{R}\}$$

has a limit point in U.

Then f has the following properties:

(i) $f(z) \in \mathbb{R}$ *for* $z \in U \cap \mathbb{R}$*;*
(ii) $\overline{f(\overline{z})} = f(z)$ *for* $z \in U$*;*
(iii) If z_0 *and* $r > 0$ *are such that* $D^0(z_0, r) \subset U$*, then* $\mathrm{ord}_{\overline{z_0}}(f) = \mathrm{ord}_{z_0}(f)$.

Proof. We first show that the function $z \mapsto \overline{f(\overline{z})}$ is analytic on U. Indeed, for $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)} = \overline{f'(\overline{z_0})}$$

exists.

Notice that for every $z \in U \cap \mathbb{R}$ with $f(z) \in \mathbb{R}$, we have $\overline{f(\overline{z})} = f(z)$. So by our assumption on f, the set of $z \in U$ with $\overline{f(\overline{z})} = f(z)$ has a limit point in U. Now Corollary 0.6.21 implies that $\overline{f(\overline{z})} = f(z)$ for $z \in U$. This implies (i) and (ii).

We finish with proving (iii). Our assumption implies that f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converging on $D^0(z_0, r)$. Then for $z \in D^0(\overline{z_0}, r)$ we have $\overline{z} \in D^0(z_0, r)$ and

$$f(z) = \overline{f(\overline{z})} = \overline{\left(\sum_{n=-\infty}^{\infty} a_n (\overline{z} - z_0)^n \right)} = \sum_{n=-\infty}^{\infty} \overline{a_n} (z - \overline{z_0})^n,$$

which clearly implies (iii). $\square$

0.6.7 Analytic functions defined by integrals

In analytic number theory, one often has to deal with complex functions that are defined by infinite series, infinite products, infinite integrals, or even worse, infinite integrals of infinite series. In this section we have collected some useful results that allow us to verify in a not too difficult manner that such complicated functions are analytic. Although all results we mention are well-known, we could not find a convenient reference for them, therefore we have included their not too exciting proofs.

We start with a general theorem on analytic functions defined by an integral, which will be frequently used in our course. In practical applications, condition (i) will always be taken for granted (in fact, in all our applications, f will be a Borel function, i.e., $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ will be Borel functions) and only (ii) and (iii) will be verified.

Theorem 0.6.25. *Let D be a measurable subset of $\mathbb{R}^m$, U an open subset of $\mathbb{C}$ and $f : D \times U \to \mathbb{C}$ a function with the following properties:*
(i) f is measurable on $D \times U$ (with U viewed as subset of $\mathbb{R}^2$);
(ii) for every fixed $x \in D$, the function $z \mapsto f(x,z)$ is analytic on U;
(iii) for every compact subset K of U there is a measurable function $M_K : D \to \mathbb{R}$ such that

$$|f(x,z)| \leqslant M_K(x) \text{ for } x \in D,\ z \in K, \qquad \int_D M_K(x)dx < \infty.$$

Then the function F given by

$$F(z) := \int_D f(x,z)dx$$

is analytic on U, and for every $k \geqslant 1$,

$$F^{(k)}(z) = \int_D f^{(k)}(x,z)dx,$$

where $f^{(k)}(x,z)$ denotes the k-th derivative with respect to z of the analytic function $z \mapsto f(x,z)$.

Proof. Fix $z \in U$. Choose $r > 0$ such that $\overline{D}(z,r) \subset U$, and let $0 < \delta < \frac{1}{2}r$. We show that F can be expanded into a Taylor series around z on $D(z,\delta)$; then it follows that F is analytic on $D(z,\delta)$ and so in particular in z. By assumption, there is a measurable function $M : D \to \mathbb{R}$ such that $|f(x,w)| \leqslant M(x)$ for $x \in D$, $w \in \overline{D}(z,r)$ and $\int_D M(x)dx < \infty$.

Let $w \in D(z,\delta)$. Then by Cauchy's integral formula (i.e., Corollary 0.6.5),

$$F(w) = \int_D f(x,w)dx = \int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \frac{f(x,\zeta)}{\zeta - w} \cdot d\zeta \right\} dx.$$

By inserting

$$\begin{aligned} \frac{f(x,\zeta)}{\zeta - w} &= \frac{f(x,\zeta)}{(\zeta - z) - (w - z)} = \frac{f(x,\zeta)}{\zeta - z}\left(1 - \frac{w-z}{\zeta - z}\right)^{-1} \\ &= \sum_{n=0}^{\infty} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} \cdot (w-z)^n \end{aligned}$$

we obtain

$$\begin{aligned} F(w) &= \int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \left(\sum_{n=0}^{\infty} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} (w-z)^n \right) d\zeta \right\} dx \\ &= \int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} (w-z)^n \right) dt \right\} dx. \end{aligned}$$

We want to interchange the summation with the two integrations, and require the Fubini-Tonelli Theorem to show that this is possible. We have to check that the conditions of that theorem are satisfied, i.e., that in the above expression for $F(w)$ we have absolute convergence. Note that since $|w - z| < \delta$ we have

$$\begin{aligned} &\int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} \left| \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} (w-z)^n \right| \right) dt \right\} dx \\ &\qquad \leqslant \int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} M(x) 2^{-n} \right) dt \right\} dx \leqslant \int_D 2M(x) dx < \infty, \end{aligned}$$

which shows that indeed, the conditions of the Fubini-Tonelli Theorem are satisfied. So in the expression for $F(w)$ derived above we can indeed interchange the summation and the two integrations and thus obtain

$$\begin{aligned} F(w) &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \left\{ \int_0^1 \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} dt \right\} dx \right) \\ &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} \cdot d\zeta \right\} dx \right) \\ &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \frac{f^{(n)}(x,z)}{n!} \cdot dx \right), \end{aligned}$$

where in the last step we have applied Corollary 0.6.10. This shows that indeed, F has a Taylor expansion around z converging on $D(z, \delta)$. So in particular, F is analytic in z. Further, $F^{(k)}(z)$ is equal to $k!$ times the coefficient of $(w - z)^k$, that is, $\int_D f^{(k)}(x, z) dx$. This proves our Theorem. □

We deduce a result, which states that under certain conditions, the pointwise limit of a sequence of analytic functions is again analytic.

Theorem 0.6.26. *Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging pointwise to a function f on U. Assume that for every compact subset K of U there is a constant $C_K < \infty$ such that*

$$|f_n(z)| \leqslant C_K \quad \textit{for all } z \in K,\ n \geqslant 0.$$

Then f is analytic on U, and $f_n^{(k)} \to f^{(k)}$ pointwise on U for all $k \geqslant 1$.

Proof. The set U can be covered by disks $D(z_0, \delta)$ with $z_0 \in U$, $\delta > 0$, such that the closed disk with center z_0 and radius 2δ, $\overline{D}(z_0, 2\delta)$ is contained in U. We fix such a disk $D(z_0, \delta)$ and prove that f is analytic on $D(z_0, \delta)$ and $f_n^{(k)} \to f^{(k)}$ pointwise on $D(z_0, \delta)$ for $k \geqslant 1$. This clearly suffices.

Let $z \in D(z_0, \delta)$, $k \geqslant 0$. Then by Corollary 0.6.10, we have

$$\begin{aligned} f_n^{(k)}(z) &= \frac{k!}{2\pi i} \oint_{\gamma_{z_0,2\delta}} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}} \cdot d\zeta \\ &= \int_0^1 k! \cdot \frac{f_n(z_0 + 2\delta e^{2\pi i t}) 2\delta e^{2\pi i t}}{(z_0 + 2\delta e^{2\pi i t} - z)^{k+1}} \cdot dt = \int_0^1 g_{n,k}(t, z) dt, \end{aligned}$$

say. By assumption, there is $C < \infty$ such that $|f_n(w)| \leqslant C$ for $w \in \overline{D}(z_0, 2\delta)$, $n \geqslant 0$. Further, for $t \in [0, 1]$ we have $|z_0 + 2\delta e^{2\pi i t} - z| > \delta$. Hence

$$|g_{n,k}(t, z)| \leqslant C \cdot k! \cdot 2\delta/\delta^{k+1} = 2C \cdot k!\delta^{-k} \quad \text{for } n, k \geqslant 0. \tag{0.6.6}$$

Notice that for $k \geqslant 0$, $t \in [0, 1]$, $z \in D(z_0, \delta)$ we have

$$\lim_{n\to\infty} g_{n,k}(t, z) = k! \cdot \frac{f(z_0 + 2\delta e^{2\pi i t}) 2\delta e^{2\pi i t}}{(z_0 + 2\delta e^{2\pi i t} - z)^{k+1}} = g^{(k)}(t, z),$$

where

$$g(t, z) := \frac{f(z_0 + 2\delta e^{2\pi i t}) 2\delta e^{2\pi i t}}{z_0 + 2\delta e^{2\pi i t} - z}$$

and $g^{(k)}(t, z)$ is the k-th derivative of the analytic function in z, $z \mapsto g(t, z)$.

Thanks to (0.6.6) we can apply the dominated convergence theorem, and obtain

$$\lim_{n\to\infty} f_n^{(k)}(z) = \int_0^1 g^{(k)}(t, z) dt \quad \text{for } z \in D(z_0, \delta),\ k \geqslant 0.$$

Applying this with $k = 0$ and using $f_n \to f$ pointwise, we obtain

$$f(z) = \int_0^1 g(t,z)dt \quad \text{for } z \in D(z_0, \delta).$$

It follows from Theorem 0.6.25 that the right-hand side, and hence f, is analytic on $D(z_0, \delta)$, and moreover,

$$f^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt \quad \text{for } z \in D(z_0, \delta),\ k \geqslant 1.$$

Indeed, $g(t,z)$ is measurable on $[0,1] \times D(z_0, \delta)$ and for every fixed t, the function $z \mapsto g(t,z)$ is analytic on $D(z_0, \delta)$. Further, by (0.6.6) and since $g_{n,0}(t,z) \to g(t,z)$, we have $|g(t,z)| \leqslant 2C$ for $t \in [0,1]$, $z \in D(z_0, \delta)$. So all conditions of Theorem 0.6.25 are satisfied.

Now it follows that

$$\lim_{n\to\infty} f_n^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt = f^{(k)}(z) \quad \text{for } z \in D(z_0, \delta),\ k \geqslant 1,$$

which is what we wanted to prove. □

Corollary 0.6.27. *Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging to a function f pointwise on U, and uniformly on every compact subset of U.*
Then f is analytic on U and $f_n^{(k)} \to f^{(k)}$ pointwise on U for every $k \geqslant 1$.

Proof. Take a compact subset K of U. Let $\varepsilon > 0$. Then there is N such that $|f_n(z) - f_m(z)| < \varepsilon$ for all $z \in K$, $m, n \geqslant N$. Choose $m \geqslant N$. Then there is $C > 0$ such that $|f_m(z)| \leqslant C$ for $z \in K$ since f_m is continuous. Hence $|f_n(z)| \leqslant C + \varepsilon$ for $z \in K$, $n \geqslant N$. Now our Corollary follows at once from Theorem 0.6.26. □

Corollary 0.6.28. *let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging to a function f pointwise on U and uniformly on every compact subset of U. Then*

$$\lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} = \frac{f'(z)}{f(z)}$$

for all $z \in U$ with $f(z) \neq 0$, where the limit is taken over those n for which $f_n(z) \neq 0$.

Proof. Obvious. □

Corollary 0.6.29. *Let $U \subset \mathbb{C}$ be a non-empty open set and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions. Assume that for every compact subset K of U there are reals $M_{n,K}$ such that $|f_n(z)| \leqslant M_{n,K}$ for $z \in K$ and $\sum_{n=0}^{\infty} M_{n,K}$ converges. Then*

(i) $\sum_{n=0}^{\infty} f_n$ is analytic on U, and $\left(\sum_{n=0}^{\infty} f_n\right)^{(k)} = \sum_{n=0}^{\infty} f_n^{(k)}$ for $k \geqslant 0$,

(ii) $\prod_{n=0}^{\infty}(1+f_n)$ is analytic on U.

Proof. Our assumption on the functions f_n implies that both the series $\sum_{n=0}^{\infty} f_n$ and the infinite product $\prod_{n=0}^{\infty}(1+f_n)$ converge uniformly on every compact subset of U (see Propositions 0.2.6 and 0.2.7). Now apply Corollary 0.6.27. □

Corollary 0.6.30. *Let U, $\{f_n\}_{n=0}^{\infty}$ be as in Corollary 0.6.29 and assume in addition that $f_n \neq -1$ on U for every $n \geqslant 0$. Then for the function $F = \prod_{n=0}^{\infty}(1+f_n)$ we have*

$$\frac{F'}{F} = \sum_{n=0}^{\infty} \frac{f_n'}{1+f_n}.$$

Proof. Let $F_m := \prod_{n=0}^{m}(1+f_n)$. Then $F_m \to F$ uniformly on every compact subset of U. Hence by Corollary 0.6.28,

$$\frac{F'}{F} = \lim_{m\to\infty} \frac{F_m'}{F_m} = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{f_n'}{1+f_n}$$

which clearly implies Corollary 0.6.30. □