DIOPHANTINE APPROXIMATION HOMEWORK II

Due November 15

Total number of points: 60. Grade: number of points/6

You may use all theorems from the lecture notes, unless stated otherwise.

8. Theorem 2.21 of the lecture notes (Kronecker's approximation theorem) can be refined as follows:
Let $\xi_1, \dots, \xi_n$ be reals such that $1, \xi_1, \dots, \xi_n$ are linearly independent over $\mathbb{Q}$. Then for every $\varepsilon > 0$, $\theta_1, \dots, \theta_n \in \mathbb{R}$, $y_0 > 0$, there are $x_1, \dots, x_n, y \in \mathbb{Z}$ such that

$$|\xi_1 y - x_1 - \theta_1| < \varepsilon, \dots, |\xi_n y - x_n - \theta_n| < \varepsilon, \quad y > y_0$$

(the condition that $y > y_0$ is new). To get this, the proof of Theorem 2.21 has to be modified as follows. On page 31, choose $\mathbf{b} = (\varepsilon^{-1}\theta_1, \dots, \varepsilon^{-1}\theta_n, 2M\varepsilon)^T$. Then if one is able to show that there is $\mathbf{x} = (x_1, \dots, x_n, y)^T \in \mathbb{Z}^{n+1}$ with $\|A\mathbf{z} - \mathbf{b}\|_2 \leqslant \varepsilon$, it follows that

$$\sum_{i=1}^{n} (x_i - \xi_i y - \theta_i)^2 + M^{-2}(y - 2M\varepsilon)^2 < \varepsilon^2.$$

This shows that $|\xi_i y - x_i - \theta_i| < \varepsilon$ for $i = 1, \dots, n$ and $|y - 2M\epsilon| < M\epsilon$, hence $y > M\epsilon$. Now the proof of Theorem 2.21 can be followed without any changes, except that on page 32, one has to choose $M > \max(R, R/\mu, y_0/\epsilon)$, where $R = (n+1) \cdot c(n+1)/2\epsilon$.

4 *a)* Deduce the following result from the above refinement of Kronecker's Theorem: Let $\xi_1, \dots, \xi_n$ be real numbers, linearly independent over $\mathbb{Q}$. Then for every $t_0 > 0$, $\varepsilon > 0$, $\theta_1, \dots, \theta_n \in \mathbb{R}$, there are $t \in \mathbb{R}$ with $t > t_0$, and $x_1, \dots, x_n \in \mathbb{Z}$, such that

$$|\xi_1 t - x_1 - \theta_1| < \varepsilon, \dots, |\xi_n t - x_n - \theta_n| < \varepsilon$$

(so compared with Kronecker's Theorem, we have weakened the condition that $\{1, \xi_1, \ldots, \xi_n\}$ is linearly independent over $\mathbb{Q}$ to $\{\xi_1, \ldots, \xi_n\}$ linearly independent over $\mathbb{Q}$, but instead of an unknown y assuming integer values we have an unknown t assuming real values).

6 *b)* A star has n planets, all whose orbits are circular with the star in the center and lie in the same plane. Each planet has a constant angular velocity with which it traverses its orbit. Prove that the planets are in almost the same direction infinitely often (i.e., for every $\varepsilon > 0$ there are arbitrarily large t such that at time t, seen from the star the directions of the planets are within an angle $\varepsilon > 0$ from each other) in each of the following two cases:
(i) they once have been in the same direction;
(ii) their angular velocities are linearly independent over $\mathbb{Q}$.

Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number of degree d.

The *denominator* of α is the smallest positive $m \in \mathbb{Z}$ such that $m\alpha$ is an algebraic integer, notation $\mathrm{den}(\alpha)$.

The *house* of α is defined by

$$\lceil\overline{\alpha}\rceil := \max(|\alpha^{(1)}|, \ldots, |\alpha^{(d)}|)$$

where $d = \deg \alpha$ and $\alpha^{(1)}, \ldots, \alpha^{(d)}$ denote the conjugates of α.

In the next exercises you are asked to prove some properties of the house.

1 **9.** *a)* Let α be a non-zero algebraic integer. Prove that $\lceil\overline{\alpha}\rceil \geqslant 1$.

3 *b)* Let α, β be algebraic integers. Prove that

$$\lceil\overline{\alpha + \beta}\rceil \leqslant \lceil\overline{\alpha}\rceil + \lceil\overline{\beta}\rceil, \quad \lceil\overline{\alpha\beta}\rceil \leqslant \lceil\overline{\alpha}\rceil \cdot \lceil\overline{\beta}\rceil, \quad \lceil\overline{\alpha^n}\rceil = \lceil\overline{\alpha}\rceil^n \text{ for } n \in \mathbb{Z}_{>0}.$$

3 *c)* Let α be a non-zero algebraic integer of degree d. Prove that $H(\alpha) \leqslant (2\lceil\overline{\alpha}\rceil)^d$ (consider the minimal polynomial of α).

3 *d)* Compute an explicit expression $f(C, d)$ depending only on C and d, such that the number of algebraic integers $\alpha \in \mathbb{C}$ with $\lceil\overline{\alpha}\rceil \leqslant C$, $\deg \alpha \leqslant d$ is at most $f(C, d)$.

5 *e)* Let α be a non-zero algebraic integer. Prove that $\overline{|\alpha|} = 1 \iff \alpha$ is a root of unity.

5 *f)* Let α be a non-zero algebraic integer of degree d which is not a root of unity. Compute an explicit expression $c(d) > 1$ depending only on d such that $\overline{|\alpha|} \geqslant c(d)$.

Hint. Consider the set $\{\alpha^n : 0 \leqslant n \leqslant n_0\}$ where n_0 is the largest integer n such that $\overline{|\alpha|}^n \leqslant 2$.

Remark. The Schinzel-Zassenhaus conjecture asserts that there is a constant $c > 0$ independent of d, such that $\overline{|\alpha|} \geqslant 1 + c/d$ for every non-zero algebraic integer α of degree d which is not a root of unity. Apart from the value of c this is best possible, since $\overline{|\sqrt[d]{2}|} = 2^{1/d}$ which is about $1 + (\log 2)/d$ for d large. In 1979, Dobrowolski proved that there is a constant $c > 0$ such that

$$\overline{|\alpha|} \geqslant 1 + \frac{c}{d} \cdot \left(\frac{\log\log 3d}{\log 3d} \right)^3 .$$

This had not been improved since. But very recently (last September!), Verger-Gaugry posted a manuscript of 164 pages on **arXiv** in which he claimed a proof of the Schinzel-Zassenhaus conjecture. Presumably, Verger-Gaugry has submitted his paper to a journal, and one or more referees are now checking it for correctness. This may take some time. To my knowledge, this has not been finished yet.

arXiv is a freely accessible preprint server on which researchers in mathematics and natural sciences can also freely post preprints of their papers, prior to their publication in a journal. arXiv does not require that preprints posted on it are refereed, so it can not be excluded that they contain errors. For mathematical preprints, go to the website `xxx.lanl.gov`, then scroll and click on mathematics. For those interested: if in the box 'Search or Artice-id' you type 'Verger-Gaugry' you will find the manuscript mentioned above.

4 **10.** *a)* Let $\alpha \in \overline{\mathbb{Q}}$ be a non-zero algebraic number of degree d and denote by $\alpha^{(1)}, \dots, \alpha^{(d)}$ the conjugates of α. Prove that

$$\mathrm{den}(\alpha)^d \cdot \alpha^{(1)} \cdots \alpha^{(d)} \in \mathbb{Z}, \quad |\alpha| \geqslant \mathrm{den}(\alpha)^{-d} \cdot \lceil\overline{\alpha}\rceil^{1-d}.$$

6 *b)* Using a), give a proof of the following inequality of Liouville (1844): let α be an algebraic number in $\mathbb{R}$ of degree $d \geqslant 2$. Then there is a constant $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{x}{y}\right| \geqslant c(\alpha) y^{-d} \text{ for all } x, y \in \mathbb{Z} \text{ with } y > 0.$$

5 *c)* Using b), prove that $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

11. The Lindemann-Weierstrass Theorem asserts that if $\beta_1, \dots, \beta_n$ are non-zero algebraic numbers and $\alpha_1, \dots, \alpha_n$ are distinct algebraic numbers, all in $\mathbb{C}$, then $\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0$. Deduce the following consequences. The functions $\sin z$, $\cos z$ and $\tan z$ are defined on $\mathbb{C}$ by $e^{iz} = \cos z + i \sin z$, $e^{-iz} = \cos z - i \sin z$ and $\tan z = \sin z / \cos z$ whenever $\cos z \neq 0$.

1 *a)* For an algebraic number $\alpha \in \mathbb{C}$ with $\alpha \notin \{0, 1\}$, prove that $\log \alpha$ is transcendental, where $\log \alpha$ is any solution of $e^z = \alpha$.

4 *b)* For a non-zero algebraic number $\alpha \in \mathbb{C}$, prove that $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ are transcendental.

5 *c)* Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ that are linearly independent over $\mathbb{Q}$. Prove that $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent.

5 *d)* Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers in $\mathbb{C}$. Denote by $\mathrm{rank}_{\mathbb{Q}}(\alpha_1, \dots, \alpha_n)$ the largest integer m such that $\alpha_1, \dots, \alpha_n$ contain m elements that are linearly independent over $\mathbb{Q}$. Prove that

$$\mathrm{trdeg}(e^{\alpha_1}, \dots, e^{\alpha_n}) = \mathrm{rank}_{\mathbb{Q}}(\alpha_1, \dots, \alpha_n).$$