Homework Voortgezette Analyse - Functional Analysis

Series 2

Deadline: *Thursday 4 October, 2007*

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1. Prove that two norms on a vector space are equivalent if and only if they have the same associated metric topology (i.e., that the associated metrics define the same open sets).

2. For $1 \le p < \infty$ the p–norm on $C[0,1]$ is defined by

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{\frac{1}{p}} \quad (f \in C[0,1]).$$

   For each such finite p, the space $C[0,1]$ is *not* a Banach space in the p–norm (in contrast of course to the case $p = \infty$). Show this. (Hint: consider piecewise linear functions which are equal to zero from 0 to slightly below $\frac{1}{2}$ and equal to one from slightly above $\frac{1}{2}$ to 1.)

3. Consider the following statement:

   Suppose that X is a normed vector space, Y is a linear subspace of X such that $Y \neq X$ and $\epsilon > 0$. Then there exists $x_\epsilon \in X$ such that $\|x_\epsilon\| = 1$ and $d(x_\epsilon, Y) = \inf_{y \in Y} \|x_\epsilon - y\| > 1 - \epsilon$.

   Is this "Riesz' Lemma without the subspace necessarily being closed" true? Proof? Counterexample?

4. Let X be a normed space and suppose $L \subset X$ is a finite dimensional subspace. Then for each x in X the distance $d(x, L)$ of x to L, defined as

$$d(x, L) = \inf_{l \in L} \|x - l\|,$$

   is realized by an element of L, i.e, there exists $l_x \in L$ such that $d(x, L) = \|x - l_x\|$. Prove this. (Hint: you will need something like the Heine-Borel or Bolzano-Weierstraß theorem in L).

5. Let X be a normed space. Suppose that X has the property that a series $\sum_{n=1}^{\infty} x_n$ is convergent in X whenever $\sum_{n=1}^{\infty} \|x_n\|$ is convergent in $\mathbb{R}$ (i.e., that every absolutely convergent series is convergent). Then X is a Banach space. Prove this converse to R&Y 2.30.
