Chapter 9

The singular integral

Our aim in this chapter is to replace the functions $\mathfrak{S}^*(n)$ and $J^*(n)$ by more convenient expressions; these will be called the singular series $\mathfrak{S}(n)$ and the singular integral $J(n)$. This will be done in section 9.1. We shall show that the order of magnitude of the singular integral is n^2 in section 9.2.

9.1 Introducing $\mathfrak{S}(n)$ and $J(n)$

Define

$$S(q) := \sum_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} S(q,a)^9 \mathrm{e}(-an/q),$$

where

$$S(q,a) = \sum_{m=1}^{q} \mathrm{e}(am^3/q).$$

Thus,

$$\mathfrak{S}^*(n) = \sum_{q \leqslant n^{1/300}} \frac{S(q)}{q^9}.$$

Lemma 9.1. *Let a, q be coprime integers and ϵ any positive real. Then*

$$|S(q,a)| \ll_\epsilon q^{\frac{3}{4}+\epsilon}.$$

Proof. The argument is an analogue of the differencing process in Chapter 7. We have

$$|S(q,a)|^2 = \sum_{m_2 (\text{mod } q)} \sum_{m_1 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(m_1^3 - m_2^3)\right).$$

The transformation $m_1 \mapsto h_1$ given by $m_1 \equiv h_1 + m_2 (\text{mod } q)$ shows that

$$|S(q,a)|^2 = \sum_{h_1 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}h_1^3\right) \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right),$$

hence the triangle inequality gives

$$|S(q,a)|^2 \leqslant \sum_{h_1 (\text{mod } q)} \Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|.$$

Now Cauchy's inequality reveals that

$$|S(q,a)|^4 \leqslant q \sum_{h_1 (\text{mod } q)} \Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|^2.$$

The inner term is

$$\Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|^2 = \sum_{m_2, m_3 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1(m_2 - m_3) + (m_2^2 - m_3^3)))\right)$$

and the substitution $m_2 \mapsto h_2$ given by $m_2 \equiv hm_3 + h_2 (\text{mod } q)$ leads to

$$\sum_{h_2 (\text{mod } q)} \mathrm{e}\left(3\frac{a}{q}(h_1^2 h_2 + h_1 h_2^2)\right) \sum_{m_3 (\text{mod } q)} \mathrm{e}\left(6\frac{a}{q}h_1 h_2 m_3\right).$$

Note that the coprimality of a, q shows that the sum over m_3 equals q when q divides $6h_1h_2$ and vanishes otherwise. We obtain that

$$|S(q,a)|^4 \leqslant q^2 \#\{1 \leqslant h_1, h_2 \leqslant q : q | 6h_1 h_2\}.$$

The integers $6h_1h_2$ lie in the range $[1, 6q^2]$ and are divisible by q. Hence there exists $i \in [1, 6q]$ such that $6h_1h_2 = iq$. Therefore

$$|S(q,a)|^4 \leqslant q^2 \sum_{i=1}^{6q} \#\{1 \leqslant h_1, h_2 \leqslant q : 6h_1 h_2 = iq\}.$$

In order to have $6h_1h_2 = iq$ both integers h_1, h_2 must divide iq and there are only

$$\tau(iq)^2 \ll_\epsilon (iq)^{\epsilon/2} \ll q^\epsilon$$

such pairs, where ϵ is any positive real. This concludes our proof. $\square$

The last lemma shows that

$$\frac{|S(q)|}{q^9} \ll_\epsilon \frac{1}{q^{1+\frac{1}{4}+\epsilon}},$$

therefore the following series, usually referred to as the *singular series*,

$$\mathfrak{S}(n) := \sum_{q=1}^{\infty} \frac{S(q)}{q^9} \tag{9.1}$$

converges absolutely and satisfies

$$\mathfrak{S}(n) - \mathfrak{S}^*(n) \ll_\epsilon \sum_{q>n^{1/300}} \frac{1}{q^{1+\frac{1}{4}+\epsilon}} \ll \int_{n^{1/300}}^{\infty} \frac{\mathrm{d}t}{t^{1+\frac{1}{4}+\epsilon}} \ll_\epsilon n^{-\frac{1}{1200}+\epsilon}.$$

This shows that

$$\mathfrak{S}(n) \ll 1 \tag{9.2}$$

and by (8.6) we obtain

$$R^*(n) = (\mathfrak{S}(n) + O_\epsilon(n^{-\frac{1}{1200}+\epsilon}))J^*(n). \tag{9.3}$$

We next replace $J^*(n)$ by a more suitable integral. For this we shall need to need the behaviour of $v(\beta) = \frac{1}{3}\sum_{m=1}^{n} \mathrm{e}(\beta m)/m^{2/3}$ in the range $|\beta| \leqslant \frac{1}{2}$.

Lemma 9.2. *Let $\beta \in \mathbb{R}$ with $|\beta| \leqslant \frac{1}{2}$. Then*

$$|v(\beta)| \ll \min\{n^{1/3}, |\beta|^{-1/3}\}.$$

Proof. If β is close to 0 then the terms $\mathrm{e}(\beta m)$ in the definition of $v(\beta)$ remain close to 1. Hence using the triangle inequality one does not loose much information,

$$|v(\beta)| \leqslant \frac{1}{3} \sum_{1\leqslant m\leqslant n} \frac{1}{m^{\frac{2}{3}}} \leqslant \frac{1}{3} \int_1^{n-1} \frac{\mathrm{d}t}{t^{\frac{2}{3}}} + O(1) \ll n^{1/3}.$$

If $|\beta| \leqslant 1/n$ then $|\beta|^{-1/3} > n^{1/3}$, hence the claim of our lemma is evident.

In the remaining case $|\beta| > 1/n$ we see that

$$\Big| \sum_{m\leqslant 1/|\beta|} \frac{\mathrm{e}(\beta m)}{m^{2/3}} \Big| \leqslant \sum_{m\leqslant 1/|\beta|} \frac{1}{m^{2/3}} \ll (1/|\beta|)^{1/3},$$

which is acceptable. We use partial summation to estimate the remaining sum

$$\sum_{1/|\beta|<m\leqslant n} \frac{\mathrm{e}(\beta m)}{m^{2/3}}.$$

For this purpose we define for $t \in \mathbb{R}$,

$$A(t) := \sum_{1\leqslant m\leqslant t} \mathrm{e}(\beta m) = \mathrm{e}(\beta m)\frac{\mathrm{e}(\beta[t]) - 1}{\mathrm{e}(\beta) - 1}$$

and observe that the inequality $|\mathrm{e}(\beta) - 1| \gg |\beta|$, valid for $|\beta| < 1/2$, yields

$$A(t) \ll \frac{1}{|\beta|},$$

with an implied constant that is independent of t. Partial summation now gives

$$\sum_{1/|\beta|<m\leqslant n} \frac{\mathrm{e}(\beta m)}{m^{2/3}} = \frac{A(n)}{n^{2/3}} - \frac{A(1/|\beta|)}{|\beta|^{-2/3}} + \int_{1/|\beta|}^{n} A(t)\frac{\mathrm{d}t}{t^{5/3}},$$

which is

$$\ll \frac{1/|\beta|}{n^{2/3}} + \frac{|\beta|^{2/3}}{|\beta|} + \frac{1}{|\beta|}\frac{1}{|\beta|^{-2/3}} \ll |\beta|^{-1/3}.$$

□

Define the following integral (which is usually called *singular integral*),

$$J(n) := \int_{-1/2}^{1/2} v(\beta)^9 \mathrm{e}(-\beta n)\mathrm{d}\beta \tag{9.4}$$

and observe that Lemma 9.2 shows that

$$J(n) \ll \int_0^{1/n} n^{9/3}\mathrm{d}\beta + \int_{1/n}^{1/2} \frac{\mathrm{d}\beta}{|\beta|^3},$$

hence

$$J(n) \ll n^2. \tag{9.5}$$

Now recall the definition of $J^*(n)$ in (8.8). We have

$$J(n) - J^*(n) = \int_{n^{-1+1/300}\leqslant|\beta|\leqslant 1/2} v(\beta)^9 \mathrm{e}(-\beta n)\mathrm{d}\beta,$$

which according to Lemma 9.2 is

$$\ll \int_{n^{1/300}/n}^{1/2} \beta^{-3} \mathrm{d}\beta \ll n^{2-\frac{1}{150}}.$$

Using (9.2), (9.3) and (9.5) we find an absolute constant $\delta > 0$ such that

$$R^*(n) = \mathfrak{S}(n)J(n) + O(n^{2-\delta}),$$

which when combined with (7.8), (7.10) and (8.5) yields the following theorem.

Theorem 9.3. *We have*

$$\lim_{n\to+\infty} \left| \frac{R(n)}{n^2} - \frac{\mathfrak{S}(n)J(n)}{n^2} \right| = 0.$$

9.2 The singular integral

The *Beta function* is defined as

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t, \quad \text{for } x, y > 0.$$

Before relating the singular integral $J(n)$ to the Beta function we need some information on sums of values at integers of monotonic functions. Let $y < x$ be integers and let $f : [y, x] \to \mathbb{R}$ be any monotonic function. Comparing the sum $\sum_{y \leqslant n \leqslant x} f(n)$ with the integral $\int_y^x f(t)\mathrm{d}t$ we see that

$$\sum_{y \leqslant n \leqslant x} f(n) = \int_y^x f(t)\mathrm{d}t + O(|f(y)| + |f(x)|).$$

This is also true if we allow x and y to be reals instead of integers, but assume that f does not increase or decrease too fast on small intervals, that is, there is $C > 1$ such that

$$(*) \qquad C^{-1} \leqslant \frac{|f(s)|}{|f(t)|} \leqslant C \quad \text{for all } s, t \in [y, x] \text{ with } |s - t| \leqslant 1.$$

More generally, if $y < x$ are reals, f satisfies (*), and $x_1, \ldots, x_k$ are reals in (y, x) such that $f : [y, x] \to \mathbb{R}$ is monotonic on each interval

$$(y, x_1), (x_1, x_2), \ldots, (x_k, x)$$

then

$$\sum_{y\leqslant m\leqslant x} f(m) = \int_y^x f(t)\mathrm{d}t + O(|f(y)| + |f(x)| + \sum_{i=1}^{k} |f(x_i)|).$$

Let $0 < \beta \leqslant 1$, $\alpha \geqslant \beta$ and consider the function $f(x) := x^{\beta-1}(n-x)^{\alpha-1}$. We want to compare $\sum_{1\leqslant m\leqslant n-1} f(m)$ with $\int_0^n f(t)\mathrm{d}t$, which is an integral that can be handled. There are some subtleties, since f tends to infinity if $\beta < 1$ and $t \downarrow 0$ or if $\alpha < 1$ and $t \uparrow n$. The function f is defined and positive on $(0, n)$, the derivative of f is $x^{\beta-2}(n-x)^{\alpha-2}(x(2-\alpha-\beta) - (1-\beta)n)$, which vanishes at $X = \frac{n(1-\beta)}{2-\alpha-\beta}$, and f satisfies (*) on $[1, n-1]$. If $X \in (1, n-1)$ then f is decreasing on $(1, X)$ and increasing on $(X, n-1)$ and thus

$$\begin{aligned}\sum_{1\leqslant m\leqslant n-1} f(m) &= \int_1^{n-1} f(t)\mathrm{d}t + O(f(1) + f(X) + f(n-1)) \\ &= \int_0^n f(t)\mathrm{d}t + O\Big(f(1) + f(X) + f(n-1) + \int_0^1 f(t)\mathrm{d}t + \int_{n-1}^n f(t)\mathrm{d}t\Big)\end{aligned}$$

and if $X \notin (1, n-1)$ then f is monotone on $(1, n-1)$ and the error term is the same but without $f(X)$. An easy computation shows that

$$\begin{aligned}&f(1) \ll n^{\alpha-1}, \quad f(n-1) \ll n^{\beta-1}, \\ &\int_0^1 f(t)\mathrm{d}t \ll n^{\alpha-1}\int_0^1 t^{\beta-1}\mathrm{d}t \ll n^{\alpha-1}, \quad \int_{n-1}^n f(t)\mathrm{d}t \ll n^{\beta-1}\int_{n-1}^n (n-t)^{\alpha-1}\mathrm{d}t \ll n^{\beta-1},\end{aligned}$$

and since $\alpha \geqslant \beta$ these quantities are all $\ll n^{\alpha-1}$. If $X \in (1, n-1)$ then $f(X) \ll n^{\alpha+\beta-2} \ll n^{\alpha-1}$ since $\beta \leqslant 1$. So in all cases,

$$\sum_{1\leqslant m\leqslant n-1} f(m) = \int_0^n f(t)\mathrm{d}t + O(n^{\alpha-1}).$$

The substitution $t \mapsto y$ given by $t = ny$ shows that

$$\int_0^n f(t)\mathrm{d}t = n^{\alpha+\beta-1}B(\beta, \alpha),$$

therefore

$$\sum_{m=1}^{n-1} m^{\beta-1}(n-m)^{\alpha-1} = n^{\alpha+\beta-1}\left(B(\beta,\alpha) + O(n^{-\beta})\right). \tag{9.6}$$

Before proceeding we need to recall a few standard facts about the *Gamma function*. It is defined as

$$\Gamma(t) := \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \quad \text{for } x > 0$$

and satisfies

$$\Gamma(1) = 1, \tag{9.7}$$

$$\Gamma(t+1) = t\Gamma(t) \quad \text{for } t > 0, \tag{9.8}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for } x, y > 0. \tag{9.9}$$

Observe that $\Gamma(x) = (x-1)!$ for every positive integer x. So $B(x,y) = \frac{x+y}{xy} \cdot \binom{x+y}{x}^{-1}$ for all positive integers x, y.

We have the following theorem.

Theorem 9.4. *We have*

$$J(n) = \Gamma\left(\frac{4}{3}\right)^9 \frac{n^2}{2}\left(1 + O(n^{-1/3})\right).$$

Proof. We begin by proving by induction that for every integer $s \geqslant 2$, that one has

$$\frac{1}{3^s} \sum_{\substack{1 \leqslant m_1, \dots, m_s \leqslant n \\ m_1 + \cdots + m_s = n}} \frac{1}{(m_1 \cdots m_s)^{\frac{2}{3}}} = \Gamma\left(\frac{4}{3}\right)^s \Gamma\left(\frac{s}{3}\right)^{-1} n^{\frac{s}{3}-1}\left(1 + O(n^{-1/3})\right). \tag{9.10}$$

For $s = 2$ this is valid due to (9.6) with $\alpha = \beta = 1/3$, as well as (9.8) and (9.9). Assuming that (9.10) is valid for some integer $s \geqslant 2$ then

$$\frac{1}{3^{s+1}} \sum_{\substack{1 \leqslant m_1, \dots, m_{s+1} \leqslant n \\ m_1 + \cdots + m_{s+1} = n}} \frac{1}{(m_1 \cdots m_{s+1})^{\frac{2}{3}}}$$

equals

$$\sum_{1 \leqslant m_{s+1} \leqslant n-1} \frac{1}{3m_{s+1}^{\frac{2}{3}}} \left(\frac{1}{3^s} \sum_{\substack{1 \leqslant m_1, \dots, m_s \leqslant n \\ m_1 + \cdots + m_s = n - m_{s+1}}} \frac{1}{(m_1 \cdots m_s)^{\frac{2}{3}}}\right),$$

which is

$$\Gamma\left(\frac{4}{3}\right)^{s}\Gamma\left(\frac{s}{3}\right)^{-1}\frac{1}{3}\sum_{1\leqslant m\leqslant n-1}\frac{m^{\frac{1}{3}}}{m}(n-m)^{\frac{s}{3}-1}+O\left(\sum_{1\leqslant m\leqslant n-1}m^{1/3-1}(n-m)^{\frac{(s-1)}{3}-1}\right)$$

due to the induction hypothesis. Using (9.6) with $\beta = \frac{1}{3}$ and $\alpha = \frac{s}{3}, \frac{(s-1)}{3} - 1$ respectively for the main and the error term, we conclude the proof of (9.10).

Combining (9.4) and the definition of v,

$$v(\beta) = \frac{1}{3}\sum_{1\leqslant m\leqslant n}\frac{\mathrm{e}(\beta m)}{m^{\frac{2}{3}}}$$

gives

$$J(n) = \frac{1}{3^9}\sum_{1\leqslant m_1,\ldots,m_9\leqslant n}\frac{1}{(m_1\cdots m_9)^{\frac{2}{3}}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{e}(\beta(n-m_1-\cdots-m_9))\mathrm{d}\beta.$$

The integral vanishes except when $n-m_1-\cdots-m_9=0$, thus obtaining

$$J(n) = \frac{1}{3^9}\sum_{\substack{1\leqslant m_1,\ldots,m_9\leqslant n\\ m_1+\cdots+m_9=n}}\frac{1}{(m_1\cdots m_9)^{\frac{2}{3}}}$$

and according to (9.10) our theorem is valid. $\square$