Chapter 6

The Prime number theorem for arithmetic progressions

6.1 The Prime number theorem

We denote by $\pi(x)$ the number of primes $\leqslant x$.

Theorem 6.1.1. *We have* $\pi(x) \sim \dfrac{x}{\log x}$ *as* $x \to \infty$.

Before giving the detailed proof, we outline our strategy. Define the functions

$$\theta(x) := \sum_{p \leqslant x} \log p, \quad \psi(x) := \sum_{k,p:\, p^k \leqslant x} \log p = \sum_{n \leqslant x} \Lambda(n),$$

where Λ is the von Mangoldt function, given by $\Lambda(n) = \log p$ if $n = p^k$ for some prime p and some $k \geqslant 1$, and $\Lambda(n) = 0$ otherwise.

• Using partial summation we prove that

$$\pi(x) \sim \frac{x}{\log x} \Longleftrightarrow \theta(x) \sim x \Longleftrightarrow \psi(x) \sim x \quad \text{as } x \to \infty.$$

• By Lemma 2.3.4 from Chapter 2 we have

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\frac{\zeta'(s)}{\zeta(s)} \quad \text{for } \operatorname{Re} s > 1.$$

By applying our Tauberian theorem for Dirichlet series (Theorem 5.3.1) to the latter we obtain

$$\frac{\psi(x)}{x} = \frac{1}{x}\sum_{n\leqslant x}\Lambda(n) \to 1 \quad \text{as } x\to\infty,$$

which by the above implies the Prime Number Theorem.

We work out in detail the strategy described above. We start with deducing some properties of $\theta(x)$ and $\psi(x)$.

Lemma 6.1.2. *(i)* $\theta(x) = O(x)$ *as* $x\to\infty$.
(ii) $\psi(x) = \theta(x) + O(\sqrt{x})$ *as* $x\to\infty$.
(iii) $\psi(x) = O(x)$ *as* $x\to\infty$.

Proof. (i) By Exercise 1.4 a), we have $\prod_{p\leqslant x} p \leqslant 4^x$ for $x\geqslant 2$. This implies

$$\theta(x) = \sum_{p\leqslant x}\log p \leqslant x\log 4 = O(x) \quad \text{as } x\to\infty.$$

(ii) We have

$$\begin{aligned}\psi(x) &= \sum_{p,k\colon p^k\leqslant x}\log p = \sum_{p\leqslant x}\log p + \sum_{p^2\leqslant x}\log p + \sum_{p^3\leqslant x}\log p + \cdots\\ &= \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots\end{aligned}$$

Notice that $\theta(t) = 0$ if $t<2$. So $\theta(x^{1/k}) = 0$ if $x^{1/k}<2$, that is, if $k > \log x/\log 2$. Hence

$$\begin{aligned}\psi(x)-\theta(x) &= \sum_{k=2}^{[\log x/\log 2]}\theta(x^{1/k}) \leqslant \theta(\sqrt{x}) + \sum_{k=3}^{[\log x/\log 2]}\theta(\sqrt[3]{x})\\ &\leqslant \theta(\sqrt{x}) + \Big(\frac{\log x}{\log 2}-3\Big)\theta(\sqrt[3]{x}) = O\big(\sqrt{x}+\sqrt[3]{x}\cdot\log x\big)\\ &= O(\sqrt{x}) \quad \text{as } x\to\infty.\end{aligned}$$

(iii) Combine (i) and (ii). □

Lemma 6.1.3. *We have*

$$\frac{\pi(x)\cdot\log x}{x} = \frac{\theta(x)}{x} + O\Big(\frac{1}{\log x}\Big) = \frac{\psi(x)}{x} + O\Big(\frac{1}{\log x}\Big) \quad \text{as } x\to\infty.$$

Proof. By partial summation, we have

$$\begin{aligned}\pi(x) &= \sum_{p\leqslant x} 1 = \sum_{p\leqslant x} \log p \cdot \frac{1}{\log p} = \theta(x)\frac{1}{\log x} - \int_2^x \theta(t)\cdot\Big(\frac{1}{\log t}\Big)' dt \\ &= \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t} dt.\end{aligned}$$

The integral is non-negative. Further, by Lemma 6.1.2 (i) there is a constant $C > 0$ such that $\theta(t) \leqslant Ct$ for all $t \geqslant 2$. Together with Exercise 1.3 a) this implies

$$0 \leqslant \int_2^x \frac{\theta(t)}{t\log^2 t}\cdot dt \leqslant C\cdot\int_2^x \frac{dt}{\log^2 t} = O\Big(\frac{x}{\log^2 x}\Big) \quad \text{as } x\to\infty.$$

Hence

$$\begin{aligned}\frac{\pi(x)\log x}{x} &= \frac{\theta(x)}{x} + O\Big(\frac{\log x}{x}\cdot\frac{x}{\log^2 x}\Big) \\ &= \frac{\theta(x)}{x} + O\Big(\frac{1}{\log x}\Big) \quad \text{as } x\to\infty,\end{aligned}$$

and then Lemma 6.1.2 (ii) gives

$$\frac{\pi(x)\log x}{x} = \frac{\psi(x)}{x} + O\Big(\frac{1}{\sqrt{x}}\Big) + O\Big(\frac{1}{\log x}\Big) = \frac{\psi(x)}{x} + O\Big(\frac{1}{\log x}\Big) \quad \text{as } x\to\infty.$$

□

Proof of Theorem 6.1.1. As a consequence of Lemma 6.1.3, $\pi(x)\log x/x \to 1$ as $x\to\infty$ if and only if

$$\frac{\psi(x)}{x} = \frac{1}{x}\sum_{n\leqslant x}\Lambda(n) \to 1 \quad \text{as } x\to\infty.$$

So it suffices to prove the latter. But this follows once we have verified that $L_\Lambda(s) = \sum_{n=1}^\infty \Lambda(n)n^{-s}$ satisfies conditions (i)–(iv) of Theorem 5.3.1 with $\alpha = 1$, $\sigma = 1$. To this end, we have to combine various results from the previous chapters.

(i), (ii) $\Lambda(n) \geqslant 0$ for all n by definition, and $\sum_{n\leqslant x}\Lambda(n) = \psi(x) = O(x)$ as $x\to\infty$ by Lemma 6.1.2 (iii).

(iii), (iv) By Lemma 2.3.4, $L_\Lambda(s) = -\zeta'(s)/\zeta(s)$ for $s\in\mathbb{C}$ with $\operatorname{Re} s > 1$, and the series converges and defines an analytic function for $s\in\mathbb{C}$ with $\operatorname{Re} s > 1$. By

Theorem 4.1.2, $\zeta(s)$ is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$, with a simple pole at $s = 1$. Further, by Corollary 4.1.4 and Theorem 4.2.1, $\zeta(s) \neq 0$ on $A := \{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\}$ (with the convention that a meromorphic function is non-zero at a pole), and hence also $\zeta(s) \neq 0$ on an open set U containing A. By taking U sufficiently small, we may conclude from Lemma 0.6.17, that $\zeta'(s)/\zeta(s)$ is analytic on $U \setminus \{1\}$, with a simple pole with residue -1 at $s = 1$; hence $L_\Lambda(s)$ has residue 1 at $s = 1$. We conclude that indeed, conditions (i)–(iv) of Theorem 5.3.1 hold with $\alpha = 1$, $\sigma = 1$. This completes the proof of the Prime Number Theorem. □

6.2 The Prime number theorem for arithmetic progressions

Let q, a be integers with $q \geqslant 2$, $\gcd(a, q) = 1$. Define

$$\pi(x; q, a) := \text{number of primes } p \leqslant x \text{ with } p \equiv a \pmod{q}.$$

Theorem 6.2.1. *We have* $\pi(x; q, a) \sim \dfrac{1}{\varphi(q)} \cdot \dfrac{x}{\log x}$ *as* $x \to \infty$.

The proof is very similar to that of the Prime number theorem. Define the quantities

$$\theta(x; q, a) := \sum_{p \leqslant x,\, p \equiv a \,(\mathrm{mod}\, q)} \log p,$$

$$\psi(x; q, a) := \sum_{p, k,\, p^k \leqslant x,\, p^k \equiv a \,(\mathrm{mod}\, q)} \log p = \sum_{n \leqslant x,\, n \equiv a \,(\mathrm{mod}\, q)} \Lambda(n).$$

Lemma 6.2.2. *(i)* $\theta(x; q, a) = O(x)$ *as* $x \to \infty$.
(ii) $\psi(x; q, a) - \theta(x; q, a) = O(\sqrt{x})$ *as* $x \to \infty$.
(iii) $\psi(x; q, a) = O(x)$ *as* $x \to \infty$.

Proof. (i) We have $\theta(x; q, a) \leqslant \theta(x) = O(x)$ as $x \to \infty$.

(ii) We have

$$\psi(x; q, a) - \theta(x; q, a) = \sum_{k, p,\, k \geqslant 2, p^k \leqslant x,\, p^k \equiv a \,(\mathrm{mod}\, q)} \log p$$

$$\leqslant \sum_{k, p,\, k \geqslant 2, p^k \leqslant x} \log p = \psi(x) - \theta(x) = O(\sqrt{x}) \quad \text{as } x \to \infty.$$

(iii) Obvious. □

Lemma 6.2.3. *We have*

$$\frac{\pi(x;q,a)\cdot\log x}{x}=\frac{\theta(x;q,a)}{x}+O\Big(\frac{1}{\log x}\Big)=\frac{\psi(x;q,a)}{x}+O\Big(\frac{1}{\log x}\Big)\quad \text{as } x\to\infty.$$

Proof. Follow the proof of Lemma 6.1.3 and replace everywhere $\pi(x)$, $\theta(x)$, $\psi(x)$ by $\pi(x;q,a)$, $\theta(x;q,a)$, $\psi(x;q,a)$. Verify the details yourself. □

Let $f(n):=\Lambda(n)$ if $n\equiv a\,(\mathrm{mod}\,q)$, $f(n)=0$ otherwise. Then

$$L_f(s)=\sum_{n=1,\,n\equiv a\,(\mathrm{mod}\,q)}^{\infty}\Lambda(n)n^{-s}.$$

Further, $\sum_{n\leqslant x}f(n)=\psi(x;q,a)$.

By Lemma 6.2.3, $\pi(x;q,a)\log x/x\to 1/\varphi(q)$ if and only if $\frac{1}{x}\sum_{n\leqslant x}f(n)=\psi(x;q,a)/x\to 1/\varphi(q)$, and the latter follows once we have verified that $L_f(s)$ satisfies conditions (i)–(iv) of Theorem 5.3.1 with $\alpha=1/\varphi(q)$ and $\sigma=1$.

We need some preparations. Let $G(q)$ denote the group of characters modulo q.

Lemma 6.2.4. *For $s\in\mathbb{C}$ with $\mathrm{Re}\,s>1$ we have*

$$L_f(s)=-\frac{1}{\varphi(q)}\cdot\sum_{\chi\in G(q)}\overline{\chi(a)}\cdot\frac{L'(s,\chi)}{L(s,\chi)}.$$

Proof. Let χ be a character modulo q and let $s\in\mathbb{C}$ with $\mathrm{Re}\,s>1$. By Exercise 2.5 we have, since χ is a strongly multiplicative arithmetic function and $L(s,\chi)$ converges absolutely,

$$\frac{L'(s,\chi)}{L(s,\chi)}=-\sum_{n=1}^{\infty}\chi(n)\Lambda(n)n^{-s}.$$

Using Theorem 3.2.1 (ii) (one of the orthogonality relations for characters) we obtain

$$\sum_{\chi\in G(q)}\overline{\chi(a)}\cdot\frac{L'(s,\chi)}{L(s,\chi)}=-\sum_{\chi\in G(q)}\overline{\chi(a)}\sum_{n=1}^{\infty}\chi(n)\Lambda(n)n^{-s}$$

$$=-\sum_{n=1}^{\infty}\Bigg(\sum_{\chi\in G(q)}\overline{\chi(a)}\chi(n)\Bigg)\Lambda(n)n^{-s}=-\varphi(q)\cdot\sum_{n=1,\,n\equiv a\,(\mathrm{mod}\,q)}^{\infty}\Lambda(n)n^{-s}.$$

□

Lemma 6.2.5. *There is an open subset U of $\mathbb{C}$ containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\}$ such that $L_f(s)$ can be continued to a function analytic $U \setminus \{1\}$, with a simple pole with residue $\varphi(q)^{-1}$ at $s = 1$.*

Proof. By Theorem 4.1.3 (iii), $L(s, \chi_0^{(q)})$ is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$, with a simple pole at $s = 1$. Further, by Corollary 4.1.4 and Theorem 4.2.1, $L(s, \chi_0^{(q)}) \neq 0$ for $s \in A := \{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\}$, and hence $L(s, \chi_0^{(q)}) \neq 0$ for s in an open set U containing A. By taking U sufficiently small, we may conclude from Lemma 0.6.17, that $L'(s, \chi_0^{(q)})/L(s, \chi_0^{(q)})$ is analytic on $U \setminus \{1\}$, with a simple pole with residue -1 at $s = 1$.

Let χ be a character mod q with $\chi \neq \chi_0^{(q)}$. By Theorem 4.1.3 (ii), $L(s, \chi)$ is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, and by Corollary 4.1.4 and Theorems 4.2.1, 4.2.3, it is non-zero on A, hence on an open set containing A, which we assume to be the above set U if we take this sufficiently small. Therefore, $L'(s, \chi)/L(s, \chi)$ is analytic on U.

Now by Lemma 6.2.4, $L_f(s)$ is analytic on $U \setminus \{1\}$, with a simple pole with residue $\chi_0^{(q)}(a)/\varphi(q) = \varphi(q)^{-1}$ at $s = 1$. □

Proof of Theorem 6.2.1. As mentioned above, we have to verify that $L_f(s)$ satisfies conditions (i)–(iv) of Theorem 5.3.1. Clearly $f(n) \geqslant 0$ for all n, while $\sum_{n \leqslant x} f(n) = \psi(x; q, a) = O(x)$ as $x \to \infty$ by Lemma 6.2.2 (iii). Hence (i) and (ii) are satisfied. Further, Lemmas 6.2.4 and 6.2.5 imply that (iii) and (iv) are satisfied with $\alpha = 1/\varphi(q)$ and $\sigma = 1$. Hence $\psi(x; q, a)/x \to 1/\varphi(q)$, and then $\pi(x; q, a) \log x/x \to 1/\varphi(q)$ by Lemma 6.2.3. This completes our proof. □

6.3 Related results

Riemann sketched a proof, and von Mangoldt gave the complete proof, of the following result, which relates the distribution of primes to the distribution of the zeros of the Riemann zeta function. Define

$$\psi_0(x) := \begin{cases} \psi(x) = \sum_{n \leqslant x} \Lambda(n) & \text{if } x \text{ is not a prime power,} \\ \psi(x) - \frac{1}{2}\Lambda(x) & \text{if } x \text{ is a prime power.} \end{cases}$$

Theorem 6.3.1. *We have for $x > 1$,*

$$\psi_0(x) = x - \lim_{T\to\infty} \sum_{|\mathrm{Im}\,\rho|<T} \frac{x^\rho}{\rho} - \log 2\pi - \tfrac{1}{2}\log(1 - x^{-2}),$$

where the sum is over all zeros ρ of $\zeta(s)$ with $0 < \mathrm{Re}\,\rho < 1$ and $|\mathrm{Im}\,\rho| < T$, and zeros are taken with order, that is, if a zero ρ of $\zeta(s)$ has order k then it is taken k times in the sum.

For a proof of this theorem, see Chapter 17 and some previous preparatory chapters in H. Davenport, Multiplicative Number Theory, Graduate texts in mathematics 74, Springer Verlag, 2nd ed., 1980.

Using information about the distribution of the zeros of $\zeta(s)$, such as knowledge of a zero-free region, Theorem 6.3.1 yields a good estimate for $|\psi(x) - x|$ and by means of partial summation techniques one can deduce a good estimate for the error term $|\pi(x) - \mathrm{Li}(x)|$, where $\mathrm{Li}(x) = \int_2^x dt/\log t$. This leads to the sharp versions of the Prime Number Theorem with error term, mentioned in Chapter 1. In Exercise 6.2 below you are asked to deduce a good estimate for $|\pi(x) - \mathrm{Li}(x)|$ under assumption of the Riemann Hypothesis, namely von Koch's estimate mentioned in Chapter 1.

There are similar refinenents of the Prime number theorem for arithmetic progressions with an estimate for the error $|\pi(x; q, a) - \mathrm{Li}(x)/\varphi(q)|$. The simplest case is when we fix q and let $x \to \infty$, but for applications it is important to have also versions where we allow q to move in some range depending on x when we let $x \to \infty$.

The following result was proved by Walfisz in 1936, with important preliminary work by Landau and Siegel.

Theorem 6.3.2. *For every $A > 0$ there is a constant $C(A) > 0$ such that for every real $x \geqslant 3$, every integer $q \geqslant 2$ and every integer a with $\gcd(q, a) = 1$, we have*

$$\left|\pi(x; q, a) - \frac{1}{\varphi(q)}\mathrm{Li}(x)\right| \leqslant C(A)\frac{x}{(\log x)^A}.$$

The constant $C(A)$ is *ineffective*, this means that by going through the proof of the theorem one cannot compute the constant, but only show that it exists.

Theorem 6.3.2 is trivial if $q > (\log x)^A$. For in this case, $\pi(x; q, a) \leqslant 1 + [x/q] \leqslant 1 + x/(\log x)^A$, while by Exercise 1.3a and Exercise 6.4 below,

$$\frac{\mathrm{Li}(x)}{\varphi(q)} \ll \frac{x}{\log x} \cdot \frac{\log\log q}{q} \ll \frac{x}{(\log x)^A}.$$

As we mentioned in Chapter 1 and above, there is an intricate connection between the zero-free region of $\zeta(s)$ and estimates for $|\pi(x) - \mathrm{Li}(x)|$. Similarly, there is a connection between zero-free regions of L-functions and estimates for $|\pi(x; q, a) - \mathrm{Li}(x)/\varphi(q)|$. We recall the following, rather complicated, result of Landau (1921) on the zero-free region of L-functions.

Theorem 6.3.3. *There is an absolute constant $c > 0$ such that for every integer $q \geqslant 2$ the following holds. Among all characters χ modulo q, there is at most one such that $L(s, \chi)$ has a zero in the region*

$$R(q) := \left\{ s \in \mathbb{C} :\ \mathrm{Re}\, s > 1 - \frac{c}{\log(q(1 + |\mathrm{Im}\, s|))} \right\}.$$

If such a character χ and a zero $\beta \in R(q)$ of $L(s, \chi)$ exist, then χ is real and non-principal, β is simple and real, and $L(s, \chi)$ does not have zeros in $R(q)$ other than β.

Moreover, if q' is any integer with $2 \leqslant q' < q$ and χ' is any character modulo q', then $L(s, \chi')$ does not have a zero in $R(q)$ either.

Any character χ modulo q having a zero in $R(q)$ is called an *exceptional character* mod q, and the zero of $L(s, \chi)$ in $R(q)$ is called an *exceptional zero* or *Siegel zero*. The *Generalized Riemann Hypothesis* (GRH) asserts that if χ is a Dirichlet character modulo an integer $q \geqslant 2$, then the zeros of $L(s, \chi)$ in the critical strip $0 < \mathrm{Re}\, s < 1$ lie in fact on the line $\mathrm{Re}\, s = \frac{1}{2}$. A consequence of GRH is that exceptional characters do not exist.

In order to obtain Theorem 6.3.2, one needs an estimate for the real part of a possible exceptional zero of an L-function. The following result was proved by Siegel (1935).

Theorem 6.3.4. *For every $\varepsilon > 0$ there is a number $c(\varepsilon) > 0$ such that for every integer $q \geqslant 2$ the following holds: if χ is an exceptional character modulo q and β an exceptional zero of $L(s, \chi)$, then $\mathrm{Re}\, \beta < 1 - c(\varepsilon) q^{-\varepsilon}$.*

Theorem 6.3.2 is proved by combining Theorems 6.3.3 and 6.3.4 with an analogue of Theorem 6.3.1 for $\psi(x; q, a)$, see H. Davenport, Multiplicative Number Theory, Graduate texts in mathematics 74, Springer Verlag, 2nd ed., 1980. It is because of the possible occurrence of an exceptional zero that the upper bound for the quantity

$|\pi(x;q,a) - \mathrm{Li}(x)/\varphi(q)|$ in Theorem 6.3.2 is much larger than the best one known for $|\pi(x) - \mathrm{Li}(x)|$.

Under assumption of GRH, one can show, similar to von Koch's result mentioned in Chapter 1, that for any integer $q \geqslant 2$, any integer a with $\gcd(a,q) = 1$ and any real $x \geqslant 2$,

$$\left|\pi(x;q,a) - \frac{1}{\varphi(q)}\mathrm{Li}(x)\right| \leqslant Cx^{1/2}\log x, \tag{6.3.1}$$

where C is an absolute constant, i.e., not depending on anything (see Davenport's book, Chapter 20). This estimate is trivial if $q > x^{1/2}/\log x$, for in this case $\pi(x;q,a)$ and $\mathrm{Li}(x)/\varphi(q)$ are both $\ll x^{1/2}\log x$. A conjecture, due to Montgomery, which does not follow from GRH, asserts that for every $\varepsilon > 0$ there is $C(\varepsilon) > 0$, such that for every integer $q \geqslant 2$, integer a with $\gcd(a,q) = 1$ and real $x \geqslant 2$,

$$\left|\pi(x;q,a) - \frac{1}{\varphi(q)}\mathrm{Li}(x)\right| \leqslant C(\varepsilon)q^{-1/2}x^{(1/2)+\varepsilon}.$$

There has been considerable interest in the average of the quantities

$$E(x;q) := \max_{\substack{1\leqslant a<q \\ \gcd(a,q)=1}} \left|\pi(x;q,a) - \frac{1}{\varphi(q)}\mathrm{Li}(x)\right|,$$

taken over particular ranges of q. Bombieri and independently A.I. Vinogradov (not to be confused with I.M. Vinogradov mentioned in Chapter 1 in connection with the estimate for $|\pi(x) - \mathrm{Li}(x)|$) proved around 1965 a result which states that on average, $E(x;q)$ has an upper bound similar to the right-hand side of (6.3.1). The following result, proved by Bombieri but similar to one proved by Vinogradov, is referred to as the "Bombieri-Vinogradov Theorem."

Theorem 6.3.5 (Bombieri, 1965)**.** *Let A be a real > 0. Then for every real $x \geqslant 3$, and every real Q with $x^{1/2}/(\log x)^A \leqslant Q \leqslant x^{1/2}$ one has*

$$\frac{1}{Q}\sum_{q\leqslant Q} E(x;q) \leqslant C(A))x^{1/2}(\log x)^4,$$

where $C(A)$ depends only on A.

For a proof of a related result, we refer to Davenport's book Multiplicative Number Theory, Chap. 28.

Knowing that an arithmetic progression contains infinitely many primes, one would like to know when the first prime in such a progression occurs, i.e., the smallest x such that $\pi(x; q, a) > 0$. The following estimate is due to Linnik (1944).

Theorem 6.3.6. *Denote by $P(q, a)$ the smallest prime number p with $p \equiv a \pmod{q}$. There are absolute constants c, L such that for every integer $q \geqslant 2$ and every integer a with $\gcd(a, q) = 1$ we have $P(q, a) \leqslant cq^L$.*

The exponent L is known as 'Linnik's constant.' Since the appearance of Linnik's paper, various people have tried to estimate it. The present record is $L = 5.18$, due to Xylouris (2011).

We would like to finish with mentioning some recent breakthroughs concerning gaps between consecutive primes. Let $p_1 < p_2 < p_3 < \cdots$ be the sequence of consecutive primes. The *twin prime conjecture* asserts that there are infinitely many n with $p_{n+1} - p_n = 2$. In 2014, Yitai Zhang made a breakthrough by showing that there are infinitely many n with $p_{n+1} - p_n \leqslant 7 \times 10^7$, that is, there are infinitely many pairs of consecutive primes that are not further than 7×10^7 apart. Maynard (2015) subsequently improved this to 600. The Bombieri-Vinogradov Theorem is an important ingredient in his proof. Several mathematicians have been working collectively to further improve Maynard's bound. Today's record is 246, see the polymath wiki,

`http://michaelnielsen.org/polymath1` $\rightarrow$ polymath8

6.4 Exercises

In the exercise below, the following is needed:

Definition. $f(x) = g(x) + O(x^{a+\varepsilon})$ as $x \to \infty$ for every $\varepsilon > 0$ means the following: for every $\varepsilon > 0$ there exist numbers C, x_0, that may depend on ε, such that
$|f(x) - g(x)| \leqslant C \cdot x^{a+\varepsilon}$ for every $x \geqslant x_0$.

Exercise 6.1. *In general, one obtains a version of the Prime Number Theorem with error term, i.e., $\pi(x) = \mathrm{Li}(x) + O(E(x))$ as $x \to \infty$ with some explicit function $E(x)$, from a zero-free region of $\zeta(s)$. Here, $\mathrm{Li}(x) = \int_2^x dt/\log t$.*
In this exercise you are asked to prove the converse:

Suppose that for all $\varepsilon > 0$ we have $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\varepsilon})$ as $x \to \infty$. Then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\frac{1}{2} < \mathrm{Re}\, s < 1$.

From Corollary 4.3.5 it follows that then also $\zeta(s) \neq 0$ for $s \in \mathbb{C}$ with $0 < \mathrm{Re}\, s < \frac{1}{2}$. That is, the Riemann Hypothesis holds.

To prove the above, perform the following steps.

a) *For $x \geqslant 2$, prove that*

$$\begin{aligned} \theta(x) &= \pi(x) \log x - \int_2^x (\pi(t)/t) dt, \\ x - 2 &= \mathrm{Li}(x) \log x - \int_2^x (\mathrm{Li}(t)/t) dt. \end{aligned}$$

b) *Assume that for every $\varepsilon > 0$ we have $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\varepsilon})$ as $x \to \infty$. Prove that for every $\varepsilon > 0$ we have*

$$\theta(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \text{ as } x \to \infty, \quad \psi(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \text{ as } x \to \infty.$$

c) *Using Exercise 2.2a, prove that for every $\varepsilon > 0$, $\zeta(s) + \big(\zeta'(s)/\zeta(s)\big)$ can be continued to a function analytic on $\{s \in \mathbb{C} : \mathrm{Re}\, s > \frac{1}{2} + \varepsilon\}$, and then that $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\frac{1}{2} < \mathrm{Re}\, s < 1$.*

Exercise 6.2. *Von Mangoldt proved the following identity, of which Theorem 6.3.1 is a consequence: there is a constant $C > 0$ such that for all reals $x, T \geqslant 2$ one has*

$$\psi_0(x) = x - \sum_{|\mathrm{Im}\, \rho| < T} \frac{x^\rho}{\rho} - \log 2\pi - \tfrac{1}{2} \log(1 - x^{-2}) + R(x, T),$$

where

$$|R(x,T)| \leqslant C \cdot \left(\frac{x(\log xT)^2}{T} + \log x \cdot \min\left(1, \frac{x}{\langle x \rangle T}\right) \right),$$

with $\langle x \rangle$ the distance of x to the closest prime power. Here the summation is over all zeros ρ of $\zeta(s)$ with $0 < \mathrm{Re}\, \rho < 1$ and $|\mathrm{Im}\, \rho| < T$, taken with order.

Another fact needed in this exercise is an estimate for the number $N(T)$ of zeros ρ of $\zeta(s)$ with $0 < \mathrm{Re}\, \rho < 1$ and $|\mathrm{Im}\, \rho| < T$, taken with order, which was stated without proof by Riemann in his memoir, and proved rigorously by von Mangoldt, *namely, $N(T) = O(T\log T)$ as $T\to\infty$. Both facts can be proved by means of complex analysis.*

The purpose of this exercise is to deduce von Koch's theorem from the Riemann Hypothesis and the expressions for $\psi_0(x)$ and $N(T)$.

a) *Assume the Riemann Hypothesis. Deduce that $\psi(x) = x + O(\sqrt{x}(\log x)^2)$ as $x\to\infty$.*
 Hint. *Apply the expression for $\psi_0(x)$ with $T = x$. By Schwarz' reflection principle, if s_0 is a zero of $\zeta(s)$ of order n_0, then so is $\overline{s_0}$. Let $\mathcal{N}(x)$ be the set of t with $0 \leqslant t < x$ such that $\frac{1}{2}\pm it$ are zeros of $\zeta(s)$, and for $t\in\mathcal{N}(x)$, let $n(t)$ be the order of $\frac{1}{2}\pm it$. Then*

$$\left|\sum_{|\mathrm{Im}\,\rho|<x}\frac{x^\rho}{\rho}\right| \leqslant 2\sum_{t\in\mathcal{N}(x)} n(t)|x^{(1/2)+it}|\cdot|\tfrac{1}{2}+it|^{-1} = 2x^{1/2}\sum_{t\in\mathcal{N}(x)} n(t)\frac{1}{\sqrt{\frac{1}{4}+t^2}}.$$

 Estimate this sum using partial summation.

b) *Deduce that $\theta(x) = x + O(\sqrt{x}(\log x)^2)$ as $x\to\infty$ and subsequently, using partial summation, that $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$ as $x\to\infty$.*

Exercise 6.3. *The purpose of this exercise is to improve theorems proved by Mertens in the 19th century.*

a) *Prove that there is $c_1\in\mathbb{R}$ such that*

$$\sum_{p\leqslant x}\frac{\log p}{p} = \log x + c_1 + o(1) \quad \text{as } x\to\infty$$

 (i.e., $\sum_{p\leqslant x}\frac{\log p}{p} = \log x + c_1 + R_1(x)$ where $\lim_{x\to\infty} R_1(x) = 0$). Work out the following steps:
 Prove that $\int_1^\infty \frac{\psi(x)-x}{x^2}dx$ converges (use an appropriate result from Chapter 5).
 Prove that $\int_1^\infty \frac{\theta(x)-x}{x^2}dx$ converges.
 Prove that $\sum_{p\leqslant x}\frac{\log p}{p} = \frac{\theta(x)}{x} + \int_1^x \frac{\theta(t)}{t^2}dt$.

b) Prove that there is $c_2 \in \mathbb{R}$ *such that*

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c_2 + o\Big(\frac{1}{\log x}\Big) \quad \text{as } x \to \infty$$

(i.e., $\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c_2 + \frac{R_2(x)}{\log x}$*, where* $\lim_{x\to\infty} R_2(x) = 0$*).*

Hint. *Write* $\sum_{p \leqslant x} \frac{1}{p} = \sum_{p \leqslant x} \frac{\log p}{p} \cdot \frac{1}{\log p}$ *and use partial summation. Show and use that if* $f(x), g(x)$ *are measurable real functions on* $[a, \infty)$ *such that* $g(x) \geqslant 0$ *for* $x \geqslant a$*,* $f(x) = o(g(x))$ *as* $x \to \infty$ *and* $\int_a^\infty g(t)dt$ *converges, then* $\int_a^x f(t)dt = \int_a^\infty f(t)dt + o\big(\int_x^\infty g(t)dt\big)$ *as* $x \to \infty$*.*

c) Prove that $\alpha := \sum_p \Big(\log(1 - p^{-1})^{-1} - \frac{1}{p}\Big)$ *converges, and that*

$$\sum_{p \leqslant x} \Big(\log(1 - p^{-1})^{-1} - \frac{1}{p}\Big) = \alpha + O\Big(\frac{1}{x}\Big) \quad \text{as } x \to \infty.$$

Subsequently, deduce that there is $c_3 \in \mathbb{R}$ *such that*

$$\sum_{p \leqslant x} \log(1 - p^{-1})^{-1} = \log\log x + c_3 + o\Big(\frac{1}{\log x}\Big) \quad \text{as } x \to \infty.$$

Hint. *Use that* $\log(1 - z)^{-1} = \sum_{n=1}^{\infty} z^n/n$ *for* $z \in \mathbb{C}$ *with* $|z| < 1$*.*

d) Prove that there is $c_4 > 0$ *such that*

$$\prod_{p \leqslant x} \Big(1 - \frac{1}{p}\Big)^{-1} = c_4 \log x + o(1) \quad \text{as } x \to \infty.$$

With a more precise analysis, it can be shown that $c_4 = e^\gamma$, where γ is the Euler-Mascheroni constant.

Mertens proved in an elementary manner the above results with error terms $O(1)$, $O\big(\frac{1}{\log x}\big)$ instead of $o(1)$, $o\big(\frac{1}{\log x}\big)$.

Exercise 6.4. *a) Prove that* $\prod_{p|n} (1 - p^{-1})^{-1} = O(\log\log n)$ *as* $n \to \infty$*, and subsequently that there is* $c_1 > 0$ *such that* $\varphi(n) \geqslant c_1 \frac{n}{\log\log n}$ *for all integers* $n \geqslant 3$*.*

b) Prove that there is a constant $c_2 > 0$ *such that* $\varphi(n) \leqslant c_2 \frac{n}{\log\log n}$ *holds for infinitely many integers* n.

c) You may even try to prove that $\liminf_{n\to\infty} \varphi(n) \log\log n/n = c_4^{-1}$, *where* c_4 *is the constant from Exercise 6.3 (so in fact the liminf is* $e^{-\gamma}$*).*

Exercise 6.5. *a) Let* q, a *be integers with* $q \geqslant 2$ *and* $\gcd(a, q) = 1$. *Prove that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leqslant x \\ n\equiv a (\mathrm{mod}\, q)}} \mu(n) = 0.$$

b) Let m *be any positive integer. Prove that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leqslant x \\ n\equiv a (\mathrm{mod}\, q) \\ \gcd(n,m)=1}} \mu(n) = 0.$$

c) Let q, a *be integers with* $q \geqslant 2$, $\gcd(q, a) = d > 1$. *Prove that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leqslant x \\ n\equiv a (\mathrm{mod}\, q)}} \mu(n) = 0.$$