Chapter 2

Dirichlet series and arithmetic functions

An *arithmetic function* is a function $f : \mathbb{Z}_{>0} \to \mathbb{C}$. To such a function we associate its *Dirichlet series*

$$L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

where s is a complex variable. It is common practice (although this doesn't make sense) to write $s = \sigma + it$, where $\sigma = \operatorname{Re} s$ and $t = \operatorname{Im} s$. It has shown very fruitful in number theory, to study an arithmetic function by means of its Dirichlet series. In this chapter, we prove some basic properties of Dirichlet series and arithmetic functions.

2.1 Dirichlet series

We want to develop a theory for Dirichlet series similar to that for power series. Every power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence R such that the series converges for all $z \in \mathbb{C}$ with $|z| < R$ and diverges for all $z \in \mathbb{C}$ with $|z| > R$. As we will see, a Dirichlet series $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has an *abscissa of convergence* $\sigma_0(f)$ such that the series converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma_0(f)$ and diverges for all $s \in \mathbb{C}$ with $\operatorname{Re} s < \sigma_0(f)$. For instance, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has abscissa of convergence 1.

We start with a simple summation result, which is extremely important in analytic number theory.

Theorem 2.1.1 (Partial summation, summation by parts)**.** *Let M, N be reals with $M < N$. Let $x_1, \ldots, x_r$ be real numbers with $M \leqslant x_1 < \cdots < x_r \leqslant N$, let $a(x_1), \ldots, a(x_r)$ be complex numbers, and put $A(t) := \sum_{x_k \leqslant t} a(x_k)$ for $t \in [M, N]$. Further, let $g : [M, N] \to \mathbb{C}$ be a differentiable function. Then*

$$\sum_{k=1}^{r} a(x_k) g(x_k) = A(N)g(N) - \int_M^N A(t) g'(t) dt.$$

Proof. Let $x_0 < M$ and put $A(x_0) := 0$. Then

$$\begin{aligned}\sum_{k=1}^{r} a(x_k) g(x_k) &= \sum_{k=1}^{r} (A(x_k) - A(x_{k-1})) g(x_k) \\ &= \sum_{k=1}^{r} A(x_k) g(x_k) - \sum_{k=1}^{r-1} A(x_k) g(x_{k+1}) \\ &= A(x_r) g(x_r) - \sum_{k=1}^{r-1} A(x_k)(g(x_{k+1}) - g(x_k)).\end{aligned}$$

Since $A(t) = A(x_k)$ for $x_k \leqslant t < x_{k+1}$ we have

$$A(x_k)(g(x_{k+1}) - g(x_k)) = \int_{x_k}^{x_{k+1}} A(t) g'(t) dt.$$

Hence

$$\begin{aligned}\sum_{k=1}^{r} a(x_k) g(x_k) &= A(x_r) g(x_r) - \sum_{k=1}^{r-1} \int_{x_k}^{x_{k+1}} A(t) g'(t) dt \\ &= A(x_r) g(x_r) - \int_{x_1}^{x_r} A(t) g'(t) dt.\end{aligned} \tag{2.1.1}$$

In case that $x_1 = M$, $x_r = N$ we are done. if $x_1 > M$, then $A(t) = 0$ for $M \leqslant t < x_1$ and thus, $\int_M^{x_1} A(t) g'(t) dt = 0$. If $x_r < N$, then $A(t) = A(x_r)$ for $x_r \leqslant t \leqslant N$, hence

$$\int_{x_r}^{N} A(t) g'(t) dt = A(N) g(N) - A(x_r) g(x_r).$$

Together with (2.1.1) this implies Theorem 2.1.1. □

Recall that an analytic (complex differentiable) function f on an open subset U of $\mathbb{C}$ is differentiable infinitely often (cf. Corollary 0.6.9 in the Prerequisites). We denote the k-th derivative of f by $f^{(k)}$. We recall the following theorem on sequences of analytic functions from the Prerequisites.

Theorem 0.6.26. *Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging pointwise to a function f on U. Assume that for every compact subset K of U there is a constant $C_K < \infty$ such that*

$$|f_n(z)| \leqslant C_K \quad \textit{for all } z \in K,\ n \geqslant 0.$$

Then f is analytic on U, and $f_n^{(k)} \to f^{(k)}$ pointwise on U for all $k \geqslant 1$.

From Theorems 2.1.1 and 0.6.26 we deduce the following important result on the convergence of Dirichlet series.

Theorem 2.1.2. *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arithmetic function with the property that there exists a constant $C > 0$ such that $|\sum_{n=1}^{N} f(n)| \leqslant C$ for every $N \geqslant 1$. Then $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges for every $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$.*
More precisely, on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ the function L_f is analytic, and for its k-th derivative we have

$$L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s}. \tag{2.1.2}$$

Proof. Notice that on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, the partial sums

$$L_{f,N}(s) := \sum_{n=1}^{N} f(n)n^{-s} = \sum_{n=1}^{N} f(n)e^{-s\log n} \quad (N = 1, 2, \dots)$$

are analytic, and $L_{f,N}^{(k)}(s) = \sum_{n=1}^{N} f(n)(-\log n)^k n^{-s}$ for $k \geqslant 0$. We have to show that the partial sums converge on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, and that analyticity and the formula for the k-th derivative are maintained if we let $N \to \infty$.

Let $s \in \mathbb{C}$, $\operatorname{Re} s > 0$. We first rewrite $L_{f,N}(s)$ using partial summation. Let $F(t) := \sum_{1 \leqslant n \leqslant t} f(n)$. By Theorem 2.1.1 (with $\{x_1, \dots, x_r\} = \{1, \dots, N\}$ and $g(t) = t^{-s}$) we have

$$L_{f,N}(s) = F(N)N^{-s} - \int_1^N F(t)(-s)t^{-s-1}dt = F(N)N^{-s} + s\int_1^N F(t)t^{-s-1}dt.$$

By assumption, there is $C > 0$ such that $|F(t)| \leqslant C$ for every $t \geqslant 1$. Further $|t^{-s-1}| = t^{-\mathrm{Re}\, s-1}$. Hence $|F(t)t^{-s-1}| \leqslant Ct^{-\mathrm{Re}\, s-1}$. Since $\mathrm{Re}\, s > 0$, the integral $\int_1^\infty t^{-\mathrm{Re}\, s-1}dt$ converges, therefore, $\int_1^\infty F(t)t^{-s-1}dt$ converges. Further, $|F(N)N^{-s}| \leqslant C \cdot N^{-\mathrm{Re}\, s} \to 0$ as $N \to \infty$. It follows that $L_f(s) = \lim_{N\to\infty} L_{f,N}(s)$ converges if $\mathrm{Re}\, s > 0$.

We apply Theorem 0.6.26 to the sequence of partial sums $\{L_{f,N}(s)\}$. Let K be a compact subset of $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$. There are $\sigma > 0, A > 0$ such that $\mathrm{Re}\, s \geqslant \sigma$, $|s| \leqslant A$ for $s \in K$. Thus, for $s \in K$ and $N \geqslant 1$, we have

$$\begin{aligned} |L_{f,N}(s)| &\leqslant |F(N)N^{-s}| + |s| \int_1^N |F(t)t^{-s-1}|dt \\ &\leqslant C \cdot N^{-\sigma} + A \int_1^N C \cdot t^{-\sigma-1}dt = C \cdot N^{-\sigma} + AC \cdot \sigma^{-1}(1 - N^{-\sigma}) \\ &\leqslant C + AC \cdot \sigma^{-1}, \end{aligned}$$

which is an upper bound independent of s, N.

Now Theorem 0.6.26 implies that for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$, the series $L_f(s) = \lim_{N\to\infty} L_{f,N}(s)$ is analytic and moreover,

$$L_f^{(k)}(s) = \lim_{N\to\infty} L_{f,N}^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s}.$$

□

Corollary 2.1.3. *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arithmetic function and let $s_0 \in \mathbb{C}$ be such that $\sum_{n=1}^\infty f(n)n^{-s_0}$ converges. Then for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \mathrm{Re}\, s_0$ the function L_f converges and is analytic, and*

$$L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s} \text{ for } k \geqslant 1. \tag{2.1.2}$$

Proof. Write $s = s' + s_0$. Then $\mathrm{Re}\, s' > 0$ if $\mathrm{Re}\, s > \mathrm{Re}\, s_0$. There is $C > 0$ such that $|\sum_{n=1}^N f(n)n^{-s_0}| \leqslant C$ for all N. Apply Theorem 2.1.2 to $\sum_{n=1}^\infty (f(n)n^{-s_0})n^{-s'}$. □

Theorem 2.1.4. *There exists a number $\sigma_0(f)$ with $-\infty \leqslant \sigma_0(f) \leqslant \infty$ such that $L_f(s)$ converges for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \sigma_0(f)$ and diverges for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s < \sigma_0(f)$.*

Moreover, if $\sigma_0(f) < \infty$*, then for* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > \sigma_0(f)$ *the function* L_f *is analytic, and*

$$L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s} \text{ for } k \geqslant 1. \tag{2.1.2}$$

Proof. If there is no $s \in \mathbb{C}$ for which $L_f(s)$ converges we have $\sigma_0(f) = \infty$. Assume that $L_f(s)$ converges for some $s \in \mathbb{C}$ and define

$$\sigma_0(f) := \inf\{\sigma : \exists s \in \mathbb{C} \text{ such that } \operatorname{Re} s = \sigma,\ L_f(s) \text{ converges}\}.$$

Clearly, $L_f(s)$ diverges if $\operatorname{Re} s < \sigma_0(f)$. To prove that $L_f(s)$ converges for $\operatorname{Re} s > \sigma_0(f)$, take such s and choose s_0 such that $\sigma_0(f) < \operatorname{Re} s_0 < \operatorname{Re} s$ and $L_f(s_0)$ converges. By Corollary 2.1.3, L_f is convergent and analytic in s, and for $L_f^{(k)}(s)$ we have expression (2.1.2). □

The number $\sigma_0(f)$ is called the *abscissa of convergence* of L_f.

There exists also a real number $\sigma_a(f)$, called the *abscissa of absolute convergence* of L_f such that $L_f(s)$ converges absolutely if $\operatorname{Re} s > \sigma_a(f)$, and does not converge absolutely if $\operatorname{Re} s < \sigma_a(f)$.

In fact, we have $\sigma_a(f) = \sigma_0(|f|)$, that is the abscissa of convergence of $L_{|f|}(s) = \sum_{n=1}^{\infty} |f(n)|n^{-s}$. For write $\sigma = \operatorname{Re} s$. Then $\sum_{n=1}^{\infty} |f(n)n^{-s}| = \sum_{n=1}^{\infty} |f(n)|n^{-\sigma}$ converges if $\sigma > \sigma_0(|f|)$ and diverges if $\sigma < \sigma_0(|f|)$.

Theorem 2.1.5. *For every arithmetic function* $f : \mathbb{Z}_{>0} \to \mathbb{C}$ *we have* $\sigma_0(f) \leqslant \sigma_a(f) \leqslant \sigma_0(f) + 1$.

Proof. It is clear that $\sigma_0(f) \leqslant \sigma_a(f)$. To prove $\sigma_a(f) \leqslant \sigma_0(f) + 1$, we have to show that $L_f(s)$ converges absolutely if $\operatorname{Re} s > \sigma_0(f) + 1$.

Take such s; then $\operatorname{Re} s = \sigma_0(f) + 1 + \varepsilon$ with $\varepsilon > 0$. Put $\sigma := \sigma_0(f) + \varepsilon/2$, so that $\operatorname{Re} s = \sigma + 1 + \varepsilon/2$. The series $\sum_{n=1}^{\infty} f(n)n^{-\sigma}$ converges, hence there is a constant C such that $|f(n)n^{-\sigma}| \leqslant C$ for all n. Therefore,

$$|f(n)n^{-s}| = |f(n)| \cdot n^{-\operatorname{Re} s} = |f(n)n^{-\sigma}| \cdot n^{-1-\varepsilon/2} \leqslant Cn^{-1-\varepsilon/2}$$

for $n \geqslant 1$. The series $\sum_{n=1}^{\infty} n^{-1-\varepsilon/2}$ converges, hence $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ converges. □

Exercise 2.1. *Show that there exist arithmetic functions f such that $\sigma_a(f) = \sigma_0(f) + 1$.*

The next theorem implies that an arithmetic function is uniquely determined by its Dirichlet series.

Theorem 2.1.6. *Let $f, g : \mathbb{Z}_{>0} \to \mathbb{C}$ be two arithmetic functions for which there is $\sigma \in \mathbb{R}$ such that $L_f(s)$, $L_g(s)$ converge absolutely and $L_f(s) = L_g(s)$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$. Then $f = g$.*

Proof. Let $h := f - g$. Our assumptions imply that $L_h(s)$ converges absolutely, and $L_h(s) = 0$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$. We have to prove that $h = 0$.

Assume that there are positive integers n with $h(n) \neq 0$, and let m be the smallest such n. Then for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$ we have

$$h(m)m^{-s} = -\sum_{n=m+1}^{\infty} h(n)n^{-s}.$$

Let $\sigma_1 > \sigma$, and let $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma_1$. Then

$$\begin{aligned} |h(m)| &\leqslant \sum_{n=m+1}^{\infty} |h(n)|(m/n)^{\operatorname{Re} s} = \sum_{n=m+1}^{\infty} |h(n)|(m/n)^{\sigma_1}(m/n)^{\operatorname{Re} s - \sigma_1} \\ &\leqslant m^{\sigma_1}\left(\sum_{n=m+1}^{\infty} |h(n)| \cdot n^{-\sigma_1}\right) \cdot (m/(m+1))^{\operatorname{Re} s - \sigma_1}. \end{aligned}$$

The series between the parentheses is convergent, hence a finite number. So the right-hand side tends to 0 as $\operatorname{Re} s \to \infty$. This contradicts that $h(m) \neq 0$. □

2.2 Arithmetic functions

A *multiplicative* function is an arithmetic function f such that $f \not\equiv 0$ and $f(mn) = f(m)f(n)$ for all positive integers m, n with $\gcd(m, n) = 1$. A *strongly multiplicative* function is an arithmetic function f with the property that $f \not\equiv 0$ and $f(mn) = f(m)f(n)$ for all integers m, n.

Notation. In expressions $p_1^{k_1} \cdots p_t^{k_t}$ it is always assumed that the p_i are distinct prime numbers, and the k_i positive integers.

We start with some simple observations.

Lemma 2.2.1. *(i) Let f be a multiplicative function. Then $f(1) = 1$. Further, if $n = p_1^{k_1} \cdots p_t^{k_t}$, then $f(n) = f(p_1^{k_1}) \cdots f(p_t^{k_t})$.*

(ii) Let f, g be two multiplicative functions such that $f(p^k) = g(p^k)$ for every prime p and $k \in \mathbb{Z}_{\geqslant 1}$. Then $f = g$.

(iii) Let f, g be two strongly multiplicative functions such that $f(p) = g(p)$ for every prime p. Then $f = g$.

Proof. Obvious. □

We define the *convolution product* $f * g$ of two arithmetic functions f, g by

$$(f * g)(n) := \sum_{d|n} f(n/d)g(d) \quad \text{for } n \in \mathbb{Z}_{>0},$$

where $'d \mid n'$ means that the sum is taken over all positive divisors of n.

Examples. Define the arithmetic functions e, E by

$$e(1) = 1, \; e(n) = 0 \text{ for all } n \in \mathbb{Z}_{>1},$$
$$E(n) = 1 \text{ for all } n \in \mathbb{Z}_{>0}.$$

Clearly, e is multiplicative, and E is strongly multiplicative. If f is any arithmetic function, then $e * f = f$, while

$$(E * f)(n) = \sum_{d|n} f(d).$$

Lemma 2.2.2. *(i) For any two arithmetic functions f, g we have $f * g = g * f$.*
(ii) For any three arithmetic functions f, g, h we have $(f * g) * h = f * (g * h)$.

Proof. Straightforward. □

Theorem 2.2.3. *(i) Let $\mathcal{A}$ be the set of arithmetic functions f with $f(1) \neq 0$. Then $\mathcal{A}$ with $*$ is an abelian group with unit element e.*
(ii) Let $\mathcal{M}$ be the set of multiplicative functions. Then $\mathcal{M}$ with $*$ is a subgroup of $\mathcal{A}$.

Proof. (i) We know already that $*$ is commutative and associative and that e is the unit element of $*$. It remains to verify that every element of $\mathcal{A}$ has a (necessarily unique) inverse with respect to $*$. Let $f \in \mathcal{A}$. Notice that for an arithmetic function g we have

$$\begin{aligned} f * g = e \iff & f(1)g(1) = 1, \quad \sum_{d|n} f(n/d)g(d) = 0 \text{ for } n > 1 \\ \iff & g(1) := f(1)^{-1}, \quad g(n) := -f(1)^{-1} \sum_{d|n,\, d<n} f(n/d)g(d) \text{ for } n > 1. \end{aligned}$$

Clearly, the function g can be defined inductively by these last two relations. This shows that f has an inverse with respect to $*$.

(ii) We first have to verify that the convolution product of two multiplicative functions is again multiplicative. Here we use that if m, n are two coprime integers and d is a positive divisor of mn, then d has a unique decomposition $d = d_1d_2$ where d_1 is a positive divisor of m and d_2 a positive divisor of n. Now let $f, g \in \mathcal{M}$ and let m, n be two coprime positive integers. Then

$$\begin{aligned} (f * g)(mn) &= \sum_{d|mn} f(mn/d)g(d) = \sum_{d_1|m,\, d_2|n} f(mn/d_1d_2)g(d_1d_2) \\ &= \sum_{d_1|m} \sum_{d_2|n} f(m/d_1)f(n/d_2)g(d_1)g(d_2) \\ &= \left(\sum_{d_1|m} f(m/d_1)g(d_1) \right) \cdot \left(\sum_{d_2|n} f(n/d_2)g(d_2) \right) \\ &= (f * g)(m) \cdot (f * g)(n). \end{aligned}$$

This shows that $f * g \in \mathcal{M}$.

It remains to show that the inverse of a multiplicative function is again multiplicative. Let $f \in \mathcal{M}$ and let f^{-1} be its inverse with respect to $*$. Define h by

$$\begin{aligned} & h(p^k) := f^{-1}(p^k) \ \text{ for any prime power } p^k, \\ & h(n) := h(p_1^{k_1}) \cdots h(p_t^{k_t}) \ \text{ if } n = p_1^{k_1} \cdots p_t^{k_t}. \end{aligned}$$

Then h is multiplicative, and $(f * h)(p^k) = (f * f^{-1})(p^k) = e(p^k)$ for every prime power p^k. Both $f * h$ and e are multiplicative, so in fact $f * h = e$, and thus, $h = f^{-1}$. This shows that indeed, f^{-1} is multiplicative. □

Example. The *Möbius function* μ is the inverse under $*$ of E, where $E(n) = 1$ for all n.

Lemma 2.2.4. *We have*

$$\mu(n) = \begin{cases} (-1)^t & \text{if } n = p_1 \cdots p_t \text{ with } p_1, \dots, p_t \text{ distinct primes,} \\ 0 & \text{if } n \text{ is divisible by the square of a prime.} \end{cases}$$

Proof. We first compute μ at the prime powers. First, $\mu(1) = 1$. Further, for every prime p and positive integer k one has

$$0 = e(p^k) = \sum_{d|p^k} E(p^k/d)\mu(d) = \mu(1) + \mu(p) + \cdots + \mu(p^k).$$

From these relations one reads off that $\mu(p) = -1$ and $\mu(p^2) = \mu(p^3) = \cdots = 0$. The expression for $\mu(n)$ for arbitrary positive integers n follows by using that μ is multiplicative. □

Theorem 2.2.5 (Möbius' Inversion Formula)**.** *Let f be an arithmetic function. Define $F(n) := \sum_{d|n} f(n)$ for $n \in \mathbb{Z}_{>0}$. Then*

$$f(n) = \sum_{d|n} \mu(n/d)F(d) \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Proof. We have $F = E * f$. Hence

$$\mu * F = \mu * (E * f) = (\mu * E) * f = e * f = f.$$

□

Examples. **1)** Define $\varphi(n) := |\{k \in \mathbb{Z} : 1 \leqslant k \leqslant n, \gcd(k,n) = 1\}|$. It is well-known that $\sum_{d|n} \varphi(d) = n$ for $n \in \mathbb{Z}_{>0}$. This implies that

$$\varphi(n) = \sum_{d|n} \mu(n/d)d,$$

or $\varphi = \mu * I_1$, where we define $I_\alpha(n) = n^\alpha$ for $n \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{C}$. As a consequence, φ is multiplicative, and for $n = p_1^{k_1} \cdots p_t^{k_t}$ we have

$$\varphi(n) = \prod_{i=1}^{t} \varphi(p_i^{k_i}) = \prod_{i=1}^{t} (p_i^{k_i} - p_i^{k_i-1}).$$

2) Let $\alpha \in \mathbb{C}$ and define $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for $n \in \mathbb{Z}_{>0}$. Then $\sigma_\alpha = E * I_\alpha$, which implies that σ_α is multiplicative. Hence for $n = p_1^{k_1} \cdots p_t^{k_t}$ we have

$$\sigma_\alpha(n) = \prod_{i=1}^{t} \sigma_\alpha(p_i^{k_i}) = \begin{cases} \displaystyle\prod_{i=1}^{t} \frac{p_i^{\alpha(k_i+1)-1}}{p_i^\alpha - 1} & \text{if } \alpha \neq 0, \\ \displaystyle\prod_{i=1}^{t} (k_i + 1) & \text{if } \alpha = 0. \end{cases}$$

The function $\sigma_0(n)$ is usually denoted by $\tau(n)$, and $\sigma_1(n)$ by $\sigma(n)$.

2.3 Convolution product vs. Dirichlet series

We investigate the relation between the convolution product of two arithmetic functions and their associated Dirichlet series.

Theorem 2.3.1. *Let f, g be two arithmetic functions. Let $s \in \mathbb{C}$ be such that $L_f(s)$ and $L_g(s)$ converge absolutely.*
Then also $L_{f*g}(s)$ converges absolutely, and $L_{f*g}(s) = L_f(s)L_g(s)$.

Proof. Since both $L_f(s)$ and $L_g(s)$ are absolutely convergent we can rearrange their product as a double series and then rearrange the terms:

$$\begin{aligned} &\Big(\sum_{m=1}^{\infty} f(m)m^{-s}\Big)\Big(\sum_{n=1}^{\infty} g(n)n^{-s}\Big) \\ &\quad = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m)g(n)(mn)^{-s} = \sum_{k=1}^{\infty}\Big(\sum_{mn=k} f(m)g(n)\Big)k^{-s} \\ &\quad = \sum_{k=1}^{\infty}(f*g)(k)k^{-s} = L_{f*g}(s). \end{aligned}$$

We now show that $L_{f*g}(s)$ converges absolutely:

$$\begin{aligned} \sum_{k=1}^{\infty} |(f*g)(k)k^{-s}| \;&\leqslant\; \sum_{k=1}^{\infty}\Big(\sum_{mn=k} |f(m)|\cdot|g(n)|\Big)\cdot|k^{-s}| \\ &= \;\Big(\sum_{m=1}^{\infty} |f(m)m^{-s}|\Big)\Big(\sum_{n=1}^{\infty} |g(n)n^{-s}|\Big) < \infty \end{aligned}$$

by following the above reasoning in opposite direction and taking absolute values everywhere. This completes our proof. □

We define $\sum_p(\cdots) = \lim_{N\to\infty}\sum_{p\leqslant N}(\cdots)$, and $\prod_p(\cdots) = \lim_{N\to\infty}\prod_{p\leqslant N}(\cdots)$ where the sums and products are taken over the primes.

Theorem 2.3.2. *Let f be a multiplicative function. let $s \in \mathbb{C}$ be such that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely. Then*

$$L_f(s) = \prod_p \Big(\sum_{j=0}^{\infty} f(p^j)p^{-js} \Big). \tag{2.3.1}$$

Further, $L_f(s) \neq 0$ as soon as $\sum_{j=0}^{\infty} f(p^j)p^{-js} \neq 0$ for every prime p.

Proof. The series $L_p(s) := \sum_{j=0}^{\infty} f(p^j)p^{-js}$ (p prime) converge absolutely, since $\sum_{j=0}^{\infty} |f(p^j)p^{-js}| \leqslant \sum_{n=1}^{\infty} |f(n)n^{-s}| < \infty$. To deal with their product we apply Proposition 0.2.5. We have

$$\sum_p |L_p(s) - 1| \leqslant \sum_p \sum_{j=1}^{\infty} |f(p^j)p^{-js}| \leqslant \sum_{n=1}^{\infty} |f(n)n^{-s}| < \infty,$$

hence the infinite product $\prod_p L_p(s)$ is defined, and it is 0 if and only if at least one of the factors $L_p(s)$ is 0.

It remains to prove that $L_f(s) = \prod_p L_p(s)$. Let $N > 1$ and let $p_1, \ldots, p_t$ be the prime numbers $\leqslant N$. Further, let S_N be the set of integers composed of prime numbers $\leqslant N$ and T_N the set of remaining integers, i.e., divisible by at least one prime $> N$. Since the series $L_p(s)$ (p prime) converge absolutely, we can rearrange terms and obtain

$$\prod_{p\leqslant N} L_p(s) = \sum_{j_1,\ldots,j_t \geqslant 0} f(p_1^{j_1}) \cdots f(p_t^{j_t})(p_1^{-j_1} \cdots p_t^{-j_t})^s = \sum_{n\in S_N} f(n)n^{-s}.$$

Now clearly,

$$\begin{aligned} \Big| L_f(s) - \prod_{p\leqslant N} L_p(s) \Big| &= \Big| \sum_{n=1}^{\infty} f(n)n^{-s} - \sum_{n\in S_N} f(n)n^{-s} \Big| = \Big| \sum_{n\in T_N} f(n)n^{-s} \Big| \\ &\leqslant \sum_{n=N+1}^{\infty} |f(n)n^{-s}| \ \to 0 \ \text{ as } N \to \infty. \end{aligned}$$

This proves (2.3.1). □

Corollary 2.3.3. *Let f be a strongly multiplicative function. Let $s \in \mathbb{C}$ be such that $L_f(s)$ converges absolutely. Then*

$$L_f(s) = \prod_p \frac{1}{1 - f(p)p^{-s}}.$$

Further, $L_f(s) \neq 0$.

Proof. Use that

$$\sum_{j=0}^{\infty} f(p^j)p^{-js} = \sum_{j=0}^{\infty} (f(p)p^{-s})^j = \frac{1}{1 - f(p)p^{-s}}$$

and that all factors $(1 - f(p)p^{-s})^{-1}$ are $\neq 0$. □

Examples. 1) For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1} \quad \text{(Euler)}.$$

2) For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, the series $L_\mu(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ converges absolutely, hence

$$\zeta(s)L_\mu(s) = \sum_{n=1}^{\infty} (E * \mu)(n)n^{-s} = \sum_{n=1}^{\infty} e(n)n^{-s} = 1.$$

That is, $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. An alternative way to prove this, is to observe that

$$\zeta(s)^{-1} = \prod_p (1 - p^{-s}) = \prod_p \Big(\sum_{j=0}^{\infty} \mu(p^j)p^{-js} \Big) = \sum_{n=1}^{\infty} \mu(n)n^{-s}.$$

3) Recall that $\varphi = \mu * I_1$. The series $L_{I_1}(s) = \sum_{n=1}^{\infty} n/n^s = \zeta(s-1)$ converges absolutely for $\operatorname{Re} s > 2$. Hence

$$\sum_{n=1}^{\infty} \varphi(n)n^{-s} = L_{\varphi(s)} = L_\mu(s)L_{I_1}(s) = \zeta(s-1)/\zeta(s)$$

and $L_\varphi(s)$ converges absolutely if $\operatorname{Re} s > 2$.

4) The (very important) *von Mangoldt function* Λ is defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

n	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$	0	$\log 2$	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0

For $n = p_1^{k_1} \cdots p_t^{k_t}$ (unique prime factorization) we have

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{t} \sum_{j=1}^{k_i} \log p_i = \sum_{i=1}^{t} k_i \log p_i = \log n.$$

Hence $E * \Lambda = \log$, where log denotes the arithmetic function $n \mapsto \log n$. So $\Lambda = \mu * \log$.

Lemma 2.3.4. *For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, the series $\sum_{n=1}^{\infty} \Lambda(n) n^{-s}$ converges absolutely, and*

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\zeta'(s)/\zeta(s).$$

Proof. We apply Theorem 2.3.1. First recall that $L_\mu(s)$ converges absolutely if $\operatorname{Re} s > 1$. Further, by Theorem 2.1.4, we have $\zeta'(s) = \sum_{n=1}^{\infty} (-\log n) n^{-s}$ for $\operatorname{Re} s > 1$. Hence

$$\sum_{n=1}^{\infty} |\log(n) n^{-s}| = \sum_{n=1}^{\infty} (\log n) n^{-\operatorname{Re} s} = -\zeta'(\operatorname{Re} s)$$

converges if $\operatorname{Re} s > 1$. That is, $L_{\log}(s)$ converges absolutely if $\operatorname{Re} s > 1$. It follows that

$$L_\Lambda(s) = L_\mu(s) L_{\log}(s) = -\zeta(s)^{-1} \zeta'(s)$$

and $L_\Lambda(s)$ converges absolutely if $\operatorname{Re} s > 1$. □

2.4 Exercises

Exercise 2.2. *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arithmetic function.*

a) Suppose that there are $C > 0$ and $\sigma \geqslant 0$ such that $\left|\sum_{n \leqslant x} f(n)\right| \leqslant C \cdot x^\sigma$ for all $x \geqslant 1$. Prove that $L_f(s)$ converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$.

b) Let $\sigma \geqslant 0$ and suppose that $L_f(s)$ converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geqslant \sigma$. Prove that there is $C > 0$ such that $\left|\sum_{n \leqslant x} f(n)\right| \leqslant C \cdot x^\sigma$ for all $x \geqslant 1$.
Hint. *Apply partial summation. Write $f(n) = \alpha(n) n^\sigma$ where $\alpha(n) = f(n) n^{-\sigma}$.*

Exercise 2.3. *Let* $f : \mathbb{Z}_{>0} \to \mathbb{C}$ *be a periodic arithmetic function, i.e., there is an integer* $q \geqslant 1$ *such that* $f(n+q) = f(n)$ *for all* $n \geqslant 1$. *Further suppose that* f *is not identically* 0. *Prove that* $\sigma_0(f) = 0$ *if* $\sum_{n=1}^{q} f(n) = 0$ *and* $\sigma_0(f) = 1$ *if* $\sum_{n=1}^{q} f(n) \neq 0$.

Exercise 2.4. *a) Let* k *be a positive integer. For a positive integer* n *define* $A_k(n) := 1$ *if* n *is* k*-th power free, i.e., not divisible by* p^k *for some prime* p, *and* $A_k(n) = 0$ *if* n *is not* k*-th power free. Prove that*

$$\sum_{n=1}^{\infty} A_k(n) n^{-s} = \frac{\zeta(s)}{\zeta(ks)} \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

Hint. *Write both the left-hand side and right-hand side as infinite products* $\prod_p (\ldots)$ *and show that the factors in these products are equal.*

b) Denote by $\Omega(n)$ *the number of prime powers dividing* n, *i.e., if* $n = p_1^{k_1} \cdots p_t^{k_t}$, *then* $\Omega(n) = k_1 + \cdots + k_t$. *Prove that*

$$\sum_{n=1}^{\infty} (-1)^{\Omega(n)} n^{-s} = \frac{\zeta(2s)}{\zeta(s)} \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

Exercise 2.5. *Let* $f : \mathbb{Z}_{>0} \to \mathbb{C}$ *be a strongly multiplicative arithmetic function such that* $|f(p)| \leqslant 1$ *for every prime* p. *Prove that for* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > 1$, *the series* $L_f(s)$ *converges absolutely and that*

$$L_f(s)^{-1} = \sum_{n=1}^{\infty} \mu(n) f(n) n^{-s}, \quad \frac{L_f'(s)}{L_f(s)} = -\sum_{n=1}^{\infty} \Lambda(n) f(n) n^{-s}.$$

Exercise 2.6. *Let* $F : [1, \infty) \to \mathbb{C}$ *be any function and define* $G(x) := \sum_{n \leqslant x} F(x/n)$ *for* $x \geqslant 1$. *Prove that* $F(x) = \sum_{n \leqslant x} \mu(n) G(x/n)$ *for* $x \geqslant 1$.

Exercise 2.7. *Denote by* $\tau(n)$ *the number of divisors of a positive integer* n. *Recall that if* $n = p_1^{k_1} \cdots p_t^{k_t}$, *where* $p_1, \ldots, p_t$ *are distinct primes and* $k_1, \ldots, k_t$ *positive integers, then* $\tau(n) = \prod_{i=1}^{t} (k_i + 1)$.

a) Let $0 < \varepsilon < 1$. *Prove that* $\tau(n) \leqslant 2 \cdot (1/\varepsilon)^{\pi(2^{1/\varepsilon})} \cdot n^{\varepsilon}$.

Hint. *For a prime* p *and a positive integer* k, *prove using basic calculus that*

$$\frac{k+1}{p^{\varepsilon k}} \leqslant \begin{cases} (e\varepsilon \log p)^{-1} p^{\varepsilon} & \text{for every prime } p, \\ 1 & \text{if } p^{\varepsilon} \geqslant 2. \end{cases}$$

b) *Prove that* $\log\tau(n) = O\big(\log n/\log\log n\big)$ *as* $n\to\infty$.
Hint. *Apply a) and choose for* ε *an appropriate decreasing function of* n.

Using the Prime Number Theorem, you may try to prove that

$$\limsup_{n\to\infty}\frac{\log\tau(n)}{\log n/\log\log n} = \log 2.$$

Exercise 2.8. *Define the arithmetic functions* f, g *by*

$$f(n) = (-1)^{n-1} \ \text{for } n\in\mathbb{Z}_{>0}, \quad g(n) = \begin{cases} 1 & \text{for } n\in\mathbb{Z}_{>0},\ n\not\equiv 0 \,(\mathrm{mod}\, 3), \\ -2 & \text{for } n\in\mathbb{Z}_{>0},\ n\equiv 0\,(\mathrm{mod}\, 3). \end{cases}$$

a) *Show that* $\sigma_0(f) = \sigma_0(g) = 0$.

b) *Show that* $L_f(s) = (1-2^{1-s})\zeta(s)$, $L_g(s) = (1-3^{1-s})\zeta(s)$ *for all* $s\in\mathbb{C}$ *with* $\mathrm{Re}\, s > 1$.

c) *Using a) and b), prove that* $\zeta(s)$ *has an analytic continuation to the set* $\{s\in\mathbb{C} : \mathrm{Re}\, s > 0\}\setminus\{1\}$, *with a simple pole with residue* 1 *at* $s=1$, *i.e., if this analytic continuation is also denoted by* $\zeta(s)$, *then* $\lim_{s\to 1}(s-1)\zeta(s) = 1$.
Hint. *Prove that* $\{s\in\mathbb{C} :\ 2^{1-s} = 1\}\cap\{s\in\mathbb{C} :\ 3^{1-s}=1\} = \{1\}$ *(both sets have infinitely many elements!) and apply Corollary 0.6.23 from the Prerequisites. As for the pole at* $s=1$, *use that* $\sum_{n=1}^{\infty}(-1)^{n-1}/n = \log 2$.

Exercise 2.9. *Recall that a positive integer* n *is called* square-free *if it is not divisible by* p^2 *for some prime* p. *The purpose of this exercise is to show that the number of square-free integers up to* x *is equal to* $\frac{1}{\zeta(2)}x + O(\sqrt{x}) = \frac{6}{\pi^2}x + O(\sqrt{x})$ *as* $x\to\infty$. *Define* $f(n) := 1$ *if* n *is square-free, and* $f(n) := 0$ *otherwise.*

a) *Prove that* $f(n) = \sum_{d^2|n}\mu(d)$, *where the summation is over all positive integers* d *such that* d^2 *divides* n.

b) *Prove that for all reals* $x\geqslant 1$ *we have*

$$\sum_{n\leqslant x} f(n) = \sum_{d\leqslant\sqrt{x}}\mu(d)[x/d^2] = x\sum_{d\leqslant\sqrt{x}}\mu(d)d^{-2} - \sum_{d\leqslant\sqrt{x}}\mu(d)\{x/d^2\},$$

where $[a]$ *denotes the largest integer* $\leqslant a$, *and* $\{a\} := a - [a]$.
Hint. *Substitute a) and interchange the summation.*

c) *Prove that*

$$\sum_{n \leqslant x} f(n) = \frac{1}{\zeta(2)} x + O(\sqrt{x}) \quad \text{as } x \to \infty.$$

Hint. *Estimate* $\sum_{d > \sqrt{x}} d^{-2}$ *using an integral.*

Exercise 2.10. *For a positive real* x*, denote by* $F(x)$ *the number of pairs of integers* (a, b) *with* $1 \leqslant a, b \leqslant x$ *and* $\gcd(a, b) = 1$.

a) *Prove that* $F(x) = \sum_{n \leqslant x} \mu(n)[x/n]^2$.

b) *Prove that* $F(x) = \frac{1}{\zeta(2)} x^2 + O(x \log x)$ *as* $x \to \infty$ *(consequently, the probability that two randomly chosen positive integers* $\leqslant x$ *are coprime, converges to* $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$ *as* $x \to \infty$*).*

Hint. *Use the tricks from Exercise 2.9, and also that* $\sum_{n \leqslant x} \frac{1}{n} = O(\log x)$ *as* $x \to \infty$.