

①

ANSWERS EXAM DIOPHANTINE APPROXIMATION 3/2/2022

① a) The convex body $\{x = (x_1, \dots, x_n) \in \mathbb{R}^{2n} : y_1^2 + y_2^2 \leq 1\}$, $y_1^2 + y_2^2 \leq A_n^2$

has volume $(\pi A^2)^n - (\pi A_n^2)^n = \pi^n (A - A_n)^2$. (Cartesian product of circles)

Minkowski's theorem asserts, if L is a lattice in \mathbb{R}^{2n} and C a central symmetric convex body in \mathbb{R}^{2n} such that $\text{vol}(C) \geq 2^{2n} \det L$, then C contains a non-zero lattice point from L .
So for our C we need $\pi^n (A - A_n)^2 \geq 2^{2n} |\det L|$ which is indeed equivalent to $A - A_n \geq (2/\sqrt{\pi})^{2n} |\det L|^{1/2}$.

b) While $x = x_1 + \sqrt{-d}y_1, y = x_2 + \sqrt{-d}y_2$ with $x_i, y_i \in \mathbb{Z}$ and $d = a+b\sqrt{-d}$ with $a, b \in \mathbb{R}$. Then the condition $\|\beta y\| \leq \frac{4\sqrt{d}}{\pi} Q^{-1}$, $|y| \in \mathbb{Q}$ translates into

$$\left| (x_1 + \sqrt{-d}y_1) - (ax_1 + by_1)(x_2 + \sqrt{-d}y_2) \right|^2 \leq \frac{16d}{\pi^2} Q^{-2}, \quad |x_2 + \sqrt{-d}y_2| \leq Q^{-1/2}$$

or

$$(x_1 - ax_2 + b\sqrt{d}y_2)^2 + (\sqrt{d}y_1 - ay_2 - bx_2)^2 \leq \frac{16d}{\pi^2} Q^{-2}$$

$$x_2^2 + (\sqrt{d}y_2)^2 \leq Q^{-2}$$

Consider the lattice $L = \{(x_1 - ax_2 + b\sqrt{d}y_2, \sqrt{d}y_1 - ay_2 - bx_2, x_2, \sqrt{d}y_2) : x_1, y_1, x_2, y_2 \in \mathbb{Z}\}$

in \mathbb{R}^4 .

This lattice has determinant

$$\begin{vmatrix} 1 & 0 & -a & b\sqrt{d} \\ 0 & \sqrt{d} & -b & -a\sqrt{d} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{d} \end{vmatrix} = d$$

According to 1a), there are $y_1, y_2 \in \mathbb{R}$, not both 0, satisfying (a)
if $\frac{4\sqrt{d}}{\pi} Q^{-1} Q \geq \left(\frac{2}{\sqrt{\pi}}\right)^2 \sqrt{d}$, which is indeed satisfied.

In fact, we must have $y \neq 0$. Otherwise, $|y| \leq \frac{4\sqrt{d}}{\pi} Q^{-1} < 1$,
which is impossible since $|y| \geq 1$ for $y \neq 0$.

(2)

(2) a) It is allowed to use exercise 4.7 (ii)

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be algebraic over \mathbb{Q} and let $\beta_1, \beta_2, \dots, \beta_m$ be transcendental over \mathbb{Q} . Then $e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}$ is transcendental. With $\beta_1 = 1, \beta_2 = 2, \beta_3 = 3$ we get that $e^{\sqrt{2}} \cdot 3^{\sqrt{3}}$ is transcendental.

b) ~~Suppose $\sin \alpha = e^{-\alpha}, \sin \beta = e^{-\beta}$~~ let $u = e^{-\alpha}, v = e^{-\beta}$

Suppose $\sin \alpha, \sin \beta$ are algebraically dependent. Then there is non-zero $P \in \mathbb{Q}(X_1, Y)$ such that $P\left(\frac{u-v}{z_1}, \frac{v-u}{z_2}\right) = 0$

Write $P = \sum_{(j,k) \in J} q_{jk} X^j Y^k$ with $q_{jk} \in \mathbb{Q}$ for $(j, k) \in J$ (finite set)

Then $\sum_{(j,k) \in J} q_{jk} \left(\frac{u-v}{z_1}\right)^j \left(\frac{v-u}{z_2}\right)^k = 0$. Let m be the maximum of the j occurring in J

and n the maximum of the k occurring in J . Then

$$\sum_{(j,k) \in J} q_{jk} u^{m_j} \left(\frac{v-u}{z_1}\right)^j \left(\frac{v-u}{z_2}\right)^k = 0$$

implying that u^{m_j}, v^{n_k} are algebraically dependent

But a theorem states: if $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over \mathbb{Q} , then $e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}$ are algebraically independent

In our case $\pi, \theta, \alpha_1, \dots, \alpha_n$ are linearly independent. This gives a contradiction.

c) Schanuel's conjecture asserts that if $x_1, x_n \in \mathbb{C}$ are linearly independent over \mathbb{Q} , then $\text{ind}_{\mathbb{Q}}(x_1, x_n, e^{\alpha_1}, \dots, e^{\alpha_n}) \geq n$

In our case, we have $\text{ind}_{\mathbb{Q}}(1, \pi, \pi i, e, e^{\pi}, e^{\pi i}) \geq 3$

which implies $\text{ind}_{\mathbb{Q}}(\pi, e, e^{\pi}) \geq 3$ since πi is algebraically dependent on π and $e^{\pi i - \pi}$. But we have to verify that $1, \pi, \pi i$ are linearly independent over \mathbb{Q} . Assume the contrary, i.e., $a + b\pi + c\pi i = 0$ for some $a, b, c \in \mathbb{Q}$, not all 0. At least one of b, c is $\neq 0$ so $a + (b+c)\pi = 0$, $\pi = \frac{-a}{b+c}$ which is impossible since $\pi \notin \mathbb{Q}$.

(3)

a) Let $P(X) = p_0 + p_1 X + \dots + p_n X^n$ with $p_i \in \mathbb{Z}$, $p_n \neq 0$.
 By exercise 3.5, $\beta(\sqrt[n]{\alpha}) \in \overline{\mathbb{D}(\sqrt[n]{\alpha})}$, $\beta(\sqrt[n]{\alpha}) \in \overline{\mathbb{D}(\sqrt[n]{\alpha})}$ for $\beta \in Q$.
 So $|P(\sqrt[n]{\alpha})| = |p_0 + p_1 \sqrt[n]{\alpha} + \dots + p_n (\sqrt[n]{\alpha})^n| \leq |p_0| + |p_1| \sqrt[n]{1} + \dots + |p_n| \sqrt[n]{1}^n$
 $\leq H(1 + \sqrt[n]{1} + \dots + \sqrt[n]{1}) \in H(1 + \sqrt[n]{1}).$

Let $m = \text{den}(\alpha)$, so $m \in \mathbb{Z}$. Then

$$m^n P(\alpha) = m^n (p_0 + p_1 \alpha + \dots + p_n \alpha^n) = m^n p_0 + m^n p_1 (\alpha) + m^n p_2 (\alpha^2) + \dots + m^n p_n (\alpha^n) \in \mathbb{Z}, \text{ since } \mathbb{Z} \text{ is a ring.}$$

b) By exercise 3.7 we have, for every algebraic number α of degree d , $|\alpha| \geq (\text{den}(\alpha))^{-d} |\sqrt[d]{1}|^{1-d}$, provided $\alpha \neq 0$.

So

$$|P(\alpha)| \geq (\text{den}(\alpha))^{-d} |P(\sqrt[d]{\alpha})|^{1-d} \geq (\text{den}(\alpha))^{-d} (H(1 + \sqrt[d]{1}))^{1-d}$$

$$= (\text{den}(\alpha))^{-d} H^{1-d} (1 + \sqrt[d]{1})^{n(d-1)}.$$

c) Apply b) to $P(X) = X \cdot X - y$. Then we get

$$|x \cdot \sqrt[d]{y}| \geq (\text{den}(\alpha))^{-d} \cdot \max(|x|, |y|)^{1-d} \cdot (1 + \sqrt[d]{1})^{n(d-1)}.$$

So

$$|x \cdot \sqrt[d]{y}| \geq (\text{den}(\alpha))^{-d} \max(|x|, |y|)^{1-d} (1 + \sqrt[d]{1})^{n(d-1)}.$$

(4)

a) Dirichlet's Theorem asserts that if $x \in \mathbb{P} \setminus \mathbb{Q}$, then there are infinitely many pairs $(x, y) \in \mathbb{Z}^2$ with $|x - y| < \epsilon |y|^2$, $y \neq 0$.

In particular, this implies that there are infinitely many pairs $(x_1, x_2) \in \mathbb{Z}^2$ with $|x_1 - \alpha x_2| < |x_2|^2$, $x_2 \neq 0$.

For such pairs we have $|x_1| \leq 1 + |\alpha| |x_2| \leq (1 + |\alpha|) |x_2|$,
 $\max(|x_1|, |x_2|) \leq (1 + |\alpha|) |x_2|$.

It follows that if $x = (x_1, x_2, 0)$, then
 $|(\bar{x}_1 - \alpha \bar{x}_2 + \beta_1 x_3)(\bar{x}_1 - \alpha \bar{x}_2 + \beta_2 x_3)| \leq |\bar{x}_1 - \alpha \bar{x}_2|^2 \leq |x_2|^2 \leq (1 + |\alpha|)^2 \|x\|^2$
 $\leq C \|x\|^{-1-\delta}$ if $\|x\|$ is sufficiently large.

These x provide infinitely many solutions of (*) with $x_3 \neq 0$.

b) If $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ is a solution of (*), then if
 $|(\bar{x}_1 - \alpha \bar{x}_2 + \beta_1 x_3)(\bar{x}_1 - \alpha \bar{x}_2 + \beta_2 x_3)| \geq C |x_1| \|x\|^{-1-\delta} \leq C \|x\|$

The three linear forms $\bar{x}_1 - \alpha \bar{x}_2 + \beta_1 x_3$, $\bar{x}_1 - \alpha \bar{x}_2 + \beta_2 x_3$, x_1 are linearly independent, since their coefficient determinant is

$$\begin{vmatrix} 1 & -\alpha & \beta_1 \\ 1 & -\alpha & \beta_2 \\ 1 & 0 & 0 \end{vmatrix} = |-\alpha(\beta_2 - \beta_1)| \neq 0.$$

Schmidt's Subspace Theorem asserts that if l_1, l_2 are linearly independent linear forms in $\mathbb{Q}(X_1, X_2)$, and C_{20}, δ_{20} , then no solutions of $|l_1(x) - l_2(x)| \leq C \|x\|^{-\delta}$, $x \in \mathbb{Z}^2$ lie in finitely many proper linear subspaces of \mathbb{Q}^2 . By applying this, we get (b).

(c) By (b) we have to show that each two-dimensional linear subspace of $\mathbb{Q}^3 \times \{x_3=0\}$ contains only finitely many solutions of (*). Consider such a subspace, say given by an equation $ax_1 + bx_2 + cx_3 = 0$ with $a, b, c \in \mathbb{Q}$ and at least one of $a, b \neq 0$.

Note that the problem is symmetric in x_1, x_2 . For we can rewrite

$$(*) \text{ as } |x_2 - \alpha' x_1 - \frac{b}{a} x_3|, |x_2 - \alpha'' x_1 - \frac{b}{a} x_3| \leq (C/|a|^2) \|x\|^{-1-\delta}.$$

So there is no loss of generality to assume that $a \neq 0$.

(5)

Then we may as well assume that $\alpha_3 \neq 0$, and thus that
 The subspace is given by $x = bx_2 + cx_3$ with $b, c \in \mathbb{Q}$. By
 substituting this into (7) we get

$$(7) \quad ((b\alpha_1)x_1 + (c\beta_1)x_3), ((b\alpha_2)x_2 + (c\beta_2)x_3) \in \left(\max(|x_1|, |x_2|) \right)^{\text{ad}}$$

These two linear forms have determinant $\begin{vmatrix} b\alpha_1 & c\beta_1 \\ b\alpha_2 & c\beta_2 \end{vmatrix} = (b\alpha_1)(c\beta_2) - (b\alpha_2)(c\beta_1)$
 since $\alpha_1 \beta_2$ and $\beta_1 \alpha_2$ are linearly independent.
 Hence the solutions of (7) lie in only finitely many one-dimensional
 linear subspaces of \mathbb{Q}^2 . But since $x = bx_2 + cx_3$ it follows that
 the solutions of (7) in T lie in only finitely many one-dimensional
 subspaces of T . So the solutions of (7) with $x_3 \neq 0$ lie in only
 finitely many one-dimensional linear subspaces altogether.
 Now let S be such a one-dimensional subspace, say
 $\mathbb{Q}\{d(x_1, x_2)\} \cap T(\mathbb{Q})$. S contains only two points (α_1, β_1)
 with $\gcd(x_1, x_2) = 1$. Hence (7) has only finitely many solutions
 with $x_3 \neq 0$, $\gcd(x_1, x_2) = 1$.