# EXAM DIOPHANTINE APPROXIMATION

**Thursday February 3, 2022, 14:00-17:00**

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• Write down your name (in CAPITALS), university and student number on each paper sheet. Take care that these are **VERY WELL READABLE**.

• To facilitate the grading, please give your answers in English.

• This exam consists of four pages, each with one exercise.

• The maximal number of points for each part of an exercise is indicated in the left margin. Grade = (total number of points)/10.

• Unless otherwise stated, you may use without proof the theorems and the exercises from the lecture notes. But you have to mention the theorems and exercises you are using (e.g., by name such as for instance Minkowski's first convex body theorem, or by the number from the lecture notes).

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10 **1.** *a)* Let $L$ be a lattice in $\mathbb{R}^{2n}$ and $A_1, \dots, A_n$ positive reals with

$$A_1 \cdots A_n \geq \left(2/\sqrt{\pi}\right)^n (\det L)^{1/2}.$$

Prove that there is a nonzero $\mathbf{u} = (u_1, \dots, u_{2n}) \in L$ with

$$u_1^2 + u_2^2 \leq A_1^2, \dots, u_{2n-1}^2 + u_{2n}^2 \leq A_n^2.$$

15 *b)* Let $R_d = \{x + y\sqrt{-d} \ : \ x, y \in \mathbb{Z}\}$ where $d$ is a positive integer. Further, let $\alpha \in \mathbb{C}$, $Q > 4\sqrt{d}/\pi$.

Prove that there are $\xi, \eta \in R_d$ with

$$|\xi - \alpha\eta| \leq \frac{4\sqrt{d}}{\pi} Q^{-1}, \quad |\eta| \leq Q, \quad \eta \neq 0.$$

In what follows, $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers in $\mathbb{C}$.

5 **2.** *a)* Prove that $e^{\sqrt{2}}3^{\sqrt{5}}$ is transcendental.

10 *b)* Let $\alpha, \beta \in \overline{\mathbb{Q}}$ with $\alpha\beta \neq 0$ and $\alpha/\beta \notin \mathbb{Q}$. Prove that $\sin\alpha$, $\sin\beta$ are algebraically independent.

10 *c)* Assuming Schanuel's conjecture, prove that $e, \pi, e^{\pi}$ are algebraically independent.

Recall that the house $\overline{|\beta|}$ of an algebraic number $\beta$ is the maximum of the absolute values of its conjugates. Further, the denominator $\mathrm{den}(\beta)$ of $\beta$ is the smallest positive integer $m$ such that $m\beta$ is an algebraic integer.

**3.** Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number of degree $d$ and $P \in \mathbb{Z}[X]$ a non-zero polynomial of degree $n$. Suppose that the coefficients of $P$ have absolute values at most $H$.

10 *a)* Prove that

$$\overline{|P(\alpha)|} \leq H \cdot (1 + \overline{|\alpha|})^n \quad \text{and } \mathrm{den}(P(\alpha)) \leq (\mathrm{den}(\alpha))^n.$$

7 *b)* Suppose that $P(\alpha) \neq 0$. Prove that

$$|P(\alpha)| \geq (\mathrm{den}(\alpha))^{-nd} H^{1-d} (1 + \overline{|\alpha|})^{n(1-d)}.$$

8 *c)* Let $x, y \in \mathbb{Z}$ with $y \neq 0$. Deduce from b) that

$$\left|\alpha - \frac{x}{y}\right| \geq (\mathrm{den}(\alpha))^{-d} (1 + \overline{|\alpha|})^{1-d} \big(\max(|x|, |y|)\big)^{-d}$$

**4.** Let $\alpha, \beta_1, \beta_2 \in \overline{\mathbb{Q}}$ be such that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta_1 \neq \beta_2$. Further, let $C > 0$, $0 < \delta < 1$. Consider the inequality

$$(*) \qquad |(x_1 - \alpha x_2 + \beta_1 x_3)(x_1 - \alpha x_2 + \beta_2 x_3)| \leq C \|\mathbf{x}\|^{-1-\delta}$$
$$\text{in } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3 \text{ with } \gcd(x_1, x_2, x_3) = 1,$$

where $\|\mathbf{x}\| := \max(|x_1|, |x_2|, |x_3|)$.

8 *a)* Prove that (*) has infinitely many solutions with $x_3 = 0$.

7 *b)* Prove that the set of solutions of (*) is contained in the union of finitely many two-dimensional linear subspaces of $\mathbb{Q}^3$.

10 *c)* Prove that (*) has only finitely many solutions with $x_3 \neq 0$.
**Hint.** You have to show that each two-dimensional linear subspace $T$ of $\mathbb{Q}^3$ different from $x_3 = 0$ contains only finitely many solutions. The subspace $T$ is given by an equation $ax_1 + bx_2 + cx_3 = 0$ with $a, b, c \in \mathbb{Q}$, where at least one of $a, b$ is non-zero. By symmetry, you may assume that $a \neq 0$.