Homework Voortgezette Analyse - Functional Analysis

Series 1

Deadline: *Thursday 20 September, 2007*

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1. Let (M, d) be a metric space. Prove that if $f, g \in C(M)$ and $\lambda \in \mathbb{C}$, then $f + g$, fg and λf are also elements of $C(M)$. (Remark: this shows that $C(M)$ is an algebra under pointwise operations, which is in fact true in the more general situation where M is a topological space).

2. Let (M, d) be a metric space. For $x \in M$ and $r > 0$ define the open ball $B_x(r)$ with center x and radius r as
$$B_x(r) = \{y \in M \mid d(x, y) < r\}$$
and define the closed ball $\overline{B_x(r)}$ with center x and radius r as
$$\overline{B_x(r)} = \{y \in M \mid d(x, y) \leq r\}.$$
It is not true in general that $\overline{B_x(r)}$ is the closure of $B_x(r)$. Show this by giving an example. (There is a mistake at this point in R&Y—listed in the errata—which leads to a somewhat confusing, if not incorrect, notation. Fortunately, in the normed linear spaces in which we will be working equality *does* hold, so we need not be worried.)

3. Let $\{f_n\}_{n=0}^{\infty} \subset C[a, b]$ be pointwise bounded, i.e., for every $x \in [a, b]$ there exists $M_x \geq 0$ such that $|f_n(x)| \leq M_x$ for all $n = 0, 1, 2, \ldots$. Prove that there exists a subinterval of $[a, b]$ on which the f_n are uniformly bounded, i.e, that there exist $a \leq c < d \leq b$ and $M \geq 0$ such that $|f_n(x)| \leq M$ for all $x \in [c, d]$ and all $n = 0, 1, 2, \ldots$.

4. Let M be a metric space and let $C_b(M)$ be the collection of all bounded continuous functions on M. We supply $C_b(M)$ with the uniform metric, i.e.,
$$d(f, g) = \sup_{x \in M} |f(x) - g(x)| \quad (f, g, \in C_b(M)).$$
Then $(C_b(M), d)$ is complete. Prove this. This result generalizes the statement that $C(M)$ is complete in the uniform metric if M is compact. Why?

5. Does the definition

$$d(f,g) = \int_a^b |f'(x) - g'(x)|\,dx \quad (f,g,\in C^1[a,b])$$

yield a metric on $C^1[a,b]$? Proof? Counterexample? Answer the same question for

$$d(f,g) = |f(c) - g(c)| + \int_a^b |f'(x) - g'(x)|\,dx \quad (f,g\in C^1[a,b]),$$

where $c \in [a,b]$ is fixed, as a candidate metric on $C^1[a,b]$.

6. Prove that a compact metric space is separable.
