Chapter 5

Tauberian theorems

5.1 Introduction

In 1826, Abel proved the following result for real power series. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with coefficients $a_n \in \mathbb{R}$ that converges on the real interval $(-1, 1)$. Assume that $\sum_{n=0}^{\infty} a_n$ converges. Then $\lim_{x \uparrow 1} f(x)$ exists, and in fact,

$$\lim_{x \uparrow 1} f(x) = \sum_{n=0}^{\infty} a_n.$$

In general, the converse is not true, i.e., if $\lim_{x \uparrow 1} f(x)$ exists one can not conclude that $\sum_{n=0}^{\infty} a_n$ converges. For instance, if $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, then $\lim_{x \uparrow 1} f(x) = \frac{1}{2}$, but $\sum_{n=0}^{\infty} (-1)^n$ diverges.

In 1897, Tauber proved a converse to Abel's Theorem, but under an additional hypothesis. Let again $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with real coefficients converging on $(-1, 1)$. Assume that

$$\lim_{x \uparrow 1} f(x) =: \alpha \text{ exists,} \tag{5.1.1}$$

and moreover,

$$\lim_{n \to \infty} n a_n = 0. \tag{5.1.2}$$

Then

$$\sum_{n=0}^{\infty} a_n \text{ converges and is equal to } \alpha. \tag{5.1.3}$$

Tauber's result led to various other "Tauberian theorems," which are are all of the following shape:
- suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $x \in \mathbb{R}$ with $|x| < 1$;
- suppose one knows something about the behaviour of $f(x)$ as $x \uparrow 1$ (such as (5.1.1));
- further suppose one knows something about the growth of a_n as $n \to \infty$ (such as (5.1.2));
- then one can conclude something about the convergence of $\sum_{n=0}^{\infty} a_n$ (such as (5.1.3)).

There is now a very general "Tauberian theory," which is about Tauberian theorems for functions defined by integrals. These include as special cases Tauberian theorems for power series and Dirichlet series.

We will prove a Tauberian theorem for Laplace transforms

$$G(z) := \int_0^{\infty} F(t) e^{-zt} dt,$$

where $F : [0, \infty) \to \mathbb{C}$ is a 'decent' function and z is a complex variable. This Tauberian theorem has the following shape.
- Assume that the integral converges for $\operatorname{Re} z > 0$;
- assume that one knows something about the limiting behaviour of $G(z)$ as $\operatorname{Re} z \downarrow 0$;
- assume that one knows something about the growth order of F;
- then one can conclude something about the convergence of $\int_0^{\infty} F(t) dt$.

With some modifications, we may view power series as special cases of Laplace transforms. Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series converging for $|x| < 1$. Define the function $F(t)$ on $[0, \infty)$ by

$$F(t) := a_n \ \text{ if } n \leqslant t < n+1 \ \ (n \in \mathbb{Z}_{\geqslant 0}).$$

Then if $\operatorname{Re} z > 0$,

$$\begin{aligned} \int_0^{\infty} F(t) e^{-zt} dt &= \sum_{n=0}^{\infty} \int_n^{n+1} F(t) e^{-zt} dt = \sum_{n=0}^{\infty} a_n \int_n^{n+1} e^{-zt} dt \\ &= \sum_{n=0}^{\infty} a_n \frac{1}{z} \big(e^{-nz} - e^{-(n+1)z} \big) \\ &= \frac{1 - e^{-z}}{z} \sum_{n=0}^{\infty} a_n e^{-nz}. \end{aligned}$$

Hence

$$g(e^{-z}) = \frac{z}{1-e^{-z}} \int_0^\infty F(t)e^{-zt}dt \quad \text{if } \operatorname{Re} z > 0.$$

Later, we show how a Dirichlet series can be expressed in terms of a Laplace transform.

Around 1930, Wiener developed a general Tauberian theory, which is now part of functional analysis. From this, in 1931, Ikehara deduced a Tauberian theorem for Dirichlet series (now known as the *Wiener-Ikehara Theorem*), with which one can give simple proofs of the Prime Number Theorem and various generalizations thereof. In 1980, Newman published a new method to derive Tauberian theorems, based on a clever contour integration and avoiding any functional analysis. This was developed further by Korevaar.

Using the ideas of Newman and Korevaar, we prove a Tauberian theorem for Laplace transforms, and deduce from this a weaker version of the Wiener-Ikehara theorem. This weaker version suffices for a proof of the Prime Number Theorem for arithmetic progressions. In Section 3.3 of Jameson's book you may find proofs along the same lines of variations on the Tauberian theorems we are proving here.

Literature: G.J.O. Jameson, *The Prime Number Theorem,* Cambridge University Press 2003, London Mathematical Society, Student Texts 53,
J. Korevaar, *Tauberian Theory, A century of developments,* Springer Verlag 2004, Grundlehren der mathematischen Wissenschaften, vol. 329.
J. Korevaar, *The Wiener-Ikehara Theorem by complex analysis,* Proceedings of the American Mathematical Society, vol. 134, no. 4 (2005), 1107–1116.

5.2 A Tauberian theorem for Laplace transforms

Lemma 5.2.1. *Let $F : [0,\infty) \to \mathbb{C}$ be a measurable function. Further, assume there is a constant M such that*

$$|F(t)| \leqslant M \quad \textit{for } t \geqslant 1.$$

Then

$$G(z) := \int_0^\infty F(t)e^{-zt}dt$$

converges, and defines an analytic function on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proof. We apply Theorem 0.6.25. We check that the conditions of that theorem are satisfied. Let $U := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. First, $F(t)e^{-zt}$ is measurable on $[0, \infty) \times U$. Second, for every fixed $t \in [0, \infty)$, the function $z \mapsto F(t)e^{-tz}$ is analytic on U. Third, let K be a compact subset of U. Then there is $\delta > 0$ such that $\operatorname{Re} z \geqslant \delta$ for $z \in K$, and thus,

$$|F(t)e^{-zt}| \leqslant Me^{-\delta t} \quad \text{for } z \in K.$$

The integral $\int_0^\infty M \cdot e^{-\delta t} dt$ converges. So indeed, all conditions of Theorem 0.6.25 are satisfied and thus, by that Theorem, $G(z)$ is analytic on U. □

We are now ready to state our Tauberian theorem.

Theorem 5.2.2. *Let $F : [0, \infty) \to \mathbb{C}$ be a function with the following properties:*
(i) F is measurable;
(ii) there is $M > 0$ such that $|F(t)| \leqslant M$ for all $t \geqslant 0$;
(iii) there is an analytic function $G(z)$ on an open set containing $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$, such that

$$\int_0^\infty F(t)e^{-zt} dt = G(z) \quad \text{for } \operatorname{Re} z > 0.$$

Then $\int_0^\infty F(t)dt$ converges and is equal to $G(0)$.

Remark. Theorem 5.2.2 may be rephrased as

$$\lim_{z \to 0, \operatorname{Re} z > 0} \int_0^\infty F(t)e^{-zt} dt = \int_0^\infty \lim_{z \to 0, \operatorname{Re} z > 0} F(t)e^{-zt} dt.$$

Although this seems plausible it is highly non-trivial. Indeed, it will imply the Prime Number Theorem!

Proof. The proof consists of several steps.

Step 1. *Reduction to the case $G(0) = 0$.*
We assume that Theorem 5.2.2 has been proved in the special case $G(0) = 0$ and deduce from this the general case.

Assume that $G(0) \neq 0$. Define new functions

$$\widetilde{F}(t) := F(t) - G(0)e^{-t}, \quad \widetilde{G}(z) := G(z) - \frac{G(0)}{z + 1}.$$

Then $\widetilde{F}$ satisfies (i),(ii), the function $\widetilde{G}$ is analytic on an open set containing $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$, we have $\widetilde{G}(0) = 0$, and for $\operatorname{Re} z > 0$ we have

$$\int_0^\infty \widetilde{F}(t)e^{-zt}dt = \int_0^\infty F(t)e^{-zt}dt - G(0)\int_0^\infty e^{-(z+1)t}dt = G(z) - \frac{G(0)}{z+1} = \widetilde{G}(z).$$

Hence $\widetilde{F}$ satisfies (iii). Now if we have proved that $\int_0^\infty \widetilde{F}(t)dt = \widetilde{G}(0) = 0$, then it follows that

$$\int_0^\infty F(t)dt = G(0)\int_0^\infty e^{-t}dt = G(0).$$

□

Henceforth we assume, in addition to the conditions (i)–(iii), that $G(0) = 0$.

Step 2. *The function G_T.*

For $T > 0$, define

$$G_T(z) := \int_0^T F(t)e^{-zt}dt.$$

We show that G_T is analytic on $\mathbb{C}$. We apply again Theorem 0.6.25 and verify the conditions of that theorem. First, $F(t)e^{-zt}$ is measurable on $[0,T] \times \mathbb{C}$. Second, for every fixed $t \in [0,T]$, $z \mapsto F(t)e^{-zt}$ is analytic on $\mathbb{C}$. To verify the third property, let K be a compact subset of $\mathbb{C}$. Then for $z \in K$, there is $A > 0$ such that $\operatorname{Re} z \geqslant -A$ for $z \in K$. Hence

$$|F(t)e^{-zt}| \leqslant Me^{At} \text{ for } 0 \leqslant t \leqslant T,\ z \in K$$

and clearly, $\int_0^T M \cdot e^{At}dt < \infty$ since we integrate over a bounded interval. So by Theorem 0.6.25, G_T is indeed analytic on $\mathbb{C}$. □

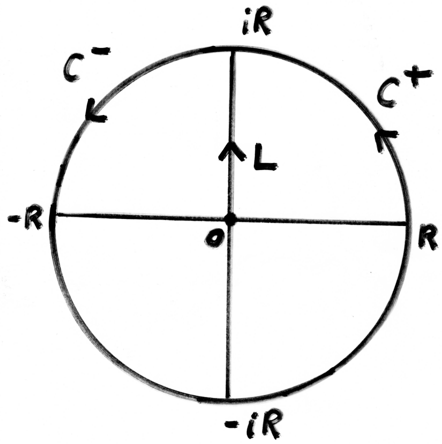
We clearly have

$$G_T(0) = \int_0^T F(t)dt.$$

So we have to prove:

$$\lim_{T\to\infty} G_T(0) = G(0) = 0. \tag{5.2.1}$$

Step 3. *An integral expression for $G_T(0)$.*



We fix a parameter $R > 0$. It will be important in the proof that R can be chosen arbitrarily large. Let:

C^+ the semi-circle $\{z \in \mathbb{C} : |z| = R, \operatorname{Re} z \geqslant 0\}$, traversed counterclockwise;
C^- the semi-circle $\{z \in \mathbb{C} : |z| = R, \operatorname{Re} z \leqslant 0\}$, traversed counterclockwise;
L the line segment from $-iR$ to iR, traversed upwards.

Define the auxiliary function (invented by Newman):

$$J_{R,T}(z) := e^{Tz}\Big(1 + \frac{z^2}{R^2}\Big) \cdot \frac{1}{z}.$$

The function $G_T(z) \cdot J_{R,T}(z)$ is analytic for $z \neq 0$, and at $z = 0$ it has a simple pole with residue $\lim_{z \to 0} G_T(z)e^{Tz}(1 + z^2/R^2) = G_T(0)$ (or a removable singularity if $G_T(0) = 0$). So by the Residue Theorem,

$$\text{(A)} \qquad \frac{1}{2\pi i} \oint_{C^+ + C^-} G_T(z)J_{R,T}(z)dz = G_T(0).$$

The function $G(z)$ is analytic on an open set containing $\{\operatorname{Re} z \geqslant 0\}$. Further, $G(z)J_{R,T}(z)$ is analytic on this open set. For it is clearly analytic if $z \neq 0$, and at $z = 0$ the simple pole of $J_{R,T}(z)$ is cancelled by the zero of $G(z)$ at $z = 0$, thanks to our assumption $G(0) = 0$. So by Cauchy's Theorem,

$$\text{(B)} \qquad \frac{1}{2\pi i} \oint_{C^+ + (-L)} G(z)J_{R,T}(z)dz = 0.$$

We derive an expression for $G_T(0)$ as a sum of three integrals by subtracting (B) from (A), splitting $\oint_{C^+ + C^-}$ in (A) into $\int_{C^+} + \int_{C^-}$ and $\oint_{C^+ + (-L)}$ in (B) into $\int_{C^+} - \int_L$

and lastly combining the two integrals over C^+, more precisely,

$$\begin{aligned}
G_T(0) &= \frac{1}{2\pi i}\oint_{C^+ + C^-} G_T(z)J_{R,T}(z)dz - \frac{1}{2\pi i}\oint_{C^+ + (-L)} G(z)J_{R,T}(z)dz \\
&= \frac{1}{2\pi i}\int_{C^+} G_T(z)J_{R,T}(z)dz + \frac{1}{2\pi i}\int_{C^-} G_T(z)J_{R,T}(z)dz \\
&\qquad\qquad - \frac{1}{2\pi i}\int_{C^+} G(z)J_{R,T}(z)dz + \frac{1}{2\pi i}\int_{L} G(z)J_{R,T}(z)dz \\
&= \frac{1}{2\pi i}\int_{C^+} (G_T(z) - G(z))J_{R,T}(z)dz + \frac{1}{2\pi i}\int_{C^-} G_T(z)J_{R,T}(z)dz \\
&\qquad\qquad + \frac{1}{2\pi i}\int_{L} G(z)J_{R,T}(z)dz \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

where I_1, I_2, I_3 denote the three integrals. To show that $G_T(0) \to 0$ as $T \to \infty$, we estimate $|I_1|$, $|I_2|$, $|I_3|$. Here we use $\left|\int_\gamma f(z)dz\right| \leqslant \text{length}(\gamma)\cdot \sup_{z\in\gamma}|f(z)|$.

Step 4. *Estimation of* $|I_1|$.

We first estimate $|(G_T(z) - G(z))J_{R,T}(z)|$ for $z \in C^+$. First assume that $z \in C^+$, $\operatorname{Re} z > 0$. Using the condition $|F(t)| \leqslant M$ for $t \geqslant 0$, we obtain

$$\begin{aligned}
|G_T(z) - G(z)| &= \left|\int_T^\infty F(t)e^{-zt}dt\right| \leqslant \int_T^\infty |F(t)|\cdot e^{-t\cdot \operatorname{Re} z}dt \\
&\leqslant \int_T^\infty M\cdot e^{-t\cdot\operatorname{Re} z}dt = \frac{M}{\operatorname{Re} z}\cdot e^{-T\cdot \operatorname{Re} z}.
\end{aligned}$$

The function $J_{R,T}(z)$ has been devised precisely for the purpose to get rid of the dependence on $\operatorname{Re} z$ in the estimate of $|I_1|$, and to make the estimate for $|I_1|$ decreasing to 0 as $R \to \infty$. Indeed, for $z \in \mathbb{C}$ with $|z| = R$ we have $z\cdot\overline{z} = R^2$, hence

$$|J_{R,T}(z)| = e^{T\cdot\operatorname{Re} z}\left|\left(1 + \frac{z^2}{z\cdot\overline{z}}\right)\frac{1}{z}\right| = e^{T\cdot\operatorname{Re} z}\left|\frac{z+\overline{z}}{z\cdot\overline{z}}\right| = 2e^{T\cdot\operatorname{Re} z}\cdot\frac{|\operatorname{Re} z|}{R^2}. \tag{5.2.2}$$

This implies that for $z \in C^+$ with $\operatorname{Re} z > 0$ we have

$$|(G_T(z) - G(z))J_{R,T}(z)| \leqslant \frac{M}{\operatorname{Re} z}\cdot e^{-T\cdot\operatorname{Re} z}\cdot 2e^{T\cdot\operatorname{Re} z}\cdot\frac{\operatorname{Re} z}{R^2} \leqslant \frac{2M}{R^2}.$$

By continuity, this is true also if $\operatorname{Re} z = 0$. Hence

$$|I_1| \leqslant \frac{1}{2\pi}\text{length}(C^+) \cdot \sup_{z\in C^+} |(G_T(z) - G(z))J_{R,T}(z)| \leqslant \frac{1}{2\pi} \cdot \pi R \cdot \frac{2M}{R^2},$$

i.e.,

$$|I_1| \leqslant \frac{M}{R}.$$

Step 5. *Estimation of* $|I_2|$.

The argument is similar to the estimation of $|I_1|$. We start with estimating $|G_T(z)J_{R,T}(z)|$ for $z \in C^-$. Using again $|F(t)| \leqslant M$ for $t \geqslant 0$, we have for $z \in C^-$ with $\operatorname{Re} z < 0$,

$$\begin{aligned} |G_T(z)| &= \left|\int_0^T F(t)e^{-zt}dt\right| \leqslant \int_0^T |F(t)| \cdot e^{-t\cdot\operatorname{Re} z}dt \\ &\leqslant \int_0^T M \cdot e^{-t\cdot\operatorname{Re} z}dt = -\frac{M}{\operatorname{Re} z}\big(e^{-T\cdot\operatorname{Re} z} - 1\big) \leqslant \frac{M}{|\operatorname{Re} z|} \cdot e^{-T\cdot\operatorname{Re} z}, \end{aligned}$$

which together with (5.2.2) implies

$$|G_T(z)J_{R,T}(z)| \leqslant \frac{2M}{R^2}.$$

Again this holds true also if $\operatorname{Re} z = 0$. So

$$|I_2| \leqslant \frac{1}{2\pi}\text{length}(C^-) \cdot \sup_{z\in C^-} |G_T(z)J_{R,T}(z)| \leqslant \frac{1}{2\pi} \cdot \pi R \cdot \frac{2M}{R^2},$$

leading to

$$|I_2| \leqslant \frac{M}{R}.$$

Step 6. *Estimation of* $|I_3|$.

We choose for L the parametrization $z = iy$, $-R \leqslant y \leqslant R$. Thus,

$$I_3 = \frac{1}{2\pi i}\int_{-R}^{R} G(iy)J_{R,T}(iy)d(iy) = \frac{1}{2\pi}\int_{-R}^{R} H_R(y)e^{iTy}dt,$$

where

$$H_R(y) := G(iy)\Big(1 - \frac{y^2}{R^2}\Big)\frac{1}{iy}.$$

Since by assumption, $G(0) = 0$, the function $G(z)/z$ is analytic on an open set containing $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$. Hence $H_R(y)$ is continuously differentiable on $[-R, R]$. Since H_R is independent of T, there is a constant $A(R)$ independent of T such that

$$|H_R(y)| \leqslant A(R), \quad |H'_R(y)| \leqslant A(R) \quad \text{for } y \in [-R, R].$$

Using integration by parts, we get

$$\begin{aligned}\int_{-R}^{R} H_R(y) e^{iTy} dy &= \frac{1}{iT} \int_{-R}^{R} H_R(y) de^{iTy} \\ &= \frac{1}{iT} \left(H_R(R) e^{iTR} - H_R(-R) e^{-iTR} - \int_{-R}^{R} H'_R(y) e^{iTy} dy \right).\end{aligned}$$

Since $|e^{iTy}| = 1$, we obtain

$$\begin{aligned}\left| \int_{-R}^{R} H_R(y) e^{iTy} dy \right| &\leqslant \frac{1}{T} \left(A(R) + A(R) + \int_{-R}^{R} |H'_R(y)| dy \right) \\ &\leqslant \frac{2A(R) + 2R \cdot A(R)}{T}.\end{aligned}$$

Hence

$$|I_3| \leqslant \frac{C(R)}{T},$$

where $C(R)$ depends on R, but is independent of T.

Step 7. *Conclusion of the proof.*

We have to prove that $\lim_{T \to \infty} G_T(0) = G(0) = 0$, in other words, for every $\varepsilon > 0$ there is T_0 such that $|G_T(0)| < \varepsilon$ for all $T \geqslant T_0$. Combining steps 3–6, we get, for every choice of R, T,

$$|G_T(0)| \leqslant |I_1| + |I_2| + |I_3| \leqslant \frac{2M}{R} + \frac{C(R)}{T}.$$

Let $\varepsilon > 0$. Then choose R such that $2M/R < \varepsilon/2$, and subsequently T_0 with $C(R)/T_0 < \varepsilon/2$. For these choices, it follows that for $T \geqslant T_0$,

$$|G_T(0)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes our proof. □

5.3 A Tauberian theorem for Dirichlet series

Let $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series. Put

$$A(x) := \sum_{n \leqslant x} f(n).$$

We prove the following Tauberian theorem.

Theorem 5.3.1. *Suppose $L_f(s)$ satisfies the following conditions:*

(i) $f(n) \geqslant 0$ for all n;
(ii) there are $C > 0, \sigma > 0$ such that $|A(x)| \leqslant Cx^{\sigma}$ for all $x \geqslant 1$;
(iii) $L_f(s)$ converges for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \sigma$;
(iv) There is an open subset U of $\mathbb{C}$ containing $\{s \in \mathbb{C} : \mathrm{Re}\, s \geqslant \sigma\}$, such that $L_f(s)$ can be continued to a function that is analytic on $U \setminus \{\sigma\}$ and for which $\lim_{s\to\sigma}(s-\sigma)L_f(s) = \alpha$.

Then

$$\lim_{x\to\infty} \frac{A(x)}{x^{\sigma}} = \frac{\alpha}{\sigma}.$$

Remarks. 1) Condition (iii) follows from (ii) (see Exercise 2.2a). Further, (iii) implies that $L_f(s)$ is analytic for $\mathrm{Re}\, s > \sigma$.
2) Condition (iv) means that $L_f(s)$ has a simple pole with residue α at $s = \sigma$ if $\alpha \neq 0$, and a removable singularity at $s = \sigma$ if $\alpha = 0$.
3) The Wiener-Ikehara Theorem is the same as Theorem 5.3.1, except that only conditions (i),(iii),(iv) are required and (ii) can be dropped.

We start with some preparations. Notice that condition (iv) of Theorem 5.3.1 implies that there is an analytic function $g(s)$ on U such that

$$L_f(s) = \frac{\alpha}{s-\sigma} + g(s) \quad \text{if } \mathrm{Re}\, s > 0. \tag{5.3.1}$$

Further, we need some lemmas.

Lemma 5.3.2. *For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \sigma$ we have*

$$L_f(s) = s\int_1^{\infty} A(x)x^{-s-1}dx.$$

Proof. Let $\operatorname{Re} s > \sigma$. Then by partial summation we have for every integer $N \geqslant 1$,

$$\sum_{n=1}^{N} f(n) n^{-s} = A(N) N^{-s} + s \int_1^N A(x) x^{-s-1} dx.$$

Since $|A(N)| \leqslant \sum_{n=1}^{N} f(n) \leqslant CN^\sigma$, we have $A(N)N^{-s} \leqslant CN^\sigma \cdot N^{-\operatorname{Re} s} \to 0$ as $N \to \infty$. By letting $N \to \infty$, the lemma follows. □

Lemma 5.3.3. $\displaystyle\int_1^\infty \frac{A(x) - (\alpha/\sigma)x^\sigma}{x^{\sigma+1}} \cdot dx = \sigma^{-1} g(\sigma) - \sigma^{-2}\alpha$ *converges.*

Proof. By substituting $x = e^t$, we see that the identity to be proved is equivalent to

$$\int_0^\infty \big(e^{-\sigma t} A(e^t) - \alpha/\sigma\big) dt = \sigma^{-1} g(\sigma) - \sigma^{-2}\alpha. \tag{5.3.2}$$

We apply Theorem 5.2.2 to $F(t) := e^{-\sigma t} A(e^t) - \alpha/\sigma$. We check that this F satisfies conditions (i),(ii),(iii) of Theorem 5.2.2.

First, $F(t)$ is measurable (e.g., it has only countably many discontinuities). Second, by condition (ii) of Theorem 5.3.1,

$$|F(t)| \leqslant C + |\alpha/\sigma| \text{ for } t \geqslant 0.$$

Hence conditions (i),(ii) of Theorem 5.2.2 are satisfied. As for condition (iii), notice that for $\operatorname{Re} z > 0$ we have

$$\begin{aligned}
&\int_0^\infty F(t) e^{-zt} dt = \int_0^\infty \big(e^{-\sigma t} A(e^t) - \alpha/\sigma\big) e^{-zt} dt \\
&= \int_1^\infty A(x) x^{-z-\sigma-1} dx - (\alpha/\sigma) \int_1^\infty x^{-z-1} dx \quad (\text{substitute } x = e^t) \\
&= \frac{1}{z+\sigma} L_f(z+\sigma) - \frac{\alpha}{\sigma z} = \frac{1}{z+\sigma}\Big(\frac{\alpha}{z} + g(z+\sigma)\Big) - \frac{\alpha}{\sigma z} \quad (\text{by Lemma 5.3.2, (5.3.1)}),
\end{aligned}$$

that is,

$$\int_0^\infty F(t) e^{-zt} dt = \frac{1}{z+\sigma}\big(g(z+\sigma) - \alpha/\sigma\big) \text{ if } \operatorname{Re} z > 0. \tag{5.3.3}$$

The right-hand side is analytic on an open set containing $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$, hence (iii) is satisfied as well. So by Theorem 5.2.2, identity (5.3.3) extends to $z = 0$, and this gives precisely (5.3.2). □

By condition (i), we have $f(n) \geqslant 0$ for all n. Hence the function $A(t)$ is non-decreasing. Now Theorem 5.3.1 follows by combining Lemma 5.3.3 with the lemma below.

Lemma 5.3.4. *Let $B:[1,\infty)\to\mathbb{R}$ be a non-decreasing function and let $\beta\in\mathbb{R}$, $\sigma>0$. Assume that*

$$\int_1^\infty \frac{B(x)-\beta x^\sigma}{x^{\sigma+1}}\cdot dx \quad \textit{converges.}$$

Then

$$\lim_{x\to\infty}\frac{B(x)}{x^\sigma} = \beta.$$

Proof. We may assume without loss of generality that $\beta=1$. Indeed, choose $\gamma>0$ such that $\beta+\gamma>0$, and replace $B(x)$ by $B^*(x):=\frac{1}{\beta+\gamma}\big(B(x)+\gamma x^\sigma\big)$. Then B^* is non-decreasing and $\int_1^\infty(B^*(x)-x^\sigma)dx/x^{\sigma+1}$ converges. If we are able to prove that $\lim_{x\to\infty}B^*(x)/x^\sigma=1$, then $\lim_{x\to\infty}B(x)/x^\sigma=\beta$ follows.

So assume that $\beta=1$. Assume that $\lim_{x\to\infty}B(x)/x^\sigma$ does not exist or is not equal to 1. Then there are two possibilities:

(a) there are $\varepsilon>0$ and an increasing sequence $\{x_n\}_{n=1}^\infty$ with $x_n\to\infty$ such that $B(x_n)/x_n^\sigma\geqslant 1+\varepsilon$ for all n;
(b) there are $\varepsilon>0$ and an increasing sequence $\{x_n\}_{n=1}^\infty$ with $x_n\to\infty$ such that $B(x_n)/x_n^\sigma\leqslant 1-\varepsilon$ for all n.

We consider only case (a); case (b) can be dealt with in the same manner. So assume (a). Then since $\int_1^\infty(B(x)-x^\sigma)dx/x^{\sigma+1}$ converges, we have

$$\lim_{y_1,y_2\to\infty}\int_{y_1}^{y_2}\frac{B(x)-x^\sigma}{x^{\sigma+1}}\cdot dx = \lim_{y_2\to\infty}\int_1^{y_2} - \lim_{y_1\to\infty}\int_1^{y_1} = 0. \tag{5.3.4}$$

We choose y_1,y_2 appropriately and derive a contradiction. Notice that for $x\geqslant x_n$ we have, since B is non-decreasing,

$$\frac{B(x)-x^\sigma}{x^{\sigma+1}} \geqslant \frac{B(x_n)-x^\sigma}{x^{\sigma+1}} \geqslant \frac{(1+\varepsilon)x_n^\sigma - x^\sigma}{x^{\sigma+1}}.$$

This is $\geqslant 0$ for $x_n\leqslant x\leqslant(1+\varepsilon)^{1/\sigma}x_n$, so there is some hope that with the choice $y_1=x_n$, $y_2=(1+\varepsilon)^{1/\sigma}x_n$ the integral in (5.3.4) becomes strictly positive and does not converge to 0 as $n \to \infty$. Indeed we have

$$\begin{aligned}\int_{x_n}^{(1+\varepsilon)^{1/\sigma}x_n} \frac{B(x)-x^\sigma}{x^{\sigma+1}}\cdot dx &\geqslant \int_{x_n}^{(1+\varepsilon)^{1/\sigma}x_n} \frac{(1+\varepsilon)x_n^\sigma - x^\sigma}{x^{\sigma+1}}\cdot dx \\ &= \int_1^{(1+\varepsilon)^{1/\sigma}} \frac{(1+\varepsilon)-u^\sigma}{u^{\sigma+1}}\cdot du \qquad (u = x/x_n) \\ &= \Big[-(1+\varepsilon)\sigma^{-1}u^{-\sigma} - \log u\Big]_1^{(1+\varepsilon)^{1/\sigma}} \\ &= \sigma^{-1}\big(\varepsilon - \log(1+\varepsilon)\big).\end{aligned}$$

This last number is independent of n and strictly positive, since $\sigma > 0$ and since $\log(1+\varepsilon) < \varepsilon$. This contradicts (5.3.4). Hence case (a) is impossible. □

5.4 Exercises

Exercise 5.1. *Work out in detail case (b) of the proof of Lemma 5.3.4.*

Exercise 5.2. *a) Prove that* $\lim_{x\to\infty} \frac{1}{x}\sum_{n\leqslant x}\mu(n) = 0$.

Hint. *Apply Theorem 5.3.1 to* $\zeta(s)+\zeta(s)^{-1}$. *Use Theorem 4.2, Corollary 4.4 and Theorem 4.5.*

b) Prove that $\lim_{x\to\infty} \frac{1}{x}\sum_{n\leqslant x}|\mu(n)| = 6/\pi^2$.

Hint. *Show first that* $L_{|\mu|}(s) = \zeta(s)/\zeta(2s)$ *for* $\mathrm{Re}\, s > 1$.

c) Let $A(x)$ *denote the number of positive integers* $n \leqslant x$ *with* $\mu(n) = 1$ *and* $B(x)$ *the number of positive integers* $n \leqslant x$ *with* $\mu(n) = -1$. *Prove that* $\lim_{x\to\infty} \frac{A(x)}{x} = \lim_{x\to\infty} \frac{B(x)}{x} = 3/\pi^2$.

Exercise 5.3. *Let* $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geqslant 0}$ *be an arithmetic function satisfying conditions (i)–(iv) of Theorem 5.3.1, with* $\alpha, \sigma \in \mathbb{R}$ *and* $\sigma > 0$.

a) Let $g : \mathbb{Z}_{>0} \to \mathbb{R}$ *be an arithmetic function, possibly assuming negative values, such that* $|g(n)| \leqslant f(n)$ *for all* n *and such that there exist* $\beta \in \mathbb{R}$ *and an open* *subset U of $\mathbb{C}$ containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant \sigma\}$ such that $L_g(s)$ can be continued to a function which is analytic on $U \setminus \{\sigma\}$ and for which $\lim_{s\to\sigma}(s-\sigma)L_g(s) = \beta$. Prove that $\lim_{x\to\infty} \frac{1}{x^\sigma} \sum_{n\leqslant x} g(n) = \beta/\sigma$.*

b) *Let now g be as in a) except that g may assume its values in $\mathbb{C}$ and not necessarily in $\mathbb{R}$. Prove again that $\lim_{x\to\infty} \frac{1}{x^\sigma} \sum_{n\leqslant x} g(n) = \beta/\sigma$.*

Hint. *Let $\overline{g} : n \mapsto \overline{g(n)}$ be the complex conjugate of g. Then $L_{\overline{g}}(s) = \overline{L_g(\overline{s})}$. In the proof of Corollary 0.6.24 from the Prerequisites it has been shown that if f is any analytic function on an open set V symmetric about the real axis (i.e., $z \in V \to \overline{z} \in V$, then $z \mapsto \overline{f(\overline{z})}$ is also analytic on V. Use this to show that $L_{\overline{g}}(s)$ is also analytic on $U \setminus \{\sigma\}$ and $\lim_{s\to\sigma}(s-\sigma)L_{\overline{g}}(s) = \overline{\beta}$ and apply a) to $\operatorname{Re} g = \frac{1}{2}(g+\overline{g})$ and $\operatorname{Im} g = \frac{1}{2i}(g-\overline{g})$.*

Exercise 5.4. *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be a multiplicative function such that $|f(n)| \leqslant 1$ for all integers $n \geqslant 1$ and such that $f(p) = -1$ for every prime number p.*

a) *Prove that*

$$\zeta(s)L_f(s) = \prod_p \left(1 + \frac{f(p^2)p^{-2s} + f(p^3)p^{-3s} + \cdots}{1-p^{-s}}\right) \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1$$

and then show, using Corollary 0.6.29 from the Prerequisites, that the right-hand side defines an analytic function on $\{s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2}\}$.

b) *Prove that $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) = 0$.*

c) *Let $A(x)$ denote the number of positive integers $n \leqslant x$ such that $\omega(n)$ is even. Prove that $\lim_{x\to\infty} \frac{A(x)}{x} = \frac{1}{2}$.*