Homework Voortgezette Analyse - Functional Analysis

Series 4

Deadline: *Thursday November 22, 2004*

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1. Let X and Y be normed spaces and suppose $T \in B(X, Y)$. Show that

$$\inf\{k \geq 0 \mid \|Tx\| \leq k\|x\| \ \forall x \in X\} = \sup_{\|x\| \leq 1} \|Tx\|$$

and that the infimum in the left hand side is in fact a minimum. Show in addition that, if $X \neq 0$, these numbers are also both equal to

$$\sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

2. (a) Let X be a normed space and suppose L is a closed linear subspace. Prove that the canonical map $\pi : X \mapsto X/L$ is bounded. What is its norm if $L \neq X$? What is its norm if $L = X$?

   (b) Let X and Y be normed spaces and suppose $T \in B(X, Y)$. If $\pi : X \mapsto X/\mathrm{Ker}\, T$ is the canonical map, then by standard linear algebra there is a unique linear map $\overline{T} : X/\mathrm{Ker}\, T \mapsto Y$ such that $T = \overline{T} \circ \pi$. Show that $\overline{T} \in B(X/\mathrm{L}, Y)$ and that $\|\overline{T}\| = \|T\|$.

3. (a) Let X be a normed space, suppose $L \neq X$ is a closed linear subspace and that $x \in X$ is not in L. Prove that there exists $f \in X'$ such that $\|f\| = 1$, $f(L) = 0$ and $f(x) = d(x, L)$. (Hint: in X/L one has $\|[x]\| = d(x, L)$. Can you use a corollary of the Hahn-Banach theorem in this context?)

   (b) As a corollary of the previous part, prove the following basic theorem: *Let X be a normed space and suppose L is a linear subspace. Then L is dense in X if and only if $\{f \in X' \mid f(L) = 0\} = 0$.*

4. (a) Two normed spaces are called *linearly homeomorphic* if there exists a bounded linear bijection between them which is boundedly invertible. If X and Y are Banach spaces and $T \in B(X, Y)$ is surjective, then $X/\mathrm{Ker}\, T$ and Y are linearly homeomorphic. Prove this; you may use the results of exercise 2. (Note the analogy with isomorphism theorems in algebra).

(b) Give an example of a normed linear space X and an operator $T \in B(X)$ which is injective and surjective, so that T is invertible as a linear map T^{-1}, but where T^{-1} is not bounded.

5. Let $n \geq 1$ be an integer and suppose we are given fixed continuous functions $p_0, \dots, p_{n-1}$ on $[a, b]$ and a fixed point $c \in [a, b]$. It is a standard result from the theory of ordinary diferential equations that, given $q \in C[a, b]$ and $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}$, there exists precisely one $y \in C^n[a, b]$ which solves the initial value problem

$$\begin{cases} y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = q; \\ y^{(k)}(c) = \alpha_k \quad (k = 0, \dots, n-1). \end{cases}$$

If one accepts this existence theorem and also the fact that $C^n[a, b]$ is a Banach space under the norm

$$\|f\| = \sum_{k=0}^{n} \|f^{(k)}\|_\infty \quad (f \in C^n[a, b]),$$

then it is easy to give a functional analytic proof of the fact that y depends continuously (in an appropriate fashion) on q and the initial values α_k ($k = 0, \dots, n$). How does this work? (Of course, one also obtains this continuous dependence as a spin-off from the existence proof.)