Bistable Reaction-Diffusion Systems with Infinite-Range Interaction

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Motivated by the study of physical structures such as crystals, grids of neurons and population patches, an increasing interest has arisen in mathematical modelling techniques that reflect the underlying spatial discreteness. The main goal of this project is to deepen our understanding of the differences and similarities between such spatially discrete systems and their traditional continuous counterparts.

Lattice Differential Equations

In this project, we will consider a bistable lattice differential equation (LDE) that can be written as

$$\dot{u}_i(t) = \sum_{j\in\mathbb{Z}} \alpha_j [u_{i+j}(t) - u_i(t)] + g\big(u_i(t);a\big), \qquad i \in \mathbb{Z}, \tag{1}$$

with prototypical cubic nonlinearity $g(u;a) = u(1-u)(u-a)$ for some $a \in (0,1)$. In particular, the coupling between lattice sites has infinite range if there are infinitely many $j \in \mathbb{Z}$ for which $\alpha_j \neq 0$.

Let us first consider the special case with homogeneous nearest-neighbour coupling, for which we have

$$\alpha_{-1} = \alpha_{+1} = \alpha > 0 \text{ and } \alpha_j = 0 \text{ for } |j| \neq 1. \tag{2}$$

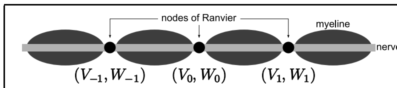

Fig. 1: Myelinated nerve fibres.

In this case, (1) reduces to

$$\dot{u}_i(t) = \alpha\big[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)\big] + g\big(u_i(t);a\big), \qquad i \in \mathbb{Z}, \tag{3}$$

which is commonly referred to as the one-dimensional discrete Nagumo equation.

The Nagumo equation is a phenomenological model in which two stable equilibria compete for dominance in a spatial domain. In modelling contexts one often thinks of these equilibria as representing material phases or chemical or biological species. For example, (3) arises naturally when studying the propagation of electrical signals through nerve fibres. Such fibres are insulated by a myeline coating that admits gaps at the so-called nodes of Ranvier [4], which are regularly spaced along the fibre; see Fig. 1. Excitations of the nerve effectively jump from one node to the next, through a process called saltatory conduction [1, 2]. The variable u_i in (1) encodes the potential at the i-th node of Ranvier and the cubic $g(u;a)$ describes the ionic interactions.

Upon choosing $\alpha = h^{-2} > 0$, the LDE (3) can be seen as the nearest-neighbour discretization of the heavily studied PDE

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + g\big(u(x,t);a\big), \qquad x \in \mathbb{R}, \tag{4}$$

on a lattice with internode distance h. Studying higher order discretizations or fractional versions of the Laplacian ∂_x^2 leads to LDEs of the form (1) with infinite range interactions.

Travelling waves

A natural place to start the analysis of (3) is to look for planar travelling wave solutions that connect the two stable equilibria $u \equiv 0$ and $u \equiv 1$. Such waves can be written as

$$u_i(t) = \Phi(i + ct); \quad \Phi(-\infty) = 0, \qquad \Phi(\infty) = 1, \tag{5}$$

Substituting this Ansatz into (1), we find that the wave profile $\Phi : \mathbb{R} \to \mathbb{R}$ necessarily satisfies the system

$$c\Phi'(\xi) \quad = \quad \alpha\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g(\Phi(\xi), a). \tag{6}$$

It is known [3] that (6) admits solutions (c, Φ) for all $0 < a < 1$. In addition, the wave speed c is unique once a is fixed, but the profile Φ is only unique (up to translations) if $c \neq 0$.

Propogation Failure

This condition on c reflects a crucial difference between the LDE (1) and its continuous counterpart (4). Indeed, the wave speed c appears in travelling wave MFDE (6) in front of the highest derivative, which should be contrasted to the travelling wave ODE

$$c\Phi'(\xi) = \Phi''(\xi) + g(\Phi(\xi); a) \tag{7}$$

that is associated to the PDE (4).

Fig. 2 illustrates the far-reaching consequences that this singular dependence can have: the wave speed c may vanish for all parameter values a in some interval $[a_*, \frac{1}{2}]$, with $a_* < \frac{1}{2}$. This phenomenon is called propagation failure and is present throughout a wide range of discrete systems. It can be interpreted as the consequence of an energy barrier caused by the gaps, which must be overcome in order to allow propagation.

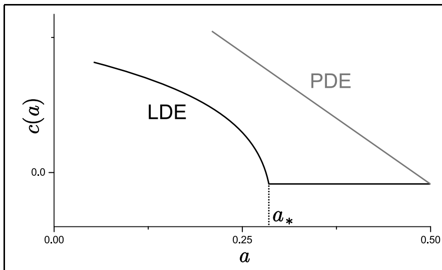

Fig. 2: Typical plot of the wave speed c as a function of a.

Infinite range coupling

In this project we intend to study travelling wave solutions to (1) with infinite range coupling. The main goals are to generalize existence and stability results for the travelling wave solutions of (3) to the setting of (1), using a number of new tools that have been recently developed. At the end of the project you will have learned about Fredholm operators, comparison principles and Green's functions, which all play a critical role in the field of dynamical systems.

References

[1] A. F. Huxley and R. Stampfli (1949), Evidence for Saltatory Conduction in Peripheral Meylinated Nerve Fibres. *J. Physiology* **108**, 315–339.

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[4] L. A. Ranvier (1878), *Lećons sur l'Histologie du Système Nerveux, par M. L. Ranvier, recueillies par M. Ed. Weber.* F. Savy, Paris.