Summer School in Statistics for Astronomers VII

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Inference II: Maximum Likelihood Estimation, the Cramér-Rao Inequality, and the Bayesian Information Criterion

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The Method of Maximum Likelihood

R. A. Fisher (1912), "On an absolute criterion for fitting frequency curves," Messenger of Math. **41**, 155–160

Fisher's first mathematical paper, written while a final-year undergraduate in mathematics and mathematical physics at Cambridge University

It's not clear what motivated Fisher to study this subject; perhaps it was the influence of his tutor, the *astronomer* F. J. M. Stratton.

Fisher's paper started with a criticism of two methods of curve fitting, least-squares and the method of moments.

X: a random variable

θ is a parameter

$f(x;\theta)$: A statistical model for X

$X_1, \ldots, X_n$: A random sample from X

We want to construct good estimators for θ

Protheroe, et al. “Interpretation of cosmic ray composition - The path length distribution,” ApJ., 247 1981

X: Length of paths

Parameter: $\theta > 0$

Model: The exponential distribution,

$$f(x;\theta) = \theta^{-1}\,\text{exp}(-x/\theta), \quad x > 0$$

Under this model,

$$E(X) = \int_0^\infty x\, f(x;\theta)\, dx = \theta$$

Intuition suggests using $\bar{X}$ to estimate θ

$\bar{X}$ is unbiased and consistent

LF for globular clusters in the Milky Way; van den Bergh's normal model,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

μ: Mean visual absolute magnitude

σ: Std. deviation of visual absolute magnitude

$\bar{X}$ is a good estimator for μ

S^2 is a good estimator for σ^2

We seek a method which produces good estimators automatically

Fisher's brilliant idea: The method of maximum likelihood

Choose a globular cluster at random; what is the chance that the LF will be *exactly* -7.1 mag? *Exactly* -7.2 mag?

For any continuous random variable X,

$$P(X = c) = 0$$

Suppose $X \sim N(\mu = -6.9, \sigma^2 = 1.21)$

X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$P(X = -7.1) = 0$, but

$$\begin{aligned} f(-7.1) &= \frac{1}{1.1\sqrt{2\pi}} \exp\left[-\frac{(-7.1+6.9)^2}{2(1.1)^2}\right] \\ &= 0.37 \end{aligned}$$

Interpretation: In one simulation of the random variable X, the “likelihood” of observing the number -7.1 is 0.37

$f(-7.2) = 0.28$

In one simulation of X, the value $x = -7.1$ is 32% more likely to be observed than the value $x = -7.2$

$x = -6.9$ is the value which has the greatest (or maximum) likelihood, for it is where the probability density function is at its maximum

Return to a general model $f(x;\theta)$

Random sample: $X_1, \ldots, X_n$

Recall that the X_i are independent random variables

The *joint* probability density function of the sample is

$$f(x_1;\theta)f(x_2;\theta)\cdots f(x_n;\theta)$$

Here the variables are the X's, while θ is fixed

Fisher's brilliant idea: Reverse the roles of the x's and θ

Regard the X's as fixed and θ as the variable

The *likelihood function* is

$$L(\theta; X_1, \ldots, X_n) = f(X_1; \theta) f(X_2; \theta) \cdots f(X_n; \theta)$$

Simpler notation: $L(\theta)$

$\widehat{\theta}$, the *maximum likelihood estimator* of θ, is the value of θ where L is maximized

$\widehat{\theta}$ is a function of the X's

Caution: The MLE is not always unique.

Example: "... cosmic ray composition - The path length distribution ..."

X: Length of paths

Parameter: $\theta > 0$

Model: The exponential distribution,

$$f(x;\theta) = \theta^{-1} \exp(-x/\theta), \quad x > 0$$

Random sample: $X_1, \ldots, X_n$

Likelihood function:

$$\begin{aligned} L(\theta) &= f(X_1;\theta) f(X_2;\theta) \cdots f(X_n;\theta) \\ &= \theta^{-n} \exp(-(X_1 + \cdots + X_n)/\theta) \\ &= \theta^{-n} \exp(-n\bar{X}/\theta) \end{aligned}$$

To maximize L, we use calculus

It is also equivalent to maximize $\ln L$:

$$\begin{aligned}
\ln L(\theta) &= -n\ln(\theta) - n\bar{X}\theta^{-1} \\
\frac{d}{d\theta}\ln L(\theta) &= -n\theta^{-1} + n\bar{X}\theta^{-2} \\
\frac{d^2}{d\theta^2}\ln L(\theta) &= n\theta^{-2} - 2n\bar{X}\theta^{-3}
\end{aligned}$$

Solve the equation $d\ln L(\theta)/d\theta = 0$:

$$\theta = \bar{X}$$

Check that $d^2\ln L(\theta)/d\theta^2 < 0$ at $\theta = \bar{X}$

$\ln L(\theta)$ is maximized at $\theta = \bar{X}$

Conclusion: The MLE of θ is $\hat{\theta} = \bar{X}$

LF for globular clusters; $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Assume that σ is known (1.1 mag, say)

Random sample: $X_1, \ldots, X_n$

Likelihood function:

$$\begin{aligned} L(\mu) &= f(X_1; \mu) f(X_2; \mu) \cdots f(X_n; \mu) \\ &= (2\pi)^{-n/2} \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right] \end{aligned}$$

Maximize $\ln L$ using calculus: $\widehat{\mu} = \bar{X}$

LF for globular clusters; $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Both μ and σ are unknown

A likelihood function of two variables,

$$\begin{aligned} L(\mu, \sigma^2) &= f(X_1; \mu, \sigma^2) \cdots f(X_n; \mu, \sigma^2) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right] \end{aligned}$$

$$\ln L = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

$$\begin{aligned} \frac{\partial}{\partial\mu} \ln L &= \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) \\ \frac{\partial}{\partial(\sigma^2)} \ln L &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} (X_i - \mu)^2 \end{aligned}$$

Solve for μ and σ^2 the simultaneous equations:

$$\frac{\partial}{\partial\mu} \ln L = 0, \qquad \frac{\partial}{\partial(\sigma^2)} \ln L = 0$$

We also verify that L is concave at the solutions of these equations (Hessian matrix)

Conclusion: The MLEs are

$$\widehat{\mu} = \bar{X}, \qquad \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\widehat{\mu}$ is unbiased: $E(\widehat{\mu}) = \mu$

$\widehat{\sigma}^2$ is not unbiased: $E(\widehat{\sigma}^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$

For this reason, we use $\frac{n}{n-1}\widehat{\sigma}^2 \equiv S^2$

Calculus cannot always be used to find MLEs

Example: "... cosmic ray composition ..."

Parameter: $\theta > 0$

Model: $f(x;\theta) = \begin{cases} \text{exp}(-(x-\theta)), & x \ge \theta \\ 0, & x < \theta \end{cases}$

Random sample: $X_1, \ldots, X_n$

$$\begin{aligned} L(\theta) &= f(X_1;\theta) \cdots f(X_n;\theta) \\ &= \begin{cases} \text{exp}(-\sum_{i=1}^{n}(X_i - \theta)), & \text{all } X_i \ge \theta \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$X_{(1)}$: The smallest observation in the sample

"all $X_i \ge \theta$" is equivalent to "$X_{(1)} \ge \theta$"

$$L(\theta) = \begin{cases} \text{exp}(-n(\bar{X} - \theta)), & \theta \le X_{(1)} \\ 0, & \text{otherwise} \end{cases}$$

Conclusion: $\widehat{\theta} = X_{(1)}$

General Properties of the MLE $\widehat{\theta}$

(a) $\widehat{\theta}$ may not be unbiased. We often can remove this bias by multiplying $\widehat{\theta}$ by a constant.

(b) For many models, $\widehat{\theta}$ is consistent.

(c) The Invariance Property: For many nice functions g, if $\widehat{\theta}$ is the MLE of θ then $g(\widehat{\theta})$ is the MLE of $g(\theta)$.

(d) The Asymptotic Property: For large n, $\widehat{\theta}$ has an approximate normal distribution with mean θ and variance $1/B$ where

$$B = nE\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right]^2$$

The asymptotic property can be used to develop confidence intervals for θ

The method of maximum likelihood works well when intuition fails and no obvious estimator can be found.

When an obvious estimator exists the method of ML often will find it.

The method can be applied to many statistical problems: regression analysis, analysis of variance, discriminant analysis, hypothesis testing, principal components, etc.

The ML Method for Linear Regression Analysis

Scatterplot data: $(x_1, y_1), \ldots, (x_n, y_n)$

Basic assumption: The x_i's are non-random measurements; the y_i are observations on Y, a random variable

Statistical model:

$$Y_i = \alpha + \beta x_i + \epsilon_i, \qquad i = 1, \ldots, n$$

Errors $\epsilon_1, \ldots, \epsilon_n$: a random sample from $N(0, \sigma^2)$

Parameters: α, β, σ^2

$Y_i \sim N(\alpha + \beta x_i, \sigma^2)$: The Y_i's are independent

The Y_i are not identically distributed, because they have differing means

The likelihood function is the joint density function of the observed data, $Y_1, \ldots, Y_n$

$$L(\alpha, \beta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(Y_i - \alpha - \beta x_i)^2}{2\sigma^2}\right]$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left[-\frac{\sum\limits_{i=1}^{n} (Y_i - \alpha - \beta x_i)^2}{2\sigma^2}\right]$$

Use partial derivatives to maximize L over all α, β and $\sigma^2 > 0$ (Wise advice: Maximize $\ln L$)

The ML estimators are:

$$\widehat{\beta} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \widehat{\alpha} = \bar{Y} - \widehat{\beta}\bar{x}$$

and

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \widehat{\alpha} - \widehat{\beta} x_i)^2$$

The ML Method for Testing Hypotheses

$X \sim N(\mu, \sigma^2)$; parameters μ and σ^2

Model: $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

Random sample: $X_1, \ldots, X_n$

We wish to test $H_0 : \mu = 3$ vs. $H_a : \mu \neq 3$

Parameter space: The space of all permissible values of the parameters

$$\Omega = \{(\mu, \sigma) : -\infty < \mu < \infty, \sigma > 0\}$$

H_0 and H_a represent restrictions on the parameters, so we are led to parameter subspaces

$$\omega_0 = \{(\mu, \sigma) : \mu = 3, \sigma > 0\}$$
$$\omega_a = \{(\mu, \sigma) : \mu \neq 3, \sigma > 0\}$$

$$\begin{aligned} L(\mu,\sigma^2) &= f(X_1;\mu,\sigma^2)\cdots f(X_n;\mu,\sigma^2) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\right] \end{aligned}$$

Maximize $L(\mu,\sigma^2)$ over ω_0 and ω_a

The *likelihood ratio test statistic* is

$$\lambda = \frac{\max_{\omega_0} L(\mu,\sigma^2)}{\max_{\omega_a \cup \omega_0} L(\mu,\sigma^2)} = \frac{\max_{\sigma>0} L(3,\sigma^2)}{\max_{\sigma>0,\mu} L(\mu,\sigma^2)}$$

Fact: $0 \le \lambda \le 1$

$L(3,\sigma^2)$ is maximized over ω_0 at

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-3)^2$$

$$\begin{aligned}\max_{\omega_0} L(3,\sigma^2) =& L\left(3, \tfrac{1}{n}\sum_{i=1}^{n}(X_i-3)^2\right)\\ &= \left[\frac{n}{2\pi e\sum_{i=1}^{n}(X_i-3)^2}\right]^{n/2}\end{aligned}$$

$L(\mu,\sigma^2)$ is maximized over ω_a at

$$\mu=\bar{X}, \quad \sigma^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

$$\begin{aligned}\max_{\omega_a\cup\omega_0} L(\mu,\sigma^2) =& L\left(\bar{X}, \tfrac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)\\ &= \left[\frac{n}{2\pi e\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]^{n/2}\end{aligned}$$

The likelihood ratio test statistic:

$$\lambda = \left[\frac{n}{2\pi e \sum_{i=1}^{n}(X_i - 3)^2}\right]^{n/2} \div \left[\frac{n}{2\pi e \sum_{i=1}^{n}(X_i - \bar{X})^2}\right]^{n/2}$$

$$= \left[\sum_{i=1}^{n}(X_i - \bar{X})^2 \div \sum_{i=1}^{n}(X_i - 3)^2\right]^{n/2}$$

λ is close to 1 iff $\bar{X}$ is close to 3

λ is close to 0 iff $\bar{X}$ is far from 3

This particular LRT statistic λ is equivalent to the t-statistic seen earlier

In this case, the ML method discovers the obvious test statistic

Given two unbiased estimators, we prefer the one with smaller variance

In our quest for unbiased estimators with minimum possible variance, we need to know how small their variances can be

Parameter: θ

X: Random variable with model $f(x;\theta)$

The "support" of f is the region where $f > 0$

We assume that the "support" of f does not depend on θ

Random sample: $X_1, \ldots, X_n$

Y: An unbiased estimator of θ

The Cramér-Rao Inequality: The smallest possible value that Var(Y) can attain is $1/B$ where

$$B = nE\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right]^2 = -nE\left[\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right]$$

Example: "... cosmic ray composition - The path length distribution ..."

X: Length of paths

Parameter: $\theta > 0$

Model: $f(x;\theta) = \theta^{-1}\,\text{exp}(-x/\theta), \quad x > 0$

$$\begin{aligned}
\ln f(X;\theta) &= -\ln\theta - \theta^{-1}X \\
\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta) &= \theta^{-2} - 2\theta^{-3}X
\end{aligned}$$

$$\begin{aligned}
E\left[\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right] &= E(\theta^{-2} - 2\theta^{-3}X) \\
&= \theta^{-2} - 2\theta^{-3}E(X) \\
&= \theta^{-2} - 2\theta^{-3}\theta \\
&= -\theta^{-2}
\end{aligned}$$

The smallest possible value of Var(Y) is θ^2/n

This is attained by $\bar{X}$. For this problem, $\bar{X}$ is *the* best unbiased estimator of θ

Y: An unbiased estimator of a parameter θ

We compare Var(Y) with $1/B$, the lower bound in the Cramér-Rao inequality:

$$\frac{1}{B} \div \text{Var}(Y)$$

This number is called the *efficiency* of Y

Obviously, $0 \le$ efficiency ≤ 1

If Y has 50% efficiency then about $1/0.5 = 2$ times as many sample observations are needed for Y to perform as well as the MVUE.

The use of Y result in confidence intervals which generally are longer than those arising from the MVUE.

If the MLE is unbiased then as n becomes large, its efficiency increases to 1.

The Cramér-Rao inequality states that if Y is any unbiased estimator of θ then

$$\mathrm{Var}(Y) \geq \frac{1}{nE\left[\frac{\partial}{\partial\theta} \ln f(X;\theta)\right]^2}$$

The Heisenberg uncertainty principle is known to be a consequence of the Cramér-Rao inequality.

Dembo, Cover, and Thomas (1991) provide a unified treatment of the Cramér-Rao inequality, the Heisenberg uncertainty principle, entropy inequalities, Fisher information, and many other inequalities in statistics, mathematics, information theory, and physics. This remarkable paper demonstrates that there is a basic oneness among these various fields.

Reference

Dembo, Cover, and Thomas (1991), "Information-theoretic inequalities," IEEE Trans. Information Theory 37, 1501–1518.

The Bayesian Information Criterion

Suppose that we have two competing statistical models

We can fit these models using residual sums of squares, the method of moments, the method of maximum likelihood, ...

The choice of model cannot be assessed entirely by these methods

By increasing the number of parameters, we can always reduce the residual sums of squares

Polynomial regression: By increasing the number of terms, we can reduce the residual sum of squares

More complicated models generally will have lower residual errors

A standard approach to hypothesis testing for *large* data sets is to use the Bayesian information criterion (BIC).

The BIC penalizes models with greater numbers of free parameters

Two competing models:
$f_1(x; \theta_1, \ldots, \theta_{m_1})$ and $f_2(x; \phi_1, \ldots, \phi_{m_2})$

Random sample: $X_1, \ldots, X_n$

Likelihood functions:
$L_1(\theta_1, \ldots, \theta_{m_1})$ and $L_2(\phi_1, \ldots, \phi_{m_2})$

Bayesian Information Criterion:

$$\text{BIC} = 2 \ln \frac{L_1(\theta_1, \ldots, \theta_{m_1})}{L_2(\phi_1, \ldots, \phi_{m_2})} - (m_1 - m_2) \ln n$$

The BIC balances any improvement in the likelihood with the number of model parameters used to achieve that improvement

Calculate all MLEs $\widehat{\theta}_i$ and $\widehat{\phi}_i$

Compute the estimated BIC:

$$\widehat{\mathsf{BIC}} = 2 \ln \frac{L_1(\widehat{\theta}_1, \ldots, \widehat{\theta}_{m_1})}{L_2(\widehat{\phi}_1, \ldots, \widehat{\phi}_{m_2})} - (m_1 - m_2) \ln n$$

General rules:

$\widehat{\mathsf{BIC}} < 2$: Weak evidence that Model 1 is superior to Model 2

$2 \le \widehat{\mathsf{BIC}} \le 6$: Moderate evidence that Model 1 is superior to Model 2

$6 < \widehat{\mathsf{BIC}} \le 10$: Strong evidence that Model 1 is superior to Model 2

$\widehat{\mathsf{BIC}} > 10$: Very strong evidence that Model 1 is superior to Model 2

Exercise: Two competing models for globular cluster LF in the Galaxy

1. A Gaussian model (van den Bergh, 1985)

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

2. A t-distn. model (Secker 1992, AJ 104)

$$g(x;\mu,\sigma,\delta) = \frac{\Gamma(\frac{\delta+1}{2})}{\sqrt{\pi\delta}\,\sigma\,\Gamma(\frac{\delta}{2})}\left[1+\frac{(x-\mu)^2}{\delta\sigma^2}\right]^{-\frac{\delta+1}{2}}$$

$-\infty < \mu < \infty, \sigma > 0, \delta > 0$

In each model, μ is the mean and σ^2 is the variance. In Model 2, δ is a shape parameter.

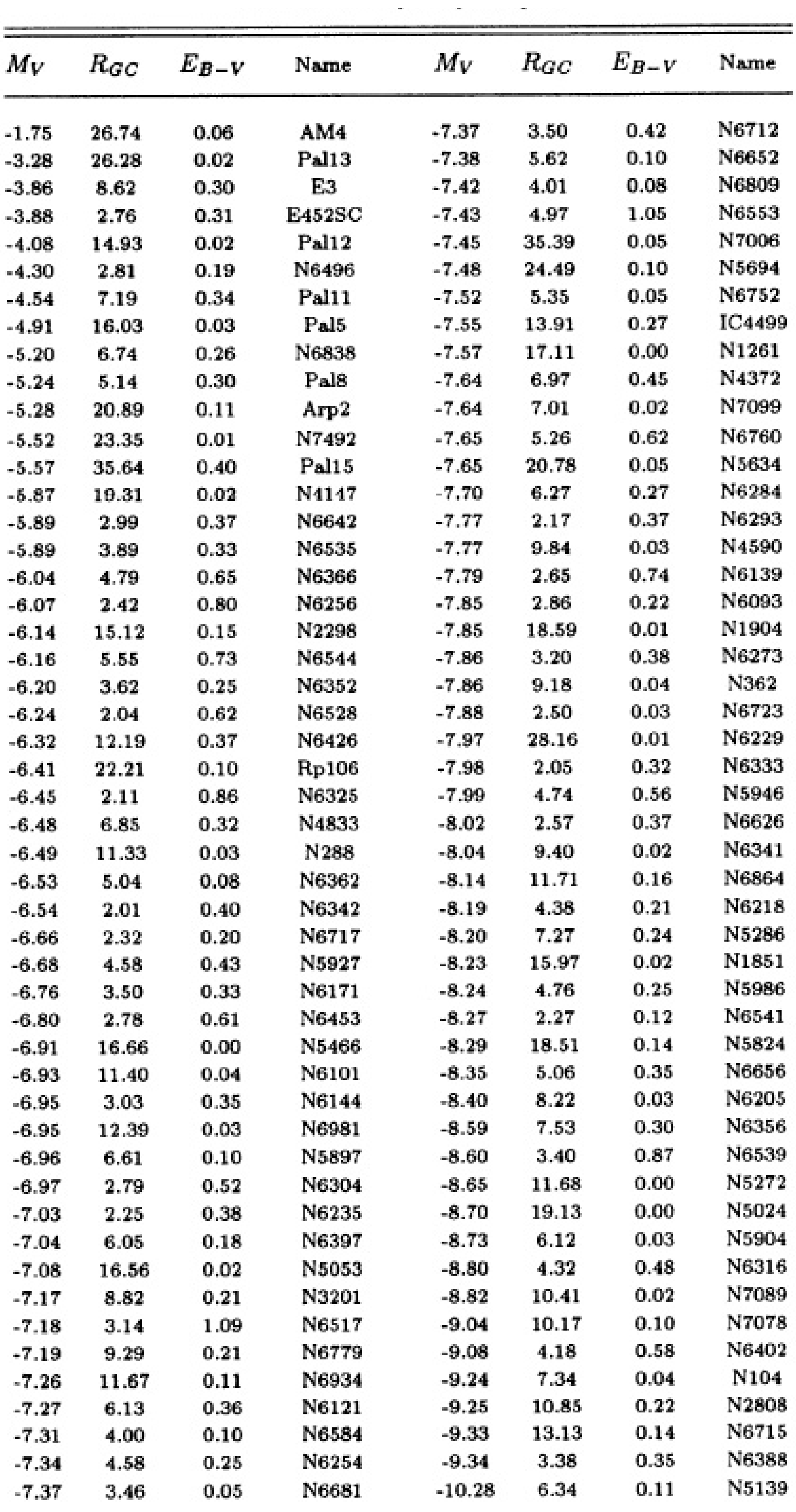
Maximum likelihood calculations suggest that Model 1 is inferior to Model 2.

Question: Is the increase in likelihood due to larger number of parameters?

This question can be studied using the BIC.

We use the data of Secker (1992), Table 1

We *assume* that the data constitute a *random sample*



M_V	R_{GC}	E_{B-V}	Name	M_V	R_{GC}	E_{B-V}	Name
-1.75	26.74	0.06	AM4	-7.37	3.50	0.42	N6712
-3.28	26.28	0.02	Pal13	-7.38	5.62	0.10	N6652
-3.86	8.62	0.30	E3	-7.42	4.01	0.08	N6809
-3.88	2.76	0.31	E452SC	-7.43	4.97	1.05	N6553
-4.08	14.93	0.02	Pal12	-7.45	35.39	0.05	N7006
-4.30	2.81	0.19	N6496	-7.48	24.49	0.10	N5694
-4.54	7.19	0.34	Pal11	-7.52	5.35	0.05	N6752
-4.91	16.03	0.03	Pal5	-7.55	13.91	0.27	IC4499
-5.20	6.74	0.26	N6838	-7.57	17.11	0.00	N1261
-5.24	5.14	0.30	Pal8	-7.64	6.97	0.45	N4372
-5.28	20.89	0.11	Arp2	-7.64	7.01	0.02	N7099
-5.52	23.35	0.01	N7492	-7.65	5.26	0.62	N6760
-5.57	35.64	0.40	Pal15	-7.65	20.78	0.05	N5634
-5.87	19.31	0.02	N4147	-7.70	6.27	0.27	N6284
-5.89	2.99	0.37	N6642	-7.77	2.17	0.37	N6293
-5.89	3.89	0.33	N6535	-7.77	9.84	0.03	N4590
-6.04	4.79	0.65	N6366	-7.79	2.65	0.74	N6139
-6.07	2.42	0.80	N6256	-7.85	2.86	0.22	N6093
-6.14	15.12	0.15	N2298	-7.85	18.59	0.01	N1904
-6.16	5.55	0.73	N6544	-7.86	3.20	0.38	N6273
-6.20	3.62	0.25	N6352	-7.86	9.18	0.04	N362
-6.24	2.04	0.62	N6528	-7.88	2.50	0.03	N6723
-6.32	12.19	0.37	N6426	-7.97	28.16	0.01	N6229
-6.41	22.21	0.10	Rp106	-7.98	2.05	0.32	N6333
-6.45	2.11	0.86	N6325	-7.99	4.74	0.56	N5946
-6.48	6.85	0.32	N4833	-8.02	2.57	0.37	N6626
-6.49	11.33	0.03	N288	-8.04	9.40	0.02	N6341
-6.53	5.04	0.08	N6362	-8.14	11.71	0.16	N6864
-6.54	2.01	0.40	N6342	-8.19	4.38	0.21	N6218
-6.66	2.32	0.20	N6717	-8.20	7.27	0.24	N5286
-6.68	4.58	0.43	N5927	-8.23	15.97	0.02	N1851
-6.76	3.50	0.33	N6171	-8.24	4.76	0.25	N5986
-6.80	2.78	0.61	N6453	-8.27	2.27	0.12	N6541
-6.91	16.66	0.00	N5466	-8.29	18.51	0.14	N5824
-6.93	11.40	0.04	N6101	-8.35	5.06	0.35	N6656
-6.95	3.03	0.35	N6144	-8.40	8.22	0.03	N6205
-6.95	12.39	0.03	N6981	-8.59	7.53	0.30	N6356
-6.96	6.61	0.10	N5897	-8.60	3.40	0.87	N6539
-6.97	2.79	0.52	N6304	-8.65	11.68	0.00	N5272
-7.03	2.25	0.38	N6235	-8.70	19.13	0.00	N5024
-7.04	6.05	0.18	N6397	-8.73	6.12	0.03	N5904
-7.08	16.56	0.02	N5053	-8.80	4.32	0.48	N6316
-7.17	8.82	0.21	N3201	-8.82	10.41	0.02	N7089
-7.18	3.14	1.09	N6517	-9.04	10.17	0.10	N7078
-7.19	9.29	0.21	N6779	-9.08	4.18	0.58	N6402
-7.26	11.67	0.11	N6934	-9.24	7.34	0.04	N104
-7.27	6.13	0.36	N6121	-9.25	10.85	0.22	N2808
-7.31	4.00	0.10	N6584	-9.33	13.13	0.14	N6715
-7.34	4.58	0.25	N6254	-9.34	3.38	0.35	N6388
-7.37	3.46	0.05	N6681	-10.28	6.34	0.11	N5139

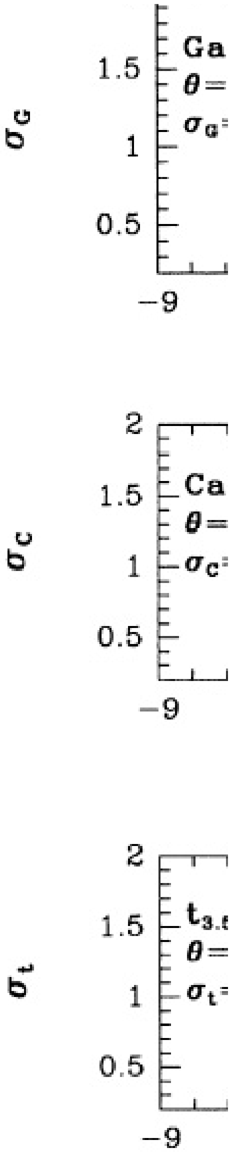


FIG. 1. Maximun
GCLF expressed
parameter space,
ing considered. Tl
eters are given in
contours represen
2.0, 2.5, and 3.0 s
the maximum-like

Model 1: Write down the likelihood function,

$$\begin{aligned} L_1(\mu, \sigma) = & \ f(X_1; \mu, \sigma) \cdots f(X_n; \mu, \sigma) \\ = & \ \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right] \end{aligned}$$

Estimate μ with $\bar{X}$, the ML estimator. Also, estimate σ^2 with S^2, a constant multiple of the ML estimator of σ^2.

Note that

$$\begin{aligned} L_1(\bar{X}, S) = & \frac{1}{(2\pi S^2)^{n/2}} \exp\left[-\frac{1}{2S^2} \sum_{i=1}^{n} (X_i - \bar{X})^2\right] \\ = & (2\pi S^2)^{-n/2} \exp(-(n-1)/2) \end{aligned}$$

Calculate $\bar{x}$ and s^2, the sample mean and variance of the Milky Way data. Use these values to calculate $L_1(\bar{x}, s)$

Secker (1992, p. 1476): $\ln L_1(\bar{x}, s) = -176.4$

Model 2: Write down the likelihood function,

$$\begin{aligned} L_2(\mu, \sigma, \delta) =& g(X_1; \mu, \sigma) \cdots g(X_n; \mu, \sigma) \\ =& \prod_{i=1}^{n} \frac{\Gamma(\frac{\delta+1}{2})}{\sqrt{\pi\delta}\,\sigma\,\Gamma(\frac{\delta}{2})} \Big[1 + \frac{(X_i - \mu)^2}{\delta\sigma^2}\Big]^{-\frac{\delta+1}{2}} \end{aligned}$$

Are the MLEs of μ, σ^2, δ unique?

No explicit formulas for the MLEs are known; we must evaluate them numerically

Substitute the Milky Way data for the X_i's in the formula for L, and maximize L numerically.

Secker (1992): $\widehat{\mu} = -7.31$, $\widehat{\sigma} = 1.03$, $\widehat{\delta} = 3.55$

Calculate $L_2(-7.31, 1.03, 3.55)$

Secker (1992, p. 1476):

$$\ln L_2(-7.31, 1.03, 3.55) = -173.0$$

Finally, calculate the estimated BIC:

$$\widehat{\text{BIC}} = 2 \ln \frac{L_1(\bar{x}, s)}{L_2(-7.31, 1.03, 3.55)} - (m_1 - m_2) \ln n$$

where $m_1 = 2$, $m_2 = 3$, $n = 100$

$$\begin{aligned}\widehat{\text{BIC}} = &\ 2[\ln L_1(\bar{x}, s) - \ln L_2(-7.31, 1.03, 3.55)] \\ &+ \ln 100 \\ = &\ 2[-176.4 - (-173.0)] + \ln 100 \\ = &\ -2.2\end{aligned}$$

Apply the General Rules on p. 30 to assess the strength of the evidence that Model 1 may be superior to Model 2.

Since $\widehat{\text{BIC}} < 2$, we have very strong evidence that the t-distribution model is superior to the Gaussian distribution Model.

We reject the null hypothesis that the t-distribution model for GCLF is superior to the Gaussian model.

Concluding general remarks on the BIC

The BIC procedure is consistent: If Model 1 is the true model then, as $n \to \infty$, the BIC will determine that it is.

Not all information criteria are consistent.

The BIC is not a panacea; some authors recommend that it be used in conjunction with other information criteria.

There are also difficulties with the BIC

Findley (1991, Ann. Inst. Statist. Math.) studied the performance of the BIC for comparing two models with different numbers of parameters: "Suppose that the log-likelihood-ratio sequence of two models with different numbers of estimated parameters is bounded in probability. Then the BIC will, with asymptotic probability 1, select the model having fewer parameters."