Chapter 9

The singular integral

Our aim in this chapter is to replace the functions $\mathcal{G}^*(n)$ and $J^*(n)$ by more convenient expressions; these will be called the singular series $\mathcal{G}(n)$ and the singular integral $J(n)$. This will be done in section 9.1. We shall show that the order of magnitude of the singular integral is $n^2$ in section 9.2.

9.1 Introducing $\mathcal{G}(n)$ and $J(n)$

Define

$$\mathcal{G}(q) := \sum_{\substack{1 \leq a \leq q \\gcd(a,q)=1}} S(q,a)^0 e(-an/q),$$

where

$$S(q,a) = \sum_{m=1}^{q} e(am^3/q).$$

Thus,

$$\mathcal{G}^*(n) = \sum_{q \leq n^{1/300}} \frac{S(q)}{q^9}.$$

Lemma 9.1. Let $a, q$ be coprime integers and $\epsilon$ any positive real. Then

$$|S(q,a)| \ll_{\epsilon} q^{\frac{3}{4} + \epsilon}.$$
Proof. The argument is an analogue of the differencing process in Chapter 7. We have

\[ |S(q,a)|^2 = \sum_{m_2 \pmod{q}} \sum_{m_1 \pmod{q}} e\left(\frac{a}{q}(m_1^3 - m_2^3)\right). \]

The transformation \(m_1 \mapsto h_1\) given by \(m_1 \equiv h_1 + m_2 \pmod{q}\) shows that

\[ |S(q,a)|^2 = \sum_{h_1 \pmod{q}} e\left(\frac{a}{q}h_1^3\right) \sum_{m_2 \pmod{q}} e\left(\frac{a}{q}(3h_1m_2 + m_2^2)\right), \]

hence the triangle inequality gives

\[ |S(q,a)|^2 \leq \sum_{h_1 \pmod{q}} \left| \sum_{m_2 \pmod{q}} e\left(\frac{a}{q}(3h_1m_2 + m_2^2)\right) \right|. \]

Now Cauchy’s inequality reveals that

\[ |S(q,a)|^4 \leq q \sum_{h_1 \pmod{q}} \left| \sum_{m_2 \pmod{q}} e\left(\frac{a}{q}(3h_1m_2 + m_2^2)\right) \right|^2. \]

The inner term is

\[ \left| \sum_{m_2 \pmod{q}} e\left(\frac{a}{q}(3h_1m_2 + m_2^2)\right) \right|^2 = \sum_{m_2, m_3 \pmod{q}} e\left(\frac{a}{q}(3h_1(m_2 - m_3) + (m_2^2 - m_3^2))\right) \]

and the substitution \(m_2 \mapsto h_2\) given by \(m_2 \equiv hm_3 + h_2 \pmod{q}\) leads to

\[ \sum_{h_2 \pmod{q}} e\left(\frac{3a}{q}(h_1^2h_2 + h_1h_2^2)\right) \sum_{m_3 \pmod{q}} e\left(\frac{6a}{q}h_1h_2m_3\right). \]

Note that the coprimality of \(a, q\) shows that the sum over \(m_3\) equals \(q\) when \(q\) divides \(6h_1h_2\) and vanishes otherwise. We obtain that

\[ |S(q,a)|^4 \leq q^2 \#\{1 \leq h_1, h_2 \leq q : 6h_1h_2\}. \]

The integers \(6h_1h_2\) lie in the range \([1, 6q^2]\) and are divisible by \(q\). Hence there exists \(i \in [1, 6q]\) such that \(6h_1h_2 = iq\). Therefore

\[ |S(q,a)|^4 \leq q^2 \sum_{i=1}^{6q} \#\{1 \leq h_1, h_2 \leq q : 6h_1h_2 = iq\}. \]

In order to have \(6h_1h_2 = iq\) both integers \(h_1, h_2\) must divide \(iq\) and there are only

\[ \tau(iq)^2 \ll \epsilon (iq)^{\epsilon/2} \ll q^\epsilon \]

such pairs, where \(\epsilon\) is any positive real. This concludes our proof. \(\square\)
The last lemma shows that
\[ \frac{|S(q)|}{q^9} \ll \epsilon \frac{1}{q^{1+\frac{1}{4}+\epsilon}}, \]
therefore the following series, usually referred to as the *singular series*,
\[ (9.1) \quad S(n) := \sum_{q=1}^{\infty} \frac{S(q)}{q^9} \]
converges absolutely and satisfies
\[ S(n) - S^*(n) \ll \epsilon \sum_{q>n^{1/300}} \frac{1}{q^{1+\frac{1}{4}+\epsilon}} \ll \int_{n^{1/300}}^{\infty} \frac{dt}{t^{1+\frac{1}{4}+\epsilon}} \ll \epsilon n^{-\frac{1}{1200}+\epsilon}. \]
This shows that
\[ (9.2) \quad S(n) \ll 1 \]
and by (8.6) we obtain
\[ (9.3) \quad R^*(n) = (S(n) + O(n^{-\frac{1}{1200}+\epsilon}))J^*(n). \]

We next replace $J^*(n)$ by a more suitable integral. For this we shall need to need the behaviour of $v(\beta) = \frac{1}{3} \sum_{m=1}^{n} \frac{e(\beta m)}{m^{2/3}}$ in the range $|\beta| \leq \frac{1}{2}$.

**Lemma 9.2.** Let $\beta \in \mathbb{R}$ with $|\beta| \leq \frac{1}{2}$. Then
\[ |v(\beta)| \ll \min\{n^{1/3}, |\beta|^{-1/3}\}. \]

**Proof.** If $\beta$ is close to 0 then the terms $e(\beta m)$ in the definition of $v(\beta)$ remain close to 1. Hence using the triangle inequality one does not lose much information,
\[ |v(\beta)| \leq \frac{1}{3} \sum_{1 \leq m \leq n} \frac{1}{m^{2/3}} \leq \frac{1}{3} \int_{1}^{n} \frac{dt}{t^{2/3}} + O(1) \ll n^{1/3}. \]
If $|\beta| \leq 1/n$ then $|\beta|^{-1/3} > n^{1/3}$, hence the claim of our lemma is evident.

In the remaining case $|\beta| > 1/n$ we see that
\[ \left| \sum_{m \leq 1/|\beta|} \frac{e(\beta m)}{m^{2/3}} \right| \leq \sum_{m \leq 1/|\beta|} \frac{1}{m^{2/3}} \ll (1/|\beta|)^{1/3}, \]
which is acceptable. We use partial summation to estimate the remaining sum

\[ \sum_{1/|\beta|< m \leq n} \frac{e(\beta m)}{m^{2/3}}. \]

For this purpose we define for \( t \in \mathbb{R} \),

\[ A(t) := \sum_{1 \leq m \leq t} e(\beta m) = e(\beta m) \frac{e(\beta [t]) - 1}{e(\beta) - 1} \]

and observe that the inequality \(|e(\beta) - 1| \gg |\beta|\), valid for \(|\beta| < 1/2\), yields

\[ A(t) \ll \frac{1}{|\beta|}, \]

with an implied constant that is independent of \( t \). Partial summation now gives

\[ \sum_{1/|\beta|< m \leq n} \frac{e(\beta m)}{m^{2/3}} = \frac{A(n)}{n^{2/3}} - \frac{A(1/|\beta|)}{|\beta|^{-2/3}} + \int_{1/|\beta|}^{n} A(t) \frac{dt}{t^{5/3}}, \]

which is

\[ \ll \frac{1/|\beta|}{n^{2/3}} + \frac{|\beta|^{2/3}}{|\beta|} + \frac{1}{|\beta|} \frac{1}{|\beta|^{-2/3}} \ll |\beta|^{-1/3}. \]

Define the following integral (which is usually called *singular integral*),

\[ J(n) := \int_{-1/2}^{1/2} v(\beta)^9 e(-\beta n) d\beta \]

and observe that Lemma 9.2 shows that

\[ J(n) \ll \int_{0}^{1/n} n^{9/3} d\beta + \int_{1/n}^{1/2} \frac{d\beta}{|\beta|^3}, \]

hence

\[ J(n) \ll n^2. \]

Now recall the definition of \( J^*(n) \) in (8.8). We have

\[ J(n) - J^*(n) = \int_{n^{-1/3+1/300} \leq |\beta| \leq 1/2} v(\beta)^9 e(-\beta n) d\beta, \]
which according to Lemma 9.2 is
\[
\ll \int_{n^{1/300}/n}^{1/2} \beta^{-3} d\beta \ll n^{2-\frac{1}{150}}.
\]
Using (9.2), (9.3) and (9.5) we find an absolute constant \(\delta > 0\) such that
\[
R^*(n) = \mathcal{G}(n)J(n) + O(n^{2-\delta}),
\]
which when combined with (7.8), (7.10) and (8.5) yields the following theorem.

**Theorem 9.3.** We have
\[
\lim_{n \to +\infty} \left| \frac{R(n)}{n^2} - \frac{\mathcal{G}(n)J(n)}{n^2} \right| = 0.
\]

### 9.2 The singular integral

The *Beta function* is defined as
\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \text{for } x, y > 0.
\]
Before relating the singular integral \(J(n)\) to the Beta function we need some information on sums of values at integers of monotonic functions. Let \(y < x\) be integers and let \(f : [y, x] \to \mathbb{R}\) be any monotonic function. Comparing the sum \(\sum_{y \leq n \leq x} f(n)\) with the integral \(\int_y^x f(t) dt\) we see that
\[
\sum_{y \leq n \leq x} f(n) = \int_y^x f(t) dt + O(|f(y)| + |f(x)|).
\]
This is also true if we allow \(x\) and \(y\) to be reals instead of integers, but assume that \(f\) does not increase or decrease too fast on small intervals, that is, there is \(C > 1\) such that
\[
(*) \quad C^{-1} \leq \frac{|f(s)|}{|f(t)|} \leq C \quad \text{for all } s, t \in [y, x] \text{ with } |s - t| \leq 1.
\]
More generally, if \(y < x\) are reals, \(f\) satisfies \((*)\), and \(x_1, \ldots, x_k\) are reals in \((y, x)\) such that \(f : [y, x] \to \mathbb{R}\) is monotonic on each interval
\[
(y, x_1), (x_1, x_2), \ldots, (x_k, x)
\]
\[
\sum_{y \leq m \leq x} f(m) = \int_y^x f(t) dt + O(|f(y)| + |f(x)| + \sum_{i=1}^k |f(x_i)|).
\]

Let \(0 < \beta \leq 1, \alpha \geq \beta\) and consider the function \(f(x) := x^{\beta-1} (n - x)^{\alpha-1}\). We want to compare \(\sum_{1 \leq m \leq n-1} f(m)\) with \(\int_0^n f(t) dt\), which is an integral that can be handled. There are some subtleties, since \(f\) tends to infinity if \(\beta < 1\) and \(t \downarrow 0\) or if \(\alpha < 1\) and \(t \uparrow n\). The function \(f\) is defined and positive on \((0, n)\), the derivative of \(f\) is \(x^{\beta-2} (n - x)^{\alpha-2} (x(2 - \alpha - \beta) - (1 - \beta)n)\), which vanishes at \(X = \frac{n(1 - \beta)}{2 - \alpha - \beta}\), and \(f\) satisfies (*) on \([1, n - 1]\). If \(X \in (1, n - 1)\) then \(f\) is decreasing on \((1, X)\) and increasing on \((X, n - 1)\) and thus

\[
\sum_{1 \leq m \leq n-1} f(m) = \int_1^{n-1} f(t) dt + O(f(1) + f(X) + f(n-1))
\]

\[
= \int_0^n f(t) dt + O\left(f(1) + f(X) + f(n-1) + \int_0^1 f(t) dt + \int_{n-1}^n f(t) dt\right)
\]

and if \(X \notin (1, n - 1)\) then \(f\) is monotone on \((1, n - 1)\) and the error term is the same but without \(f(X)\). An easy computation shows that

\[
f(1) \ll n^{\alpha-1}, \quad f(n-1) \ll n^{\beta-1},
\]

\[
\int_0^1 f(t) dt \ll n^{\alpha-1} \int_0^1 t^{\beta-1} dt \ll n^{\alpha-1}, \quad \int_{n-1}^n f(t) dt \ll n^{\beta-1} \int_{n-1}^n (n - t)^{\alpha-1} dt \ll n^{\beta-1},
\]

and since \(\alpha \geq \beta\) these quantities are all \(\ll n^{\alpha-1}\). If \(X \in (1, n - 1)\) then \(f(X) \ll n^{\alpha+\beta-2} \ll n^{\alpha-1}\) since \(\beta \leq 1\). So in all cases,

\[
\sum_{1 \leq m \leq n-1} f(m) = \int_0^n f(t) dt + O(n^{\alpha-1}).
\]

The substitution \(t \mapsto y\) given by \(t = ny\) shows that

\[
\int_0^n f(t) dt = n^{\alpha+\beta-1} B(\beta, \alpha),
\]

therefore

\[
\sum_{m=1}^{n-1} m^{\beta-1} (n - m)^{\alpha-1} = n^{\alpha+\beta-1} \left(B(\beta, \alpha) + O(n^{-\beta})\right).
\]
Before proceeding we need to recall a few standard facts about the Gamma function. It is defined as
\[
\Gamma(t) := \int_0^\infty t^{x-1}e^{-t} dt \quad \text{for } x > 0
\]
and satisfies
\[
\Gamma(1) = 1, \quad (9.7)
\]
\[
\Gamma(t + 1) = t\Gamma(t) \quad \text{for } t > 0, \quad (9.8)
\]
\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for } x, y > 0. \quad (9.9)
\]
Observe that \( \Gamma(x) = (x-1)! \) for every positive integer \( x \). So \( B(x, y) = \frac{x+y}{xy} \cdot \left( \frac{x+y}{x} \right)^{-1} \)
for all positive integers \( x, y \).

We have the following theorem.

**Theorem 9.4.** We have
\[
J(n) = \Gamma \left( \frac{4}{3} \right)^9 \frac{n^2}{2} \left( 1 + O(n^{-1/3}) \right).
\]

**Proof.** We begin by proving by induction that for every integer \( s \geq 2 \), that one has
\[
\frac{1}{3^s} \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{3}{s}}} = \Gamma \left( \frac{4}{3} \right)^s \Gamma \left( \frac{s}{3} \right)^{-1} n^{s-1} \left( 1 + O(n^{-1/3}) \right). \quad (9.10)
\]

For \( s = 2 \) this is valid due to (9.6) with \( \alpha = \beta = 1/3 \), as well as (9.8) and (9.9). Assuming that (9.10) is valid for some integer \( s \geq 2 \) then
\[
\frac{1}{3^{s+1}} \sum_{1 \leq m_1, \ldots, m_{s+1} \leq n} \frac{1}{(m_1 \cdots m_{s+1})^{\frac{3}{s+1}}} = \frac{1}{3^s} \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{3}{s}}} \left( \frac{1}{3^{s+1}} \sum_{1 \leq m_1, \ldots, m_s \leq n} \frac{1}{(m_1 \cdots m_s)^{\frac{3}{s+1}}} \right).
\]

for all positive integers \( x, y \).
which is
\[
\Gamma \left( \frac{4}{3} \right)^s \Gamma \left( \frac{s}{3} \right)^{-1} \frac{1}{3} \sum_{1 \leq m \leq n-1} \frac{m^\frac{1}{3}}{m} (n-m)^{s-1} + O \left( \sum_{1 \leq m \leq n-1} m^{1/3-1}(n-m)^{(s-1)-1} \right)
\]
due to the induction hypothesis. Using (9.6) with \( \beta = \frac{1}{3} \) and \( \alpha = \frac{s}{3}, \frac{(s-1)}{3} - 1 \) respectively for the main and the error term, we conclude the proof of (9.10).

Combining (9.4) and the definition of \( v \),
\[
v(\beta) = \frac{1}{3} \sum_{1 \leq m \leq n} \frac{e(\beta m)}{m^\frac{2}{3}}
\]
gives
\[
J(n) = \frac{1}{3^9} \sum_{1 \leq m_1, \ldots, m_9 \leq n} \frac{1}{(m_1 \cdots m_9)^\frac{2}{3}} \int_{-\frac{1}{2}}^{1/2} e(\beta (n-m_1 - \cdots - m_9)) d\beta.
\]
The integral vanishes except when \( n - m_1 - \cdots - m_9 = 0 \), thus obtaining
\[
J(n) = \frac{1}{3^9} \sum_{1 \leq m_1, \ldots, m_9 \leq n} \frac{1}{(m_1 \cdots m_9)^\frac{2}{3}}
\]
and according to (9.10) our theorem is valid. \( \square \)